Integrability in AdS/CFT
Status and Lessons

Shota Komatsu (CERN)

Strings 2021 in Brazil
What I cannot create, I do not understand.

To learn:
Bethe Ansatz Prob.

Know how to solve every problem that has been solved.

\[ f = u(y, r, a) \]

\[ g = 4(r - z) u(r, z) \]

\[ f = 2|y, z| u(y, a) \]
Disclaimer

I will focus on **N=4 super Yang-Mills in 4d.**

- ABJM, AdS$_3$/CFT$_2$, deformations, **N=2 SCFT**…..
  
  2106.08449 by Pomoni, Rabe, Zoubos
  Dynamical Yang-Baxter eq.

Goals are

- To give a “feeling” of what the **Quantum Spectral Curve** is (and explain applications)
- To explain recent developments involving **D-branes** (e.g. determinant ops)

Things that I will NOT discuss (or will only mention briefly)

- Hexagons for correlation functions, Pentagons for amplitudes / form factor
  talk by Coronado
  review by Vieira in Strings 2017, [Sever, Tumanov, Wilhelm]
- Yangian from defects in large N SQFT
  cf. talk by Dedushenko
  Ishtiaque, Moosavian, Zhou, Costello, Gaiotto, Dedushenko, Oh…..
- Conformal fishnet theories: non-unitary log CFT obtained by deforming N=4 SYM
  Gurdogan, Kazakov, Caetano, Mamroud, Torrents…..
What do we mean by “N=4 SYM is integrable”?

- $\mathcal{O}_k(x_k)$: single-trace non-BPS op

$$\left\langle \prod_k \mathcal{O}_k(x_k) \right\rangle = \left( \cdot + \frac{1}{N^2} \cdot + \frac{1}{N^4} \cdot + \cdots \right) + e^{-N}\left( \cdot + \frac{1}{N^2} \cdot + \cdots \right) + \cdots$$

$$+ e^{-N^2\star} \left( \star + \frac{1}{N^2} \star + \cdots \right) + \cdots + e^{-e^{N^2\star}} + \cdots$$

**Expectation:** we can compute $\cdot$ terms at finite $\lambda$ but not $\star$ terms.

- **Heavier operators:**
  - Some correlation functions involving $\Delta \sim O(N)$ are computable.
  - But operators with $\Delta \sim O(N^2)$ are NOT*.

- As long as operators do not back-react $\text{AdS}_5 \times S^5$, there’s a chance we can compute things using integrability.

*See however attempts [de Mello Koch, Kim, Zyl]
Why should we care?

- First example of solvable interacting gauge theory in 4d.
- String theory on RR-flux background.
  
  Alternative approaches: [Berenstein, Leigh], [Cho, Collier, Yin]  
  [Berkovits, Vafa, Witten], String Field Theory

- Qualitative/quantitative similarities with pure Yang-Mills: e.g. BFKL physics
- Starting point for conformal perturbation/Hamiltonian truncation (at large N).

- Might tell us “how the world sheet theory emerges from gauge theory”
  
  Simpler version of the question: How do bulk/gravity emerge from QFT/QM?

- Hopefully give us hints about the worldsheet dual of pure Yang-Mills?  
  talk by Gaberdiel
Radial direction from fermion bilinear

- Fluctuations on GKP string (= null Wilson lines)
  [Gubser, Klebanov, Polyakov]

\[ \partial \text{AdS} \]

\[ z \]

@ Weak coupling (=gauge), No mode corresponding to the radial fluctuation.

- Radial mode = 2 fermion (threshold) bound state
  [Basso], [Basso, Sever, Vieira]

\[ \bar{\psi} \psi \rightarrow z \]

- A similar mechanism for probe branes discussed in [Ferrari et al, '12-’16]
Status: $\mathcal{O} = \text{tr}(XZ \cdots ZX)^{L}$

Spin chain

2-pt

2002
[Minahan, Zarembo]

$\lambda \ll 1$

S-matrix, asymptotic spectrum

2005
[Beisert]

$\lambda : \text{finite}$
$L : \gg 1$

Thermodynamic Bethe ansatz

2009

$\lambda : \text{finite}$
$L : \text{finite}$

Quantum Spectral Curve

2013
[Gromov, Kazakov, Leurent, Volin]

Hexagon

2010
[Gromov, Escobedo, Sever, Vieira]

Cf. [Okuyama, Tseng] [Alday, David, Gava, Narain] [Roiban, Volovich]

2015
[Basso, SK, Vieira]

Generalized to higher pt & Nonplanar

Determinant operators

2019-2020
[Jiang, SK, Vescovi]

2023?
Main message

N=4 SYM is....

Spin chain

Large N gauge theory / matrix model
So, let’s start with basics….
It all started from here….  

• Minahan and Zarembo found a relation between a 1-loop dilatation operator and a Hamiltonian of spin chain.

\[ \mathcal{O} = \text{tr} \left[ Y Z Y Z Z \cdots \right] \]

\[ D_{1\text{-loop}} \]

\[ \Delta \]

\[ H_{\text{Heisenberg}} \propto \sum_{j} \vec{S}_j \cdot \vec{S}_{j+1} \]

\[ \text{Energy} \]

• The spin chain turns out to be solvable by Bethe ansatz.

N=4 SYM

spin chain

\[ \begin{array}{c}
\downarrow \uparrow \downarrow \uparrow \uparrow \cdots \\
\end{array} \]
Bethe ansatz

\[ H_{\text{Heisenberg}} \propto \sum_j \vec{S}_j \cdot \vec{S}_{j+1} \]

- **Step 1:** Write an ansatz

\[ |p_1, p_2, p_3\rangle = \sum_{n_1 < n_2 < n_3} \Psi_{n_1, n_2, n_3} |\uparrow \cdots \downarrow \cdots \downarrow \cdots\rangle \]

\[ \Psi_{n_1, n_2, n_3} = e^{i(p_1 n_1 + p_2 n_2 + p_3 n_3)} + S(p_1, p_2) e^{i(p_2 n_1 + p_1 n_2 + p_3 n_3)} + S(p_1, p_2) S(p_1, p_3) e^{i(p_2 n_1 + p_3 n_2 + p_1 n_3)} + \text{(permutations)} \]

- **Step 2:** Impose

\[ H |p_1, p_2, p_3\rangle = E |p_1, p_2, p_3\rangle \]

1. \( S(p_1, p_2) = \frac{u_1 - u_2 - i}{u_1 - u_2 + i} \quad \left( e^{ip} \equiv \frac{u + i/2}{u - i/2} \right) \) \text{ Rapidity variable}  
2. \( e^{ip_j L} \prod_{k \neq j} S(p_j, p_k) = 1 \) \text{ Bethe equation}  
3. \( E = \sum_j \frac{1}{u_j^2 + \frac{1}{4}} \) \text{ Energy = sum over energies of magnons}
Finite $\lambda$

\[ O = \text{tr}(XZ \cdots ZX)_L \]

- **Assume** integrability persists at finite $\lambda$: For $L >> 1$, we can write the Bethe equation at finite $\lambda$.

\[ e^{ip_j L} \prod_{k \neq j} S^{(\lambda)}(p_j, p_k) = 1 \]

- $S^{(\lambda)}(p, q)$ can be determined by imposing the **centrally-extended** $\text{SU}(2 \mid 2)^2$ symmetry. \[ [j, S^{(\lambda)}] = 0, \quad j \in \text{SU}(2 \mid 2)^2 + \overline{P + K} \subseteq \text{superconformal sym.} \]
Central charge = **gauge transf** = translation

- $P, K$ are **field-dependent** ("large") gauge transformation
  
  \[ Q(\text{boson}) = (\text{fermion}), \quad Q'(\text{fermion}) = \cdots + g_{YM}[Z, (\text{boson})] \]

  \[ \{Q, Q'\} = P \]

- $P, K$ are **discrete translations** on the spin chain

  \[ P |Z\cdots Y\cdots\rangle \propto |Z\cdots ZY\cdots\rangle - |Z\cdots YZ\cdots\rangle \]

  \[ n \quad n + 1 \quad n \]

Spin chain

Large N gauge theory / matrix model
Finite $L$

\[ O = \operatorname{tr}(XZ\cdots ZX)_L \]

- $S^{(\lambda)}$ determine all perturbative $1/L$ corrections at finite $\lambda$.

- $e^{-L\cdot}$ corrections “can” be computed by **Thermodynamic Bethe ansatz (TBA)** (later)

\[ \text{TBA} \]

\[ \text{Strong coupling Bethe ansatz} \]

- Works for some operators, but practically hard for most operators.....
Quantum Spectral Curve

[Gromov, Kazakov, Leurent, Volin]
Reformulation of Bethe equation

\[ e^{ip_j L} \prod_{k \neq j} S(p_j, p_k) = 1 \iff \left( \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L \prod_{k \neq j} \frac{u_j - u_k - i}{u_j - u_k + i} = 1 \]

- Introduce \( Q(u) \equiv \prod_j (u - u_j) \) Q-function

- Bethe eq is equivalent to
  \[ T(u)Q(u) = (u - i/2)^L Q(u + i) + (u + i/2)^L Q(u - i) \]

  \[ 0 = T(u_j)Q(u_j) = (u_j - i/2)^L Q(u_j + i) + (u_j + i/2)^L Q(u_j - i) \]

- It can be further reformulated into
  \[ u^L \propto Q(u + i/2)\tilde{Q}(u - i/2) - Q(u - i/2)\tilde{Q}(u + i/2) \]

  with \( T(u) \propto Q(u + i)\tilde{Q}(u - i) - Q(u - i)\tilde{Q}(u + i) \)

QQ-relation
(Plucker identities of Grassmannian)
QQ-relation for N=4 SYM

- **QQ-relations** can be generalized to the full N=4 SYM.

\[ Q_{a|j}(u + i/2) - Q_{a|j}(u - i/2) = P_a(u)Q_j(u) \quad (a, j = 1, \ldots, 4) \]

\( e.g. \)

\[ Q_j(u) = Q_{a|j}(u \pm i/2)P^a(u) \quad P_a(u) = Q_{a|j}(u \pm i/2)Q^j(u) \]

- **QQ-relations** do not depend on \( \lambda \). Same relations hold for spin chain at weak coupling.

- To incorporate \( \lambda \)-dependence, we need to use the fact that it is a large N gauge theory / matrix model.
Quantum Spectral Curve for Matrix Model

\[ \int dM \exp \left( -\frac{N}{2g^2} \text{tr}(M^2) \right) = \int \prod_j dm_j \prod_{j<k} (m_j - m_k)^2 e^{\frac{N}{8g^2} \sum_j m_j^2} \]

- Two-point functions of \( \text{tr}(M^L) \) are non-diagonal:

\[ \langle \text{tr}(M^L) \text{tr}(M^{L'}) \rangle_{\text{large } N, \text{ connected}} \neq 0 \quad \text{for } L \neq L' \]

- Diagonalization \quad \text{[Rodriguez-Gomez, Russo ’16]}

\[ \text{tr}(M^L) = \sum_j m_j^L \quad \Rightarrow \quad \sum_j T_L(m_j/g) \]

- “Q-function/Quantum Spectral Curve”

\[ T_L(u/g) = x^L + \frac{1}{x^L} =: Q^{(L)}(u) \quad \text{Chebyshev polynomial} \]

\[ u = g(x + 1/x) \]

Zhukovsky variable
Properties of Q-functions for Matrix Model

\[ Q^{(L)}(u) = x^L + \frac{1}{x^L} \]

1. Most naturally defined on \( x \)-plane, which is a double cover of \( u \).

\[ u = g(x + 1/x) \iff x(u) = \frac{u + \sqrt{u^2 - 4g^2}}{2g} \left( = \frac{1}{\text{resolvent}} \right) \]

2. Charges (=Length of operators) can be read off from \( u \to \infty \).

\[ Q^{(L)}(u) \xrightarrow{u \to \infty} u^L \quad \text{degree L branched covering} \]

cf. [Mulase, Penkava], [Razamat], [Gopakumar]

3. Satisfy the gluing condition (monodromy property).

\[ Q(u) = Q(\tilde{u}) \]
Gluing condition for N=4 SYM

\[ x \rightarrow \frac{1}{x} \]

\[ Q_1(\tilde{u}) \propto (Q_3(u))^* \]
\[ Q_2(\tilde{u}) \propto (Q_4(u))^* \]

- Together with QQ-relations, they determine Q-functions.

- \( \Delta \) can be read off from \( u \rightarrow \infty \):
  \[ Q_1(u) \xrightarrow{u \rightarrow \infty} u^{(\Delta - S)/2} \]
11-loop anomalous dimension of Konishi

\[
\gamma_{11} = -242508705792 + 1076639666208\zeta_3 + 70251466752\zeta_3^2 - 12468142080\zeta_3^3
\]
\[
+ 1463132160\zeta_3^4 - 71663616\zeta_3^5 + 180173002752\zeta_5 - 16655486976\zeta_3\zeta_5
\]
\[
- 24628230144\zeta_3^2\zeta_5 - 2895575040\zeta_3^3\zeta_5 + 19278176256\zeta_5^2 - 9619845120\zeta_3\zeta_5^2
\]
\[
+ 2504494080\zeta_3^2\zeta_5^2 + \frac{882108048384}{175}\zeta_3^3 + 45602231040\zeta_7 + 14993482752\zeta_3\zeta_7
\]
\[
- 12034759680\zeta_3^2\zeta_7 + 1406730240\zeta_3^3\zeta_7 + 30605033088\zeta_5\zeta_7 + 21217637376\zeta_3\zeta_7
\]
\[
- 1309941061632\zeta_7^2 - 13215327552\zeta_5^2 - 4059901440\zeta_3\zeta_5^2 - 69762034944\zeta_9
\]
\[
+ 23284599552\zeta_3\zeta_9 - 3631889664\zeta_3^2\zeta_9 - 11032374528\zeta_5\zeta_9 - 6666706944\zeta_3\zeta_5\zeta_9
\]
\[
- 23148129024\zeta_7\zeta_9 - 10024051968\zeta_3\zeta_9^2 - 54555179184\zeta_7\zeta_9 + \frac{10048541184}{5}\zeta_3\zeta_9
\]
\[
- 726029568\zeta_3^2\zeta_9 - 8975463552\zeta_5\zeta_9 - 22529041920\zeta_7\zeta_9 - \frac{143799322496}{175}\zeta_3\zeta_9
\]
\[
+ \frac{1504385419392}{35}\zeta_3\zeta_13 - 30324602880\zeta_5\zeta_13 - \frac{1511300039581392}{875}\zeta_15 - 41375093760\zeta_3\zeta_15
\]
\[
- \frac{19648414723712}{275}\zeta_7 + 309361358592\zeta_9 - 1729880064\zeta_3^2\zeta_11 - \frac{162093984}{5}\zeta_3\zeta_11
\]
\[
- 131383296\zeta_5\zeta_11^2 + \frac{138107420928}{175}\zeta_3\zeta_11^2 + \frac{3541365344}{35}\zeta_3\zeta_11^2 - \frac{5716780416}{7}\zeta_3\zeta_11^2
\]
\[
- 674832384\zeta_3\zeta_13^3 + 48227088384\zeta_15^2 + \frac{3581880576}{25}\zeta_3\zeta_13^2 + 754974720\zeta_3\zeta_15
\]
\[
- \frac{854924544}{11}\zeta_7 + \frac{4963244544}{25}\zeta_5\zeta_17 + \frac{818159616}{275}\zeta_3\zeta_17 + \frac{175363688448}{1925}\zeta_3\zeta_17
\].

\[Z_\ast^{(*)} = \text{multiple zeta values}\]
Correlation functions from QSC?

- In Gaussian MM, correlation functions are given by integrals of Q-functions:

\[
\langle \mathcal{O}_{L_1} \mathcal{O}_{L_2} \rangle_c = \oint d\mu_2 \, Q^{(L_1)}(u)Q^{(L_2)}(u) \propto \delta_{L_1,L_2}
\]

\[
\langle \mathcal{O}_{L_1} \mathcal{O}_{L_2} \mathcal{O}_{L_3} \rangle_c = \oint d\mu_3 \, Q^{(L_1)}(u_1)Q^{(L_2)}(u_2)Q^{(L_3)}(u_3)
\]

\[
\mathcal{O}_L = \text{tr}(M^L) + \ldots \quad d\mu_2 = \frac{dx}{2\pi i x} \quad d\mu_3 = \frac{dx_1 \, dx_2 \, dx_3}{2\pi i \, 2\pi i \, 2\pi i} \frac{(x_1 + x_2 + x_3 + x_1x_2x_3)(1 + x_1x_2 + x_2x_3 + x_3x_1)}{\prod_j (1 - x_j^2)^2}
\]

They describe a topological subsector of N=4 SYM. cf. [Drukker, Plefka]

- Hope for the same in N=4 SYM….?

\[
\langle \mathcal{O}_1 \mathcal{O}_2 \rangle \sim \oint d\mu_2 \, Q_1(u)Q_2(u)
\]

\[
\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \sim \oint d\mu_3 \, Q_1(u)Q_2(u)Q_3(u)
\]
Correlation functions from QSC?

- Great progress for $d\mu_2$ at **weak coupling** (= rational spin chain) 
  Cavaglia, Gromov, Levkovich-Maslyuk, Ryan, Volin, Grabner, Julius,…
  Derkachov, Olivucci, Manashov, Kozlowski,…

  $\int d\mu_2 \mathcal{Q} \left((\text{charge}) \cdot \mathcal{Q}'\right) = \int d\mu_2 \left((\text{charge}) \cdot \mathcal{Q}\right) \mathcal{Q}' \propto \delta_{\mathcal{Q}\mathcal{Q}'}$

  $GL(n)$, non-compact spin chain, general representations.…

- Simplest case (SL(2) spin chain of length L) 
  Derkachov, Korchemsky, Manashov

  \[
  \langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \int_{-\infty}^{\infty} \left( \prod_{k=1}^{L-1} \frac{dx_k}{2\pi} \frac{Q_1(x_k)Q_2(x_k)}{(\cosh \pi x_k)^L} \right) \prod_{j<k} \sinh \left( \pi(x_j - x_k) \right) (x_j - x_k)
  \]

  Can we derive/generalize using Bethe/gauge correspondence? 
  Nekrasov, Shatashvili

- Next steps: 1. Finite $\lambda$. 2. $d\mu_3$.

  Difficulties: 1. # of integration variables increases with loops.
  2. $d\mu_3 \neq d\mu_2$ (because of double traces)

Results also in fishnet limits and from localization.

Cavaglia, Gromov, Levkovich-Maslyuk 
Giombi, Jiang, SK
D-branes, determinants, g-function

Jiang, SK, Vescovi, Yang, Wu, Wang

de Leeuw, Kristjansen, Zarembo, Wilhelm, Buhr-Mortensen, Ispen, Linardopoulos, Muller

Bajnok, Gombor
Determinant correlators

Jiang, SK, Vescovi ‘19

- A new class of solvable correlation functions.

\[ \langle \det(Z)(x_1) \det(\bar{Z})(x_2) \mathcal{O}(x_3) \rangle \]

non-BPS single-trace

Analog of “baryon-baryon-meson vertex”

D-brane in AdS (giant graviton)

Matrix product state at weak coupling

Worldsheet g-function at finite coupling

- Analogy with FZZT brane in minimal string: \( \det(E - H) \)

\[ \phi^*(E) \]

\( \det(E - H) \): FZZT brane

\[ \det(x - Z) \]: giant graviton
**Weak coupling: Matrix Product State**

- Represent det’s by fermion integrals: \[ \det(Z(x_k)) = \int d\bar{\chi}_k d\chi_k \exp(\bar{\chi}_k Z(x_k) \chi_k) \]
  
  cf. talk by Shenker

- Integrate out N=4 SYM fields (\(Z\))

  \[ \rightarrow \text{effective action for "mesons" } \rho_{kj} \sim (\bar{\chi}_k \chi_j) : \int d\rho \exp(-NS_{\text{eff}}[\rho]) \]
  
  \(2 \times 2\) matrix
  
  cf. [Budzik, Gaiotto today]
  
  Saddles in chiral algebra sector = D-branes in BCOV

- Single-trace operator: \(\mathcal{O} = \text{tr}_{N \times N}(YZYZ\cdots)\)

  \[ \langle \det Z \det \bar{Z} \mathcal{O} \rangle \mapsto \text{tr}_{2 \times 2}(M_Y M_Z M_Y M_Z \cdots) \]

  \[ = \left\langle \text{MPS} \mid \mathcal{O} \right\rangle \]

  \[ \langle \text{MPS} \mid = \sum_{s=Y,Z} \langle \cdot_1 \cdot_2 \cdots \mid \text{tr} \left( M_{\cdot_1} M_{\cdot_2} \cdots \right) \]

  \[ 2 \times 2 \]

  cf. [Chen, de Mello Koch, Kim, Van Zyl]
Weak coupling: Result \( \langle \text{MPS} \mid p_1, \ldots \rangle \)

- Studied in stat-mech in the context of quench dynamics:

  Tuchiya, Brockmann, De Nardis, Wouters, Caux, Pozsgay, Vernier, Calabrese, Piroli, …
  
  de Leeuw, Kristjansen, Zarembo, Foda, Wilhelm, Buhr-Mortensen, Ispen, Linardopoulos, Muller

1. \( \langle \text{MPS} \mid p_1, \ldots \rangle \neq 0 \) iff \( \{p_1, \ldots\} = \{p_1, -p_1, p_2, -p_2, \ldots\} \)

   \( \Rightarrow \langle \text{MPS} \mid Q_{2s+1} = 0 \rightarrow \langle \text{MPS} \mid : \text{integrable bdy state} \)

2. \( \langle \text{MPS} \mid p_1, \ldots \rangle \sim \prod_j F(p_j) \times \sqrt{\frac{\det G_+}{\det G_-}} \)

   \( G_{ab}^\pm \sim \frac{\partial \log(\text{Bethe eq for } p_a)}{\partial p_b} \pm \frac{\partial \log(\text{Bethe eq for } p_a)}{\partial (-p_b)} \)

- N=4 SYM also gives generalizations of known formulae.

  \( SL(2) \) spin chain, supergroup spin chain….
Finite coupling: Worldsheet g-function

\[ \langle B \mid \begin{array}{c} \text{L} \\ \text{O} \end{array} \mid \mathcal{O} \rangle \]

(excited state) g-function

Step 1: Determine reflection amplitudes using (integrable) bootstrap

\[ R(p) := \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \]

\[ \text{SU}(2|2)_{\text{diag}} \subset \text{SU}(2|2)^2 \]

Symmetry

Boundary Yang-Baxter

Boundary unitarity

cf. Hofman, Maldacena, …
Finite coupling: Worldsheet g-function

Step 2: Consider a cylinder partition function.

\[
\text{closed string } \sum_{\psi_n} |\langle B | \psi_n \rangle|^2 e^{-RE_n}
\]

• Open string: Thermal partition function at \( \beta = L \):

\[
Z_{\text{open}}(L) \xrightarrow{R \gg 1} \int D\rho_{\text{open}} e^{-L R s_{\text{eff}}[\rho_{\text{open}}]} \]

Density of excitations determined by reflection amplitude

different saddles = \( \sum_{\psi_n} \)

Saddle pt eq = TBA eq

1-loop det = \( |\langle B | \psi_n \rangle|^2 \)
Finite coupling: Result and Fredholm det

\[ \langle B \mid \psi_n \rangle = (\text{prefactor}) \frac{\sqrt{\text{Det} 1 + \hat{G}}}{\text{Det} 1 + \hat{G}_-} \]

depends on reflection amplitudes

\[ \hat{G} \cdot \tilde{f}(u) = \int_{-\infty}^{\infty} \frac{dv}{2\pi i} \frac{\partial_v \log S^{(\lambda)}(u, v)}{1 + 1/Y(v)} \tilde{f}(v) \]

cf. talk by Johnson

- Reproduces and generalizes the weak-coupling answer.
- A variety of quantities can (in principle) be studied by the same methods:
  - 1-point function in the presence of domain wall
    SK, Wang, Bajnok, Gombor, de Leeuw, Kristjansen, Zarembo, Foda, Wilhelm, Buhr-Mortensen, Ispen, Linardopoulos, Muller
  - D-instanton
    Caetano, SK, Wang to appear
  - \( \langle \mathcal{O}_W \rangle \) : Wilson loop
    Jiang, SK, Vescovi, to appear
  - 1-pt function in the Coulomb branch
    Cordova, Coronado, SK, Zarembo, Ivanovsky, Mishnyakov, Terziev, Zaigreev, in progress
  - Other theories.
    Jiang, SK, Wu, Yang
  - And more….
Intriguing observation

Weak coupling (spin chain)  Finite coupling (String)

Bond dimension $d = \# \text{ of boundary bound states}$

- (A part of) world sheet d.o.f can be seen already at weak coupling!

*Determinant $d = 2$, D-instanton $d = 1$, Wilson loop $d = \infty$, domain wall $d \geq 1$, ...
Lessons and Future....
General lessons

• Lesson 1

Large N Feynman diagrams = String world sheet in AdS

Can we see the same at finite temperature? How do we see a horizon from large N diagrams?

• Lesson 2: $\lambda$ expansion has a finite radius of convergence.

From [Dijkgraaf, Heidenreich, Jefferson, Vafa '18]

Double-scaling near the singularity at $-\pi^2$? Connection to de Sitter? [Polyakov]
Open questions

• Can we see a horizon from large N Feynman diagrams?
  \[\text{[Festuccia, Liu '05, '06]} \quad \text{(obstacle: [Linde '80])}\]
  \[\text{cf. [Jafferis, Schneider '21]}\]

• Analyticity in $\lambda$: double scaling, connection to de Sitter?
  \[\text{cf. [Basso, Dixon, Kosower, Krajenbrink, Zhong '21]} \quad \text{[Polyakov '07], [Dijkgraaf, Heidenreich, Jefferson, Vafa '18]}\]

• Rewrite g-function in terms of Q-function / Quantum Spectral Curve (straightforward)

• Quantum Spectral Curve from 4d Chern-Simons? (How do we derive gluing conditions?)
  \[\text{[Costello, Gaiotto, Yagi '21]}\]

• Single-trace three-point functions from Quantum Spectral Curve?
  \[\text{(= Plucker coordinates of Grassmannian)}\]

• Quantum Spectral Curve in chiral algebra sector?

• N=2 SCFT? Veneziano limit? \[\text{[Pomoni, Rabe, Zoubos '21]}\]

• How to efficiently compute non-planar corrections? Analog of topological recursion?
  \[\text{(Perhaps start from topological subsectors)} \quad \text{[Giombi, SK '19]}\]