Quantum Gravity Microstates from Fredholm Determinants

- Clifford V. Johnson (USC)

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- We’re unlikely to fully understand black holes (...to everyone’s satisfaction using approaches too linked to spacetime geometry.
- There have been many hints that geometry is likely to be emergent in a full quantum theory of gravity.

Any tractable model of emergent geometry should be learned from.

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JT gravity, when treated non-perturbatively, is a rich example of one.
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2d quantum gravity, black holes, wormholes...

Non-geometrical

JT gravity on $\mathcal{M}$

Low $T$ quantum dynamics of higher dim black holes
JT gravity (briefly):

\[ I = -\frac{1}{2} \int_\mathcal{M} \sqrt{g} \phi (R+2) - \int_\mathcal{M} \sqrt{h} \phi (K-1) - \frac{S_0}{2\pi} \left( \frac{1}{2} \int_\mathcal{M} \sqrt{g} R + \int_\mathcal{M} \sqrt{h} K \right) \]

Partition function: \( b=1 \)

- \( Z(\beta) = \sum_q Z_g(\beta) + \ldots \)

- \( \int \mathcal{D} \phi \rightarrow R = -2 \); Schwarzian boundary dynamics

- \( Z_0(\beta) = e^{S_0} \frac{e^{\pi^2/\beta}}{4\sqrt{\pi} \beta^{3/2}} \Rightarrow \text{density of states} \]

\[ \rho_0(E) = e^{S_0} \sin h(2\pi\beta E) \frac{1}{4\pi^2} \]

Maldacena, Stanford, Yang, Jensen, Engelsøy-Mertens-Verlinde \( \{16\} \)
Landmark result:

**JT path integral on any $M$**

$e^{iS_0}$

**double-scaled large $N$ random Hermitian matrix model**

$(\frac{1}{N})^{-\chi}$

e.g. $Z(\beta) \leftrightarrow \langle \text{Tr} e^{\beta M} \rangle_{\text{DSL}}$

**Comments:**

- Inherently perturbative, using "topological recursion" properties (Mirzakhani, Eynard-Orantin)

- $V(M)$ not explicitly known; recursion seeded by $\delta(E)$

- Generalization (classification) to other JT (Stanford-Witten '19)

- Some partial non-pert insights, but...
RMM prototype: Gaussian Unitary Ensemble

- Sample hermitian matrices $M$ with probability
  \[ p(M) = e^{-\frac{\text{Tr} M^2}{2}} \]

- Compute eigenvalues, $\lambda_i, i=1 \ldots N$; histogram them:
  \[ \lambda \rightarrow \frac{\lambda}{\sqrt{N}} \quad \text{Wigner semi-circle} \]

- Focus on an endpoint and zoom in:
  \[ \lambda = -2 + s^2 E \frac{E^{1/2}}{N^{2/3}} \]
  \[ \rho_0(E) \sim \frac{1}{N^2} \frac{E^{1/2}}{\pi \hbar} \]
  \[ \hbar = \frac{1}{N^2} \]
  \[ s \rightarrow \infty \quad \text{as} \quad N \rightarrow \infty \]

- The magnification reveals undulations/wiggles:
  \[ \rho(E) = \hbar^{-2/3} \left( \text{Ai}(s)^2 - s \text{Ai}(s)^2 \right) \quad \text{for} \quad s = -\hbar^{-2/3} E \]
  \[ = \rho_0(E) + \rho_1(E) + \rho_2(E) + \ldots \quad \text{(non-pert.)} \]

This is now the “Airy model”. Why? Orthogonal polynomials
The Airy Model

- Rewrite matrix computations in terms of $P_n(\lambda)$, orthogonal w.r.t. $d\lambda e^{-V(\lambda)}$

\[ \int P_n(\lambda) P_m(\lambda) e^{-V(\lambda)} d\lambda = h_n \delta_{nm} ; \quad |n\rangle = \frac{|P_n\rangle}{h_n} \quad \int_0^\infty \langle n|m \rangle = h_{nm} \]

- \[ \lambda P_n(\lambda) = P_{n+1}(\lambda) + R_n P_{n-1}(\lambda) \]

\( \text{recursion relation (even } \nu) \)

\( \text{Gaussian case: } P_n \text{ are Hermite polynomials} \)

\[ 2 \alpha H_n = H_{n+1} + n H_{n-1}, \quad R_n = n \]

(\text{beware convention})

- Everything can be computed from the $R_n$.

- At large $N$: \[ \frac{n}{N} \to x ; \quad P_n(\lambda) \to P(x_1 \lambda) ; \quad R_n \to R(x) \]

- In DSL (zoom an endpoint) scaling pieces survive:

\[ \lambda \to \epsilon \]

\[ x \to x \in \mathbb{R} \]

\[ \frac{1}{N} \to \hbar \]

\[ P \to \psi(E, x) \]

\[ R \to w(x) \]

- \[ H \psi(x, \epsilon) = \epsilon \psi(x, \epsilon) \]

- \[ \rho(E) = \int_0^\infty |\psi(E, x)|^2 dx \]

- Gaussian: \[ w(x) = -x : \quad \psi(E, x) = \hbar^{-\frac{1}{2}} A_i \left(-\frac{x}{\hbar^2 \epsilon^2}\right) \]

\( \text{Brezin-Kazakov, Douglas-Shekov} \)

\( \text{Gross-Migdal, Banks et al., Moore} \)

\( '89, '90 \)
\[ u(E) = -x \]
\[ \psi(E, x) = \hbar^{-\frac{2}{3}} \text{Ai}(-(E+\imath)\hbar^{-\frac{2}{3}}) \]

\[ \rho(E) = \int_{-\infty}^{\infty} |\psi(E, x)|^2 \, dx \]

\[ = \hbar^{-\frac{2}{3}} (\text{Ai}(s)^2 - s \text{Ai}'(s)^2) \quad \text{for} \quad s = -\hbar^{-\frac{2}{3}} E \]

**Comments:**
- Not a theory of surfaces, but illustrates the power of this approach.
- Airy has precursor of problem for SSS model:

States at \( E < 0 \)

non-perturbative instability for full SSS

The picture suggests the cure...
Achieving Non-Perturbative Stability

- Identical perturbation theory (large $E$ regime)
- Achievable with model of complex matrices $M$
- Potential is $V(MM^T)$, so hermitian matrices with $\lambda > 0$
- We instead solve: $u R^2 - \frac{\hbar^2}{2} RR'' + \frac{\hbar^2}{4}(R')^2 = 0$
  ($R$ is polynomial in $x, u, u', u''...$)
- Resulting $\phi(E)$ similar to Airy, but ends at $E=0$
Challenge: Extend this to full JT gravity!

\[ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + u(x) \]

↑ find eqn for this
↓ solve it

↑ solve this
↓ then compute this

\[ H \psi(E, x) = E \psi(E, x) \]

\[ \rho(E) = \int_{-\infty}^{0} |\psi(E, x)|^2 \, dx \]
Comments:

- This is a full non-perturbative completion of JT gravity
- Explicit results, not merely formal
- Can compute useful things with it:
- Same methodology used to tackle many SJT Models of Stanford–Witten.
  CVJ 2005.01893
  CVJ 2008.13120

... and study the JT deformations of Witten, Vaid, Maxfield–Turiaci
CVJ + Rosso 2011.06026

... and complete a different SJT model using a two-cut matrix model
CVJ + Rosso + Svesko 2102.02227

Note: Other NP definitions discussed in SSSS and Gao–Taffers–Kolchemeyer 2104.01184 although so far difficult to work with...

Also note: connections with minimal string + Liouville work of Okuyama et al. Mertens et al. Betsio et al.

Is that it? No!
Revisit the Gaussian/Airy sampling...

- If also keep track of the order of the $\lambda_i$ for each sample:

  - Undulations in $f(E)$ actually built from individual underlying energy levels' statistics!
  - Sharper and more dense as $E \to \infty$ a continuum forms
  - In the full gravity model, these are the microstates!

Recall $e^{-S_0} = h \sim \frac{1}{N}$

i.e., $S_0 \sim \ln N$

Smaller $h$, more “classical”, larger $S_0$
New tool: Fredholm Determinant

- Already have the components: \( \psi(E, x) = \hbar^{-2/3} \text{Ai}(-(E + x) \hbar^{-2/3}) \)

- Assemble into "Kernel" \( K(E', E) = \int_{-\infty}^{0} \psi(x, E) \psi(x, E') \, dx \)
  \[ = \frac{\psi(E) \psi(E') - \psi(E') \psi(E)}{E - E'} \]

- "Airy Kernel" for Airy case.

- JT and variants give new kernels!

- \( f(E) - \int_{a}^{b} dE' K(E, E') f(E') = g(E) \) (Fredholm 1903)

- \( \text{det}(I - K) \) is a natural object
RMM Literature:

Probability of not finding 1st level in \((-\infty, s)\) is:

Probability distribution:

- Other levels found recursively from this.
- Challenging problem, det of \(\infty\)-dim object.
- Bornemann '10 helpful.

\[ E(0; s) = \det \left[ 1 - K \right] \]

\[ F(0; s) = -\frac{dE(0; s)}{ds} \]

Can do this for JT gravity!
JT Gravity Microstates! CVJ 2106.09048

(Also, higher-D black hole microstates!)

- If we have all microstates and their statistics, can compute anything about the model...
JT Gravity Quenched Free Energy

- Need $F_q(T) = -\beta^{-1} \langle \log Z(\beta) \rangle$
to follow thermo to low $T$
- Computation needs wormholes.
- Should need NP physics too. (Matrix model?)
- Partial results at low $T$...
- But can now just reverse engineer the spectrum data to compute by direct ensemble computation...
Comments:

- $F_Q(0) = \langle E_0 \rangle$
- Anticipated in Okuyama 2009-02840
- $F_Q(T) \approx -\frac{1}{2} T^2$...
  Low T prediction Janssen + Mirbabayi 2103.03896
- No replica symmetry breaking. cvi 2008-13120
Final Remarks

Non-perturbative sector captures JT/Black hole microstates!

This is an explicit illustration of transition from geometry to non-geometry in quantum gravity.

Fredholm determinant is a D-brane probe...

What other Random Matrix Model tools might be useful here?

What other quantum gravity questions can be answered with this remarkable tool box?

Thank You!

-cvj