Black holes in AdS5 and phases of 4d SYM

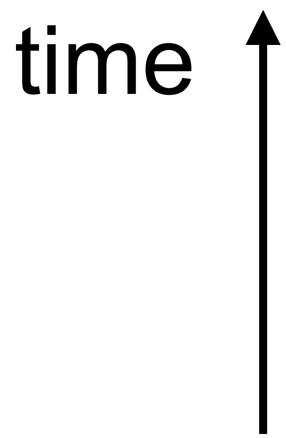
Sameer Murthy
King's College London

European Research Council
Established by the European Commission

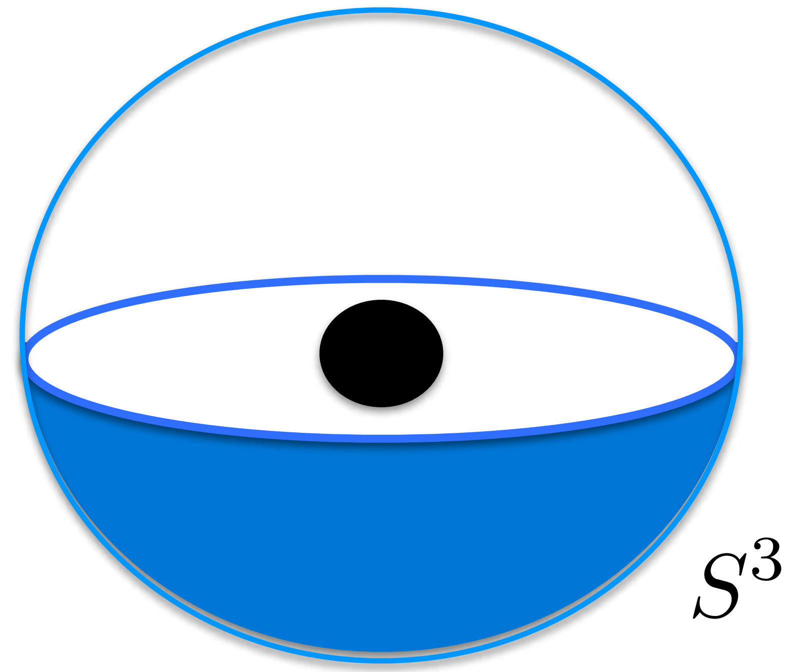
Strings 2021
São Paolo, June 22, 2021

Does the most general susy index in $\mathcal{N}=4$ SYM capture the susy BH entropy in $\mathrm{AdS}_5 \times \mathrm{S}^5$?

$SU(N)$ $\mathcal{N}=4$ SYM



$\frac{1}{N^2} = G_N$



$\mathrm{AdS}_5\,(\times S^5)$

$\frac{1}{16}$-BPS states

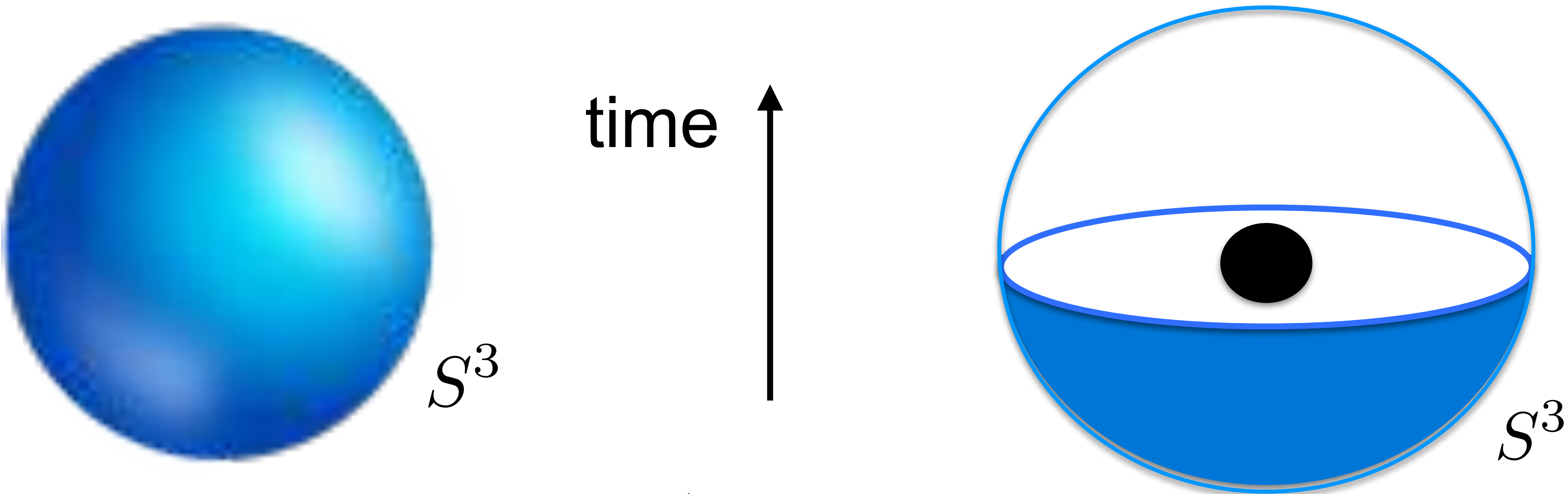
$\frac{1}{16}$-BPS BH

$\mathcal{I}_N(\tau) = \mathrm{Tr}\,(-1)^F \mathrm{e}^{2\pi \mathrm{i}\tau q}$ $\log \mathcal{I}_N \overset{?}{=} O(N^2)$	$S_{\mathrm{BH}} = \dfrac{A_H}{4G_N} = O(N^2)$

[Sundborg '99; Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk '03; Kinney, Maldacena, Minwalla, Raju '05]

[Gutowski, Reall '04; Chong, Cvetic, Lu, Pope '05; Kunduri, Lucietti, Reall '06]

Does the most general susy index in $\mathcal{N}=4$ SYM capture the susy BH entropy in $\mathrm{AdS}_5 \times \mathrm{S}^5$?



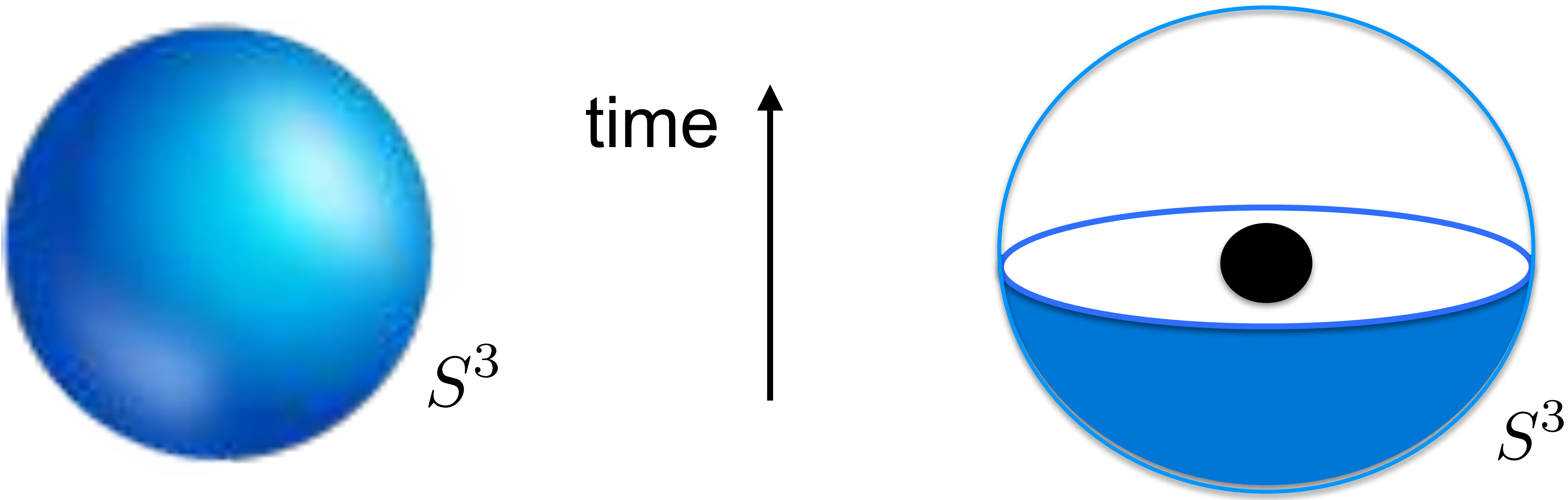
$SU(N)$ $\mathcal{N}=4$ SYM $\qquad \frac{1}{N^2} = G_N \qquad \mathrm{AdS}_5\,(\times S^5)$

$\frac{1}{16}$-BPS states $\qquad \frac{1}{16}$-BPS BH

$\mathcal{I}_N(\tau) = \mathrm{Tr}\,(-1)^F \mathrm{e}^{2\pi \mathrm{i}\tau q}$ $\log \mathcal{I}_N \overset{?}{=} O(N^2)$	$S_{\mathrm{BH}} = \dfrac{A_H}{4G_N} = O(N^2)$

[New progress: Cabo-Bizet, Cassani, Martelli, S.M. '18; Choi, Kim, Kim, Nahmgoong '18; Benini, Milan '18, ….]

Does the most general index in $\mathcal{N}=1$ SCFT4 capture the susy BH entropy in dual AdS5 ?



$\mathcal{N}=1\,\mathrm{SCFT}$ | $\mathrm{AdS}_5\,(\times M_5)$

$\frac{1}{4}$-BPS states | $\frac{1}{4}$-BPS BH

$\mathcal{I}_N(\tau) = \mathrm{Tr}\,(-1)^F \mathrm{e}^{2\pi \mathrm{i}\tau q}$ $\log \mathcal{I}_N \overset{?}{=} O(N^2)$	$S_{\mathrm{BH}} = \dfrac{A_H}{4G_N} = O(N^2)$

[New progress: Cabo-Bizet, Cassani, Martelli, S.M. ’18; Choi, Kim, Kim, Nahmgoong ’18; Benini, Milan ’18, ….]

In the simplest setting, sup.conformal index & BHs are labelled by one quantum number

$\mathcal{N} = 1\ \mathrm{SCFT}$ on S^3

$U(1)_R\ Q$ $\quad SO(4)$

$(2J = J_1 + J_2)\quad (J_1, J_2)$

$$\mathcal{I}_N(\tau) = \mathrm{Tr}\,(-1)^F\, \mathrm{e}^{2\pi \mathrm{i} \tau (2J+Q)}$$
$$= \sum_{n \in \mathbb{Z}} d_N(n)\, \mathrm{e}^{2\pi \mathrm{i}\, \tau\, n/3}$$

$$S_{\mathrm{BH}} = N^2 \sqrt{3q^2 - 4j}$$
$$= N^2\, s(n/N^2)$$

Minimal AdS_5 sugra

$g_{\mu\nu}, A_\mu \qquad G_N = 1/N^2$

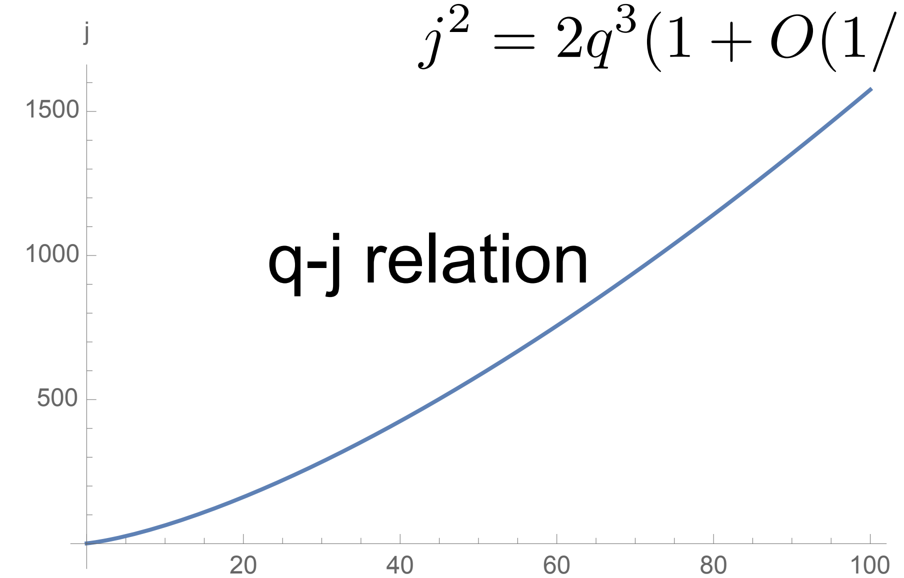
$\frac{1}{4}$-BPS BH charges

$$(J_1, J_2, Q) = N^2 (j_1, j_2, q)$$

Simplest: $j_1 = j_2 = j \neq 0$

[Gutowski, Reall '04]

$j^2 = 2q^3(1 + O(1/\sqrt{q}))$



What is the meaning of BH/exponential growth of states in the index?

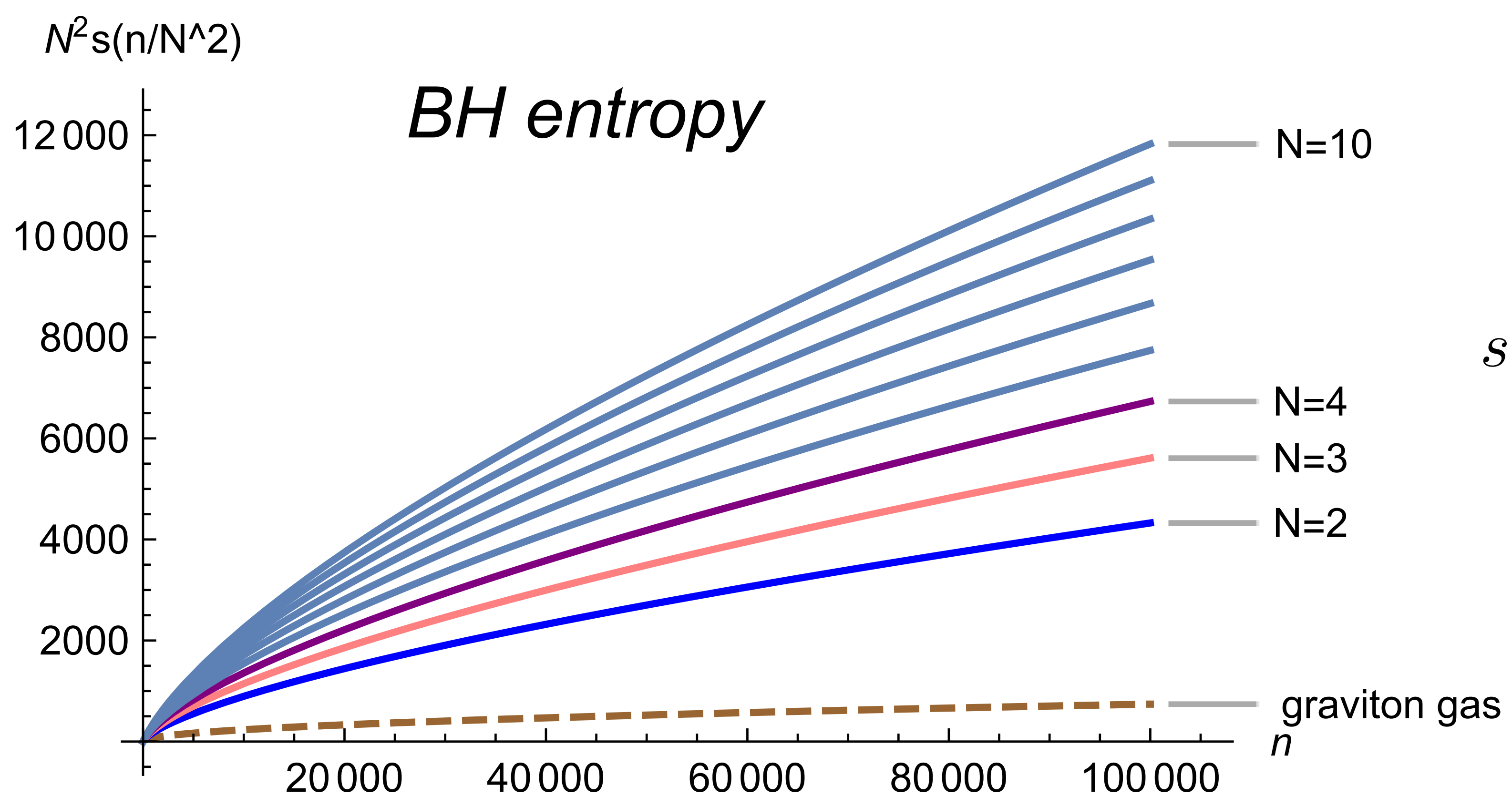
$$\log|d_N(n)| \overset{?}{\sim} N^2 s(n/N^2)$$

Large-N limit

$$n, N \to \infty \quad \text{fixed } \nu = n/N^2$$

Cardy-like limit

$$n/N^2 \to \infty \quad \text{fixed } N$$



$$s(\nu) \underset{\nu\to\infty}{\sim} \frac{\pi}{2\cdot 3^{1/6}}\,\nu^{2/3} + \dots$$

The N=4 superconformal index is an integral over unitary matrices

[Romelsberger '05;
Kinney, Maldacena, Minwalla, Raju '05]

$$\mathcal{I}_N(\tau) = \int [DU] \, \exp\Bigg(\sum_{n=1}^{\infty} \frac{1}{n} f(n\tau) \, \mathrm{tr}\, U^n \, \mathrm{tr}\, (U^\dagger)^n \Bigg)$$

Eigen-
values

$$\{ \mathrm{e}^{2\pi \mathrm{i} u_i} \}_{i=1,\ldots,N}$$

"Single-letter index"

$$f(\mathrm{x}) = \frac{3\,\mathrm{x}^2 - 3\,\mathrm{x}^4 - 2\,\mathrm{x}^3 + 2\,\mathrm{x}^6}{(1-\mathrm{x}^3)^2}$$

$$\underline{u} \equiv \{u_i\}_{i=1,\ldots,N} \qquad \int [D\underline{u}] = \frac{1}{N!} \prod_{i=1}^{N} \int_0^1 \mathrm{d}u_i$$

$$\mathcal{I}_N(\tau) = \int [D\underline{u}] \exp\Bigg(- \sum_{i,j=1}^{N} V_\tau(u_i - u_j) \Bigg)$$

The potential is governed by the *elliptic gamma function*

$$\exp\big(-V_\tau(z)\big) \;=\; \gamma_{\rm e}(z+2\tau;\tau)\;\gamma_{\rm e}\big(z+\tfrac{2}{3}\tau;\tau\big)^3$$

[Dolan-Osborn ’08]

$$\gamma_{\rm e}(z;\tau) \;=\; \prod_{n=1}^{\infty} \frac{(1-q^{n+1}/\zeta)^n}{(1-q^{n-1}\,\zeta)^n}$$

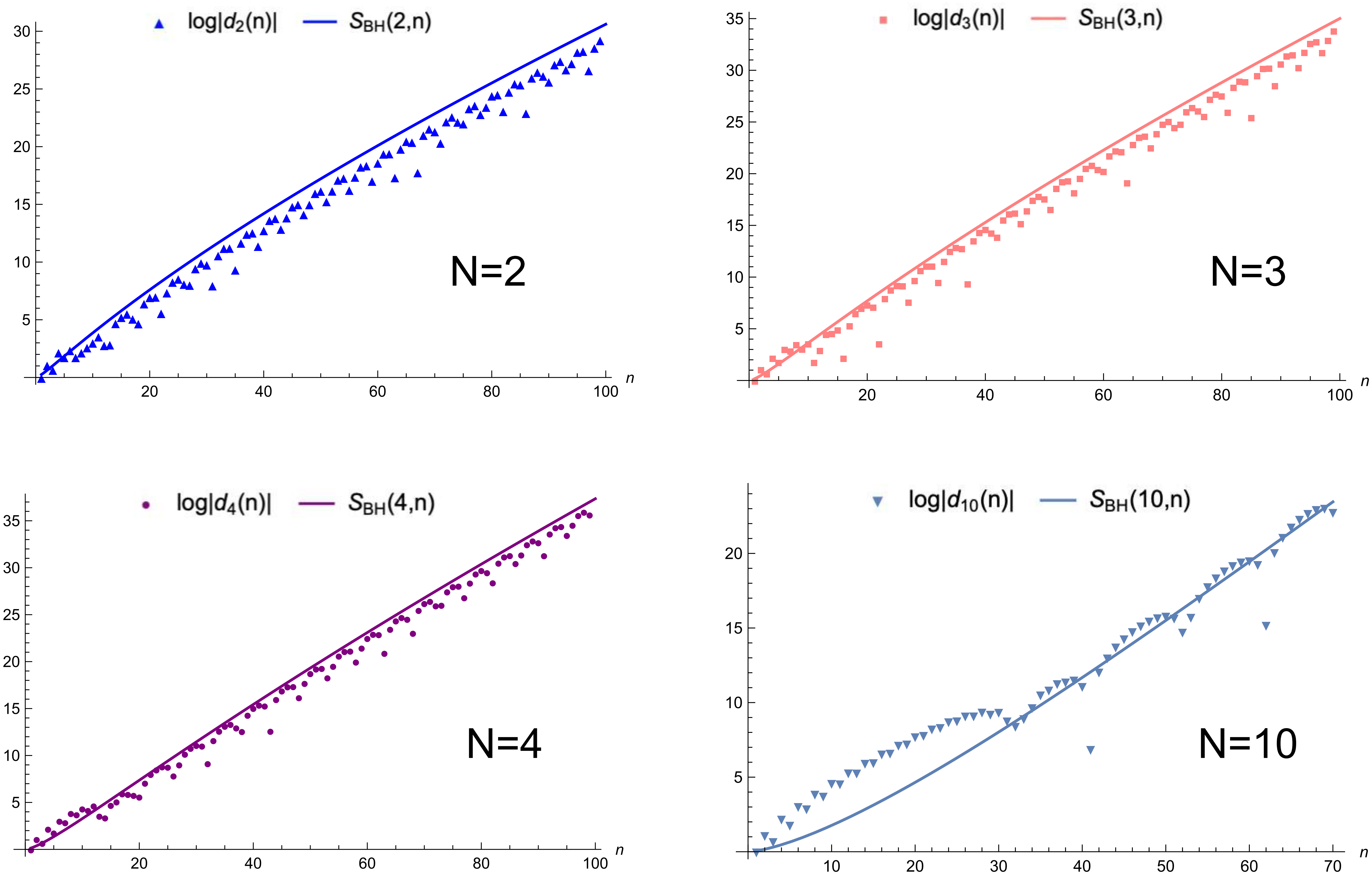
$\tau \in \mathbb{H}$

$q = {\rm e}^{2\pi {\rm i}\tau}$

$\zeta = {\rm e}^{2\pi {\rm i} z}$

Comparison for N=4 SYM (Cardy-like limit)

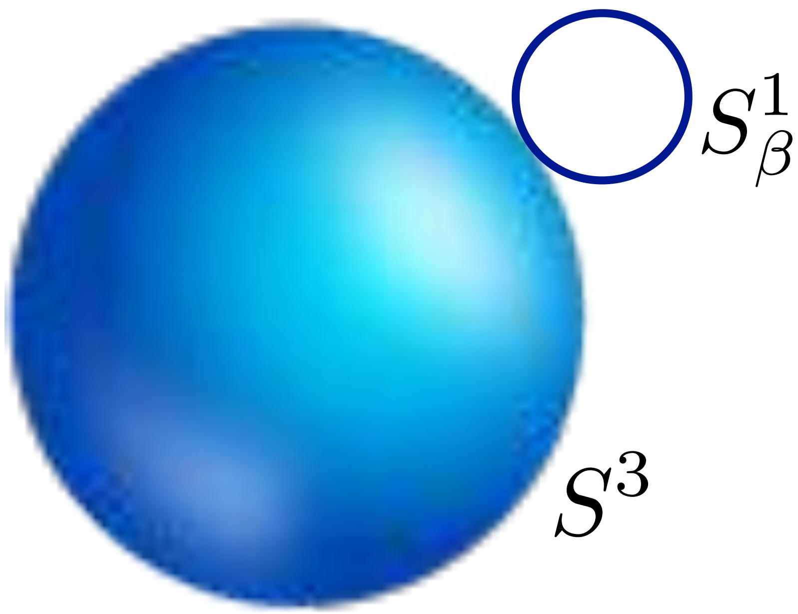
[S.M. '20]



[related numerical work in Agarwal, Choi, Kim, Kim, Nahmgoong '20]

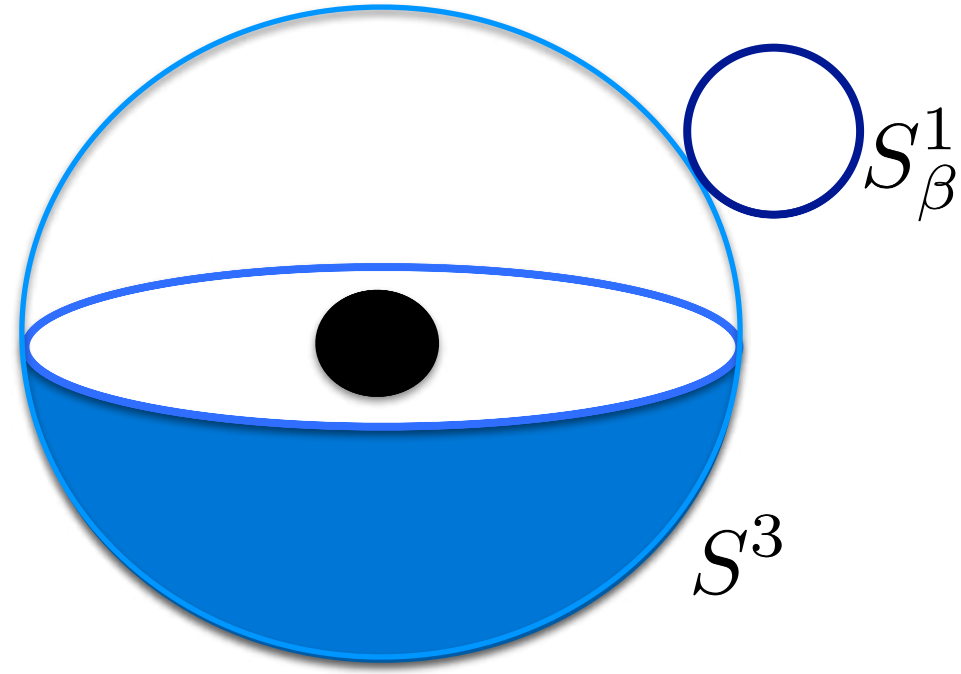
AdS boundary conditions lead to the canonical ensemble $\Longrightarrow$ phase structure

[Gibbons-Hawking-Page '80s]



$$\lambda = \ell_{\mathrm{AdS}}^4$$

$$\frac{1}{N^2} = G_N$$



$$Z_{\mathrm{CFT}}(\lambda, N; \beta) = Z_{\mathrm{AdS}}(\ell_{\mathrm{AdS}}, G_N; \beta)$$

$$= \mathrm{Tr}\, e^{-\beta H} \qquad = \sum_{\alpha \in \mathrm{Saddles}} \exp(-\mathcal{F}^\alpha)$$

$$= \sum_n d_{\mathrm{micro}}(n)\, e^{-\beta E_n} \quad \longleftrightarrow \quad = e^{-\mathcal{F}^{\mathrm{AdS}}} + e^{-\mathcal{F}^{\mathrm{BH}}} + \ldots$$

[Witten '98]

Does susy BH contribute to the gravitational path integral w/ susy boundary conditions?

$$\begin{aligned}
\mathcal{I}_N(\tau) &= \mathrm{Tr}_{\mathcal{H}_{\mathrm{SYM}}}\,(-1)^F\,\mathrm{e}^{2\pi\mathrm{i}\tau n}\,\mathrm{e}^{-\beta H^{\mathrm{susy}}} \\
&= \int_{S^3\times_\tau S^1} DA_\mu D\lambda\,\exp\big(-S_{\mathrm{SYM}}(A_\mu,\lambda)\big) \\
&= \int_{\partial(\mathrm{AdS})\,=\,S^3\times_\tau S^1} Dg_{\mu\nu} D\psi_\mu\,\exp\big(-S_{\mathrm{grav}}(g_{\mu\nu},\psi_\mu)\big)
\end{aligned}$$

- Euclidean BH: infinite throat (in the interior of AdS)

  Temperature and chemical potentials all frozen!

  ⇨ Gibbons-Hawking procedure is singular

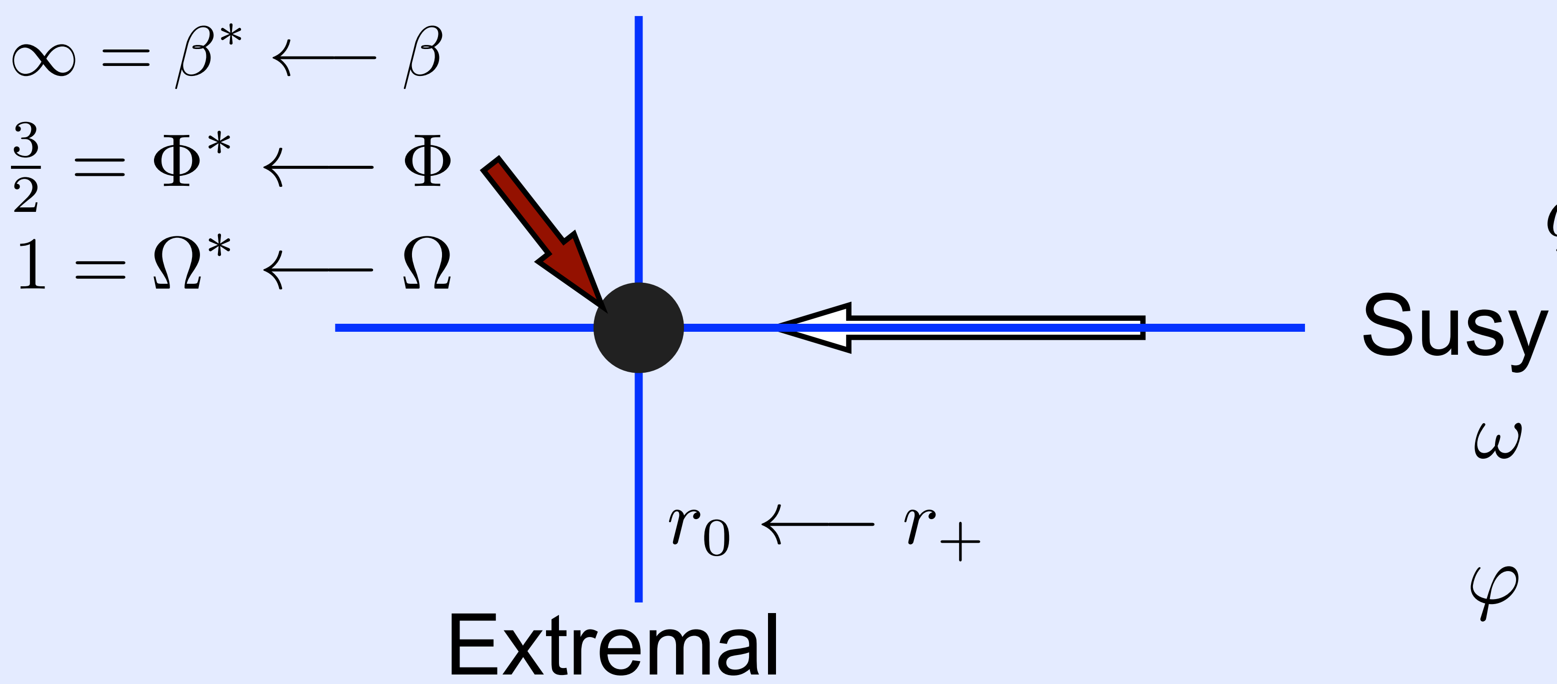
Deform susy extremal BHs to susy non-extremal solutions, …

- Minimal 5d sugra

$$\mathcal{L} = R + \Lambda - F^2 + A \wedge F \wedge F$$

- Four-parameter family of BH solutions: $(a = b, m, q)$

[Gutowski, Reall, ’04] [Chong, Cvetic, Lu, Pope, ’05; Kunduri, Lucietti, Reall, ’06]



$$q = \frac{m}{1 + a + b}$$

$$\omega := \beta(\Omega - \Omega^*)$$

$$\varphi := \beta(\Phi - \Phi^*)$$

$$\omega - \varphi = -\pi \mathrm{i}$$

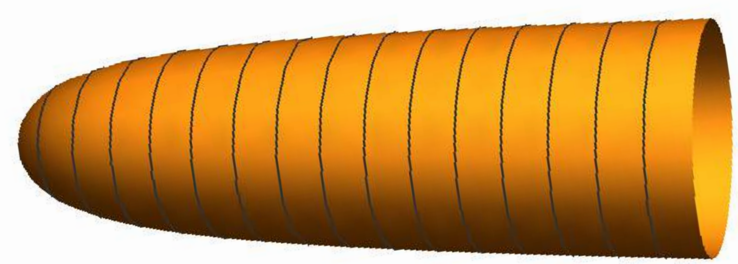
[cf Silva ’06.]

[Cabo-Bizet, Cassani, Martelli, S.M. ’18]

… then take a limit towards the susy BH

- Smooth Euclidean geometry (complex!)

[See Aharony, Benini, Mamroud, Milan, ’21]



$$\oint A^R = \pi \mathrm{i}$$

Note: rules for complex solutions not clear, but this clarifies a bit why index is compared to BH entropy.

- Killing spinor (with R-charge = 1) $\Longrightarrow$ susy preserved in bulk and on boundary

- Gibbons-Hawking procedure now well-defined. BH contribution to index = (limit of) on-shell action.

- Action is β-independent

$$\omega = 2\pi \mathrm{i}\tau$$

$$\mathcal{F}^{\mathrm{BH}} = \frac{\pi \mathrm{i}\, N^2}{27}\frac{(2\tau+1)^3}{\tau^2}$$

[cf Hosseini, Hristov, Zaffaroni ’17 where this function was noted to be the Legendre transform of the BH entropy]

On-shell action of AdS BH = Free energy of matrix model? Other phases?

$$\mathcal{I}_N(\tau) \;=\; \sum_{n\in\mathbb{Z}} d_N(n)\, \mathrm{e}^{2\pi \mathrm{i}\,\tau\, n/3}$$

$$\log|d_N(n)| \;\overset{?}{\sim}\; N^2\, s(n/N^2)$$

$n \to \infty$, fixed N

$n, N \to \infty$, fixed $j = \frac{n}{N^2}$

$\Updownarrow$

$$\log \mathcal{I}_N(\tau) \;\overset{?}{\sim}\; \frac{\pi \mathrm{i}\, N^2}{27}\, \frac{(2\tau+1)^3}{\tau^2}$$

Other singularities?

$\tau \to 0$, fixed N

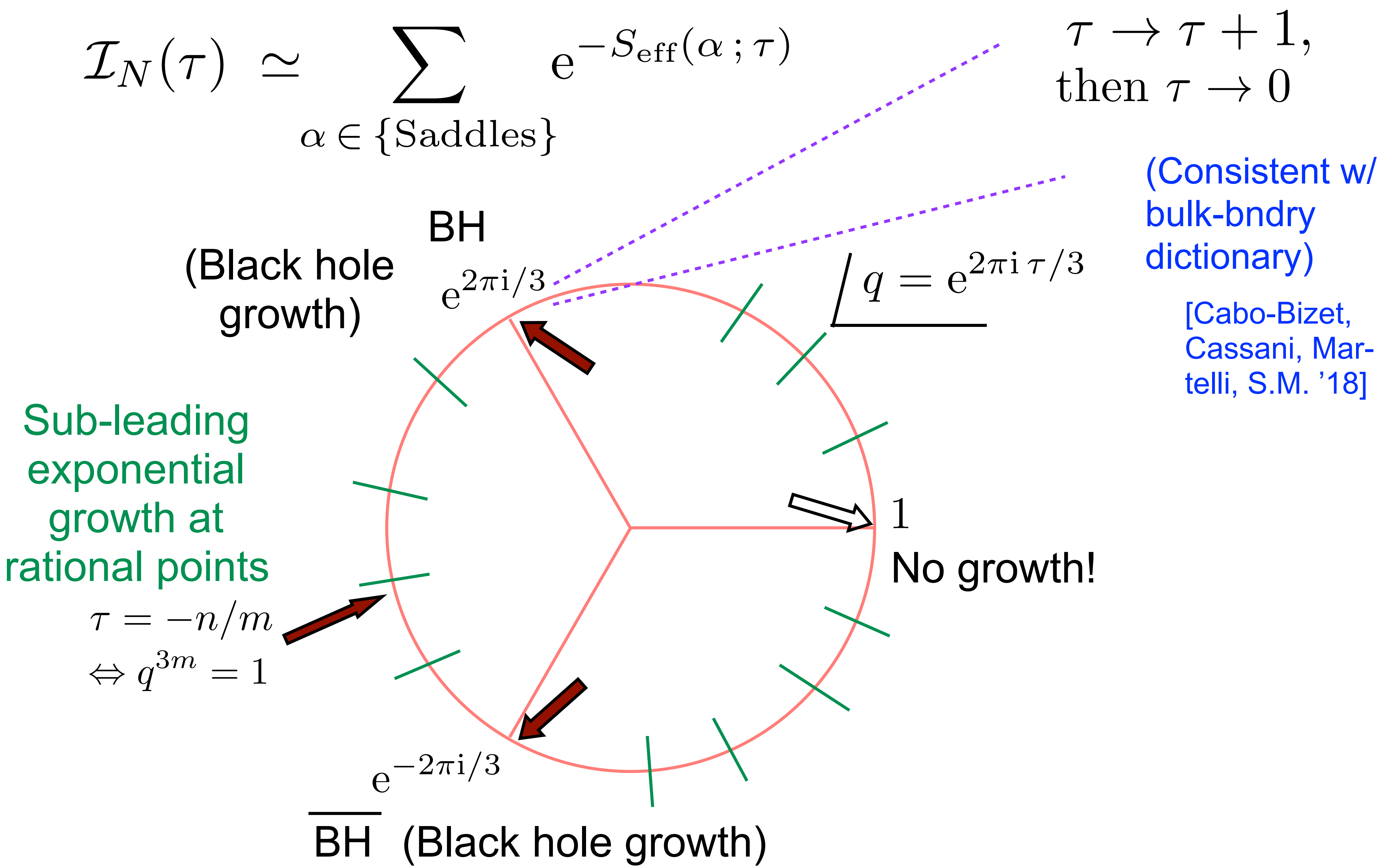
Cardy-like limit

$N \to \infty$, fixed τ

Large-N limit

Saddles lie close to singularities of the index labelled by rational points/roots of unity!

$$\mathcal{I}_N(\tau) \simeq \sum_{\alpha \in \{\text{Saddles}\}} e^{-S_{\text{eff}}(\alpha\,;\,\tau)}$$



1. Perturbation theory around each saddle is calculated from asymptotic analysis

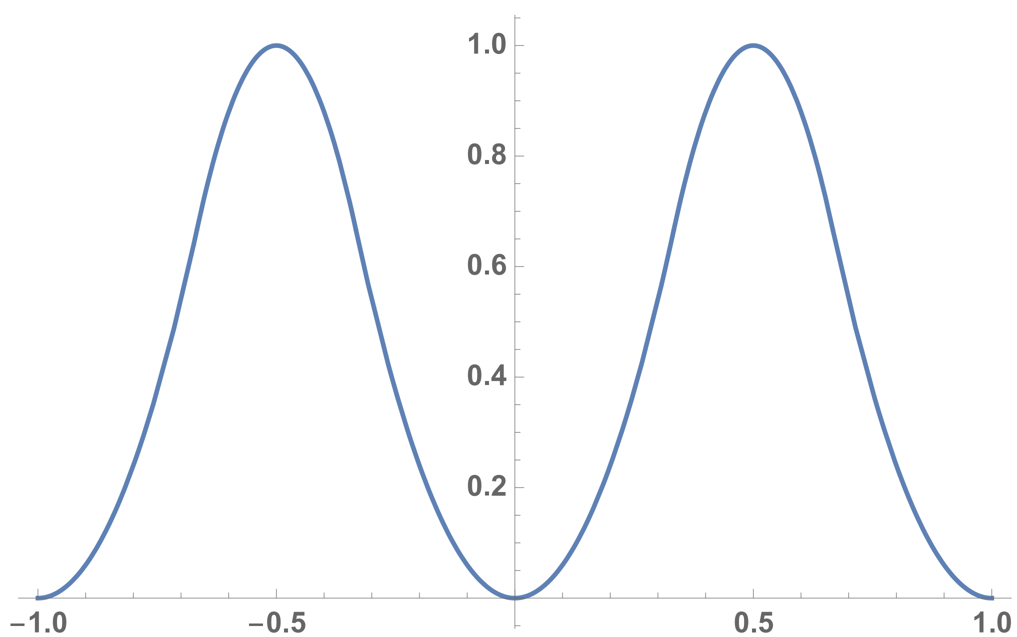
$$\mathcal{I}_N(\tau) \;=\; \int [D\underline{u}] \exp\Big(- \sum_{i,j=1}^{N} V_\tau(u_i - u_j) \Big)$$

$$e^{-V_\tau(z)} = \gamma_e(z + 2\tau; \tau) \times \gamma_e\big(z + \tfrac{2}{3}\tau; \tau\big)^3$$

As $\tau \to 0$,

$$-\frac{1}{2\pi i} \operatorname{Log} \gamma_e(z + r\tau) \;\sim\; \frac{1}{6\tau^2} \overline{B}_3(z) + \frac{1}{6\tau} (r-1) \overline{B}_2(z) + \cdots$$

Periodic Bernoulli polynomials



[Kim, Kim, Song '19]

[Cabo-Bizet, Cassani, Martelli, S.M. '19]

[Flavor: Honda '19; Ardehali, '19]

[Ardehali, Hong, Liu, '19], [Goldstein + Jejjala, Lei, van Leuven, Li '20, '21], …

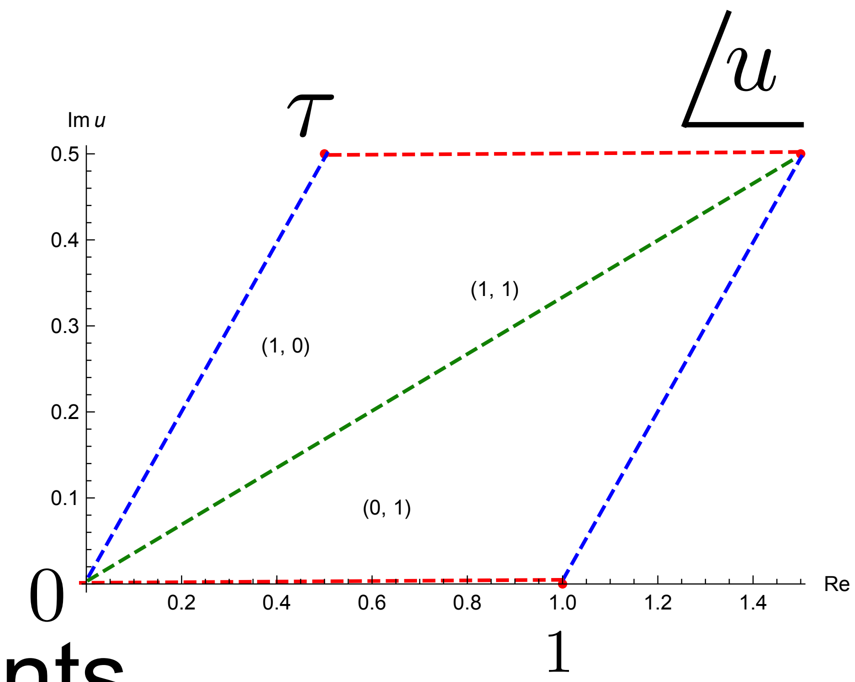
[Gonzalez-Lezcano, Hong, Liu, Pando-Zayas '21]

All-orders

[Ardehali, S.M. '21]

2. Relation to number-theoretic function allows for control at large-N

[S. Bloch, D. Wigner '78]



- Doubly-periodic extension of $V_\tau(u)$ to complex plane (torus) allows us to easily find saddle-point solutions.

Family of saddles labelled by lattice points

$$(m,n) \quad m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \quad \gcd(m,n) = 1$$

- Action of saddle calculated as the integral along a given direction is easy — use Fourier series (Bloch formula)!

[Cabo-Bizet, S.M. '19]

- Meromorphic (multi-valued) version and doubly-periodic (single-valued) version [S. Garoufalidis, S.M., D. Zagier, in progress]

[cf Method 3 (Bethe ansatz) Benini-Milan'18,…]. Answers agree when valid.

The perturbative action of the (m,n) saddle is a simple rational function of τ

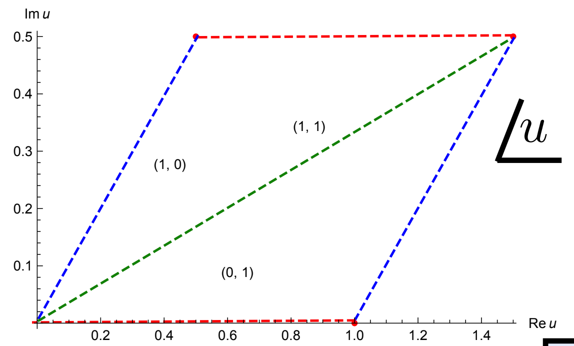
$m \neq 0$

$$S_{\text{eff}}(m,n;\tau) \;=\; \frac{N^2 \pi \mathrm{i}}{27\,m}\,\frac{\left(2\widetilde{\tau}+\chi_1\right)^3}{\widetilde{\tau}^2} + \text{const}\,,\quad \widetilde{\tau} \;=\; m\tau + n$$

$$S_{\text{eff}}(0,1;\tau) \;=\; 0$$

$m+n \pmod 3$	0	1	2
χ_1	0	1	-1

Dirichlet character mod 3.



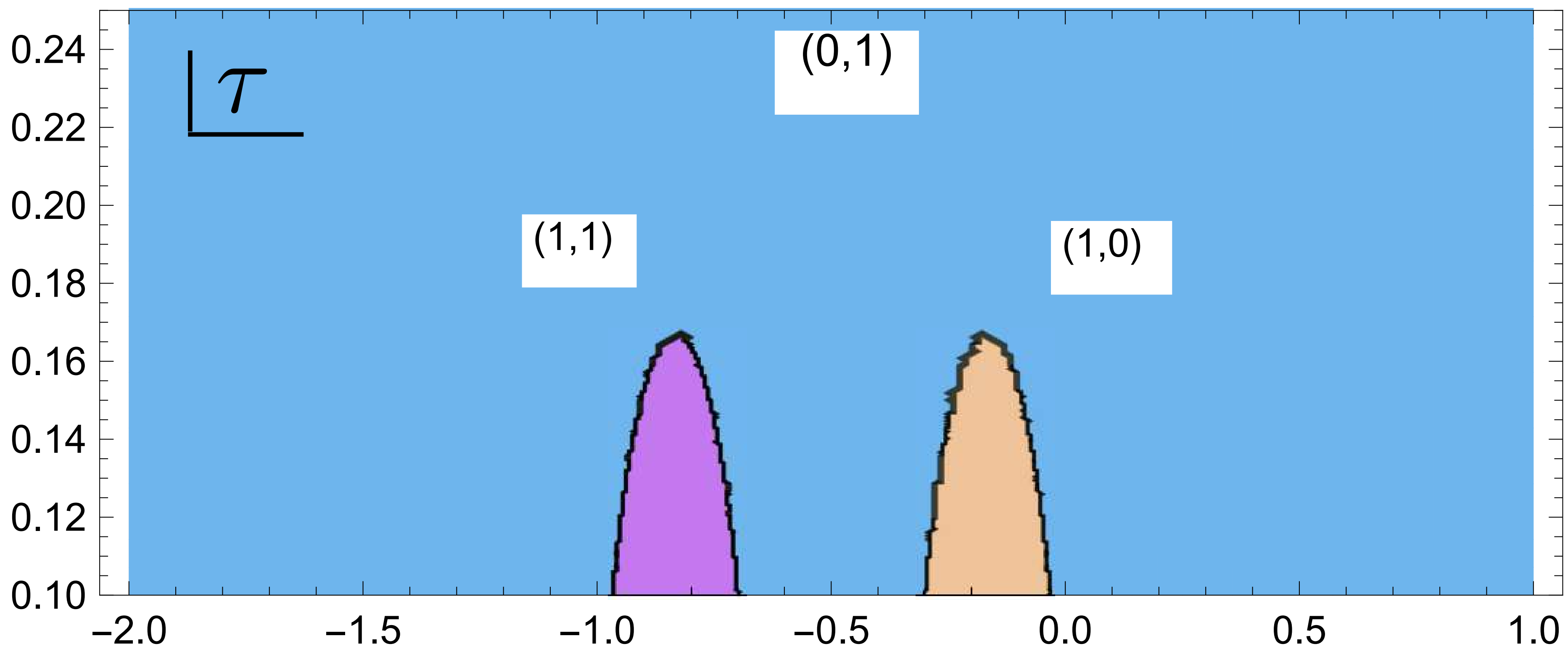
In other words, perturbation theory terminates!

In particular,

$$S_{\text{eff}}(1,0;\tau) \;=\; \frac{N^2 \pi \mathrm{i}}{27}\,\frac{\left(2\tau+1\right)^3}{\tau^2} \;=\; \mathcal{F}^{\text{BH}}(\tau)$$

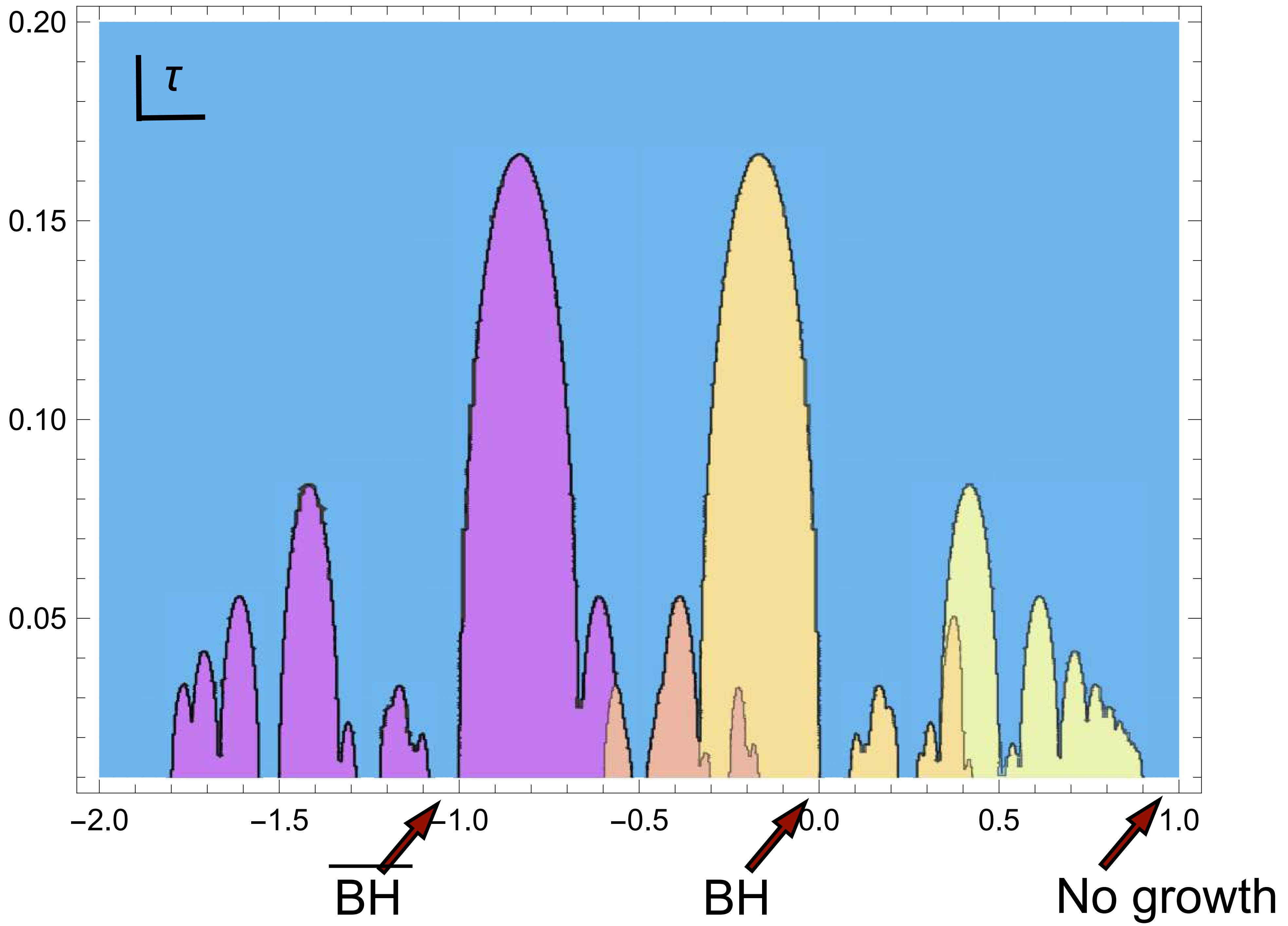
Phase structure: pure AdS dominates near $\tau = \mathrm{i}\infty$, black hole dominates near $\tau = 0$, …

[Cabo-Bizet, S.M. ’19]



… and new phases appear near rational points

[Cabo-Bizet, S.M. ’19]



Comments on formula for index

$$\mathcal{I}_N(\tau) \simeq \sum_{m,n} \exp\big(-S_{\text{eff}}(m,n;\tau)\big)$$

[Cabo-Bizet, S.M. '19]
[" + Cassani, Martelli '20]
[Cabo-Bizet '20]

- Complete asymptotics (perturbation exp) around each saddle. Only chamber-dependence remains as $N \to \infty$
- Prediction for family of gravitational solns (orbifolds)
  See [Aharony, Benini, Mamroud, Milan, '21]
  [cf AdS3 Maldacena-Strominger '98 and Dijgraaf-Maldacena-Moore-Verlinde '00]
- Each term can be derived completely from 3d effective CS field theory! [Ardehali, S.M. '21 (m,n)] [Cassani, Komargodski '21 (1,0)]
  [cf. Choi, Kim, Kim, Nahmgoong '18]

Things I said and would have liked to say

- BH states clearly present. Many phases.
Full perturbative expansion understood.

- *Complex* saddles of matrix model and of gravity are important for full understanding of AdS/CFT.

- In each phase, perturbation theory explained by 3d effective Chern-Simons theory. [I could only touch upon this point.]

- Finite-N index interpolates between gravitons and BHs! cf. [Chang, Yin '13] .
Index = sum over random partitions. cf. [Gross, Taylor '93]
[I did not have time to discuss this point.]

Thank you very much!