On global anomalies of heterotic string theories

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Introduction

String theorists often say

> *String theory is miraculously free of inconsistency*

and when they feel particularly arrogant say

> *String theory is the only consistent theory of quantum gravity*

Are you really sure?

Really?

For example, take Witten's $SU(2)$ anomaly.

In 4d, there is a nontrivial large gauge transformation associated to

$$\pi_4(SU(2)) = \mathbb{Z}_2.$$

If you have N doublet Weyl fermions, this produces a phase

$$(-1)^N.$$

The theory is inconsistent if N is odd.

Let us now ask:

> *Is Witten's* $\boldsymbol{SU(2)}$ *anomaly*
> *absent in 4d heterotic compactifications?*

Somewhat surprisingly, this rather fundamental question is still open!

I learned this question from my colleagues in IPMU
during the teatime slightly before the pandemic.

They studied this question assuming that
there are 4d $\mathcal{N}=2$ spacetime SUSY.

They showed that the anomaly vanishes in many cases,
but they didn't find a universal proof either.

[Enoki-Sato-Watari 2005.01069]

(Texts in purple is hyperlinked if you download the slides.)

Due to a technical reason,
I discuss a related but slightly different global anomaly:

a $\mathbb{Z}_{24}$ global anomaly in 2d heterotic compactifications.

The aim of this talk is to show that it vanishes.

We need to use the mathematical theory of **topological modular forms** and the associated **Segal-Stolz-Teichner conjecture**.

Let me start.

Perturbative heterotic strings

Let us consider heterotic compactification to spacetime dimension d.

The worldsheet theory consists of

	c_L	c_R
(super)ghosts	-26	-15
$X^{0,\dots,d-1}, \bar{\psi}^{0,\dots,d-1}$	d	$\frac{3}{2}d$
internal CFT	$26-d$	$15-\frac{3}{2}d$

In particular, when $d = 10$, the internal CFT should have

$$(c_L, c_R) = (16, 0).$$

There are only two choices: $SO(32)$ and $E_8 \times E_8$.
In both cases, the **perturbative anomaly cancels**.

[Green-Schwarz (1984)].

It then follows that it also cancels automatically for arbitrary smooth **geometric compactifications**.

But there can be **non-geometric compactifications**.

It is known that **perturbative anomaly always cancels** irrespective of whether the internal CFT is semi-classical or not.

[Lerche-Nilsson-Schellekens-Warner (1988)]

What about the **global anomalies**?

In 10d $E_8 \times E_8$, it was shown to vanish
in Witten's "Topological tools in ten dimenions" (1986).

This implies that it also vanishes in all smooth geometric compactifications of $E_8 \times E_8$ heterotic strings.

But there can be non-geometric compatifications,
where the internal CFT does not come from classical geometry.

A global anomaly in 2d

To be concrete, let us consider compactifications to 2d.

The internal CFT has

$$(c_L, c_R) = (24, 12).$$

Massless fermions in the spacetime theory
$\leftrightarrow$ R-sector vacuum states of the internal CFT

Spacetime chirality $\leftrightarrow$ $(-1)^{F_R}$ of the internal CFT

$\Rightarrow$ The elliptic genus of the internal CFT encodes the fermion anomaly.

The elliptic genus of the internal CFT is

$$Z_{\text{ell}}(q) = \mathrm{tr}_R(-1)^{F_R} q^{L_0 - c_L/24}$$
$$= a q^{-1} + b + O(q^1)$$

where

$$a = N^+_{\text{gravitino}} - N^-_{\text{gravitino}}, \qquad b = N^+_{1/2} - N^-_{1/2}.$$

The total fermion anomaly polynomial of the spacetime theory is

$$I_4 = \Big(-24a + b\Big)(-\frac{p_1}{48})$$

where $p_1 = \frac{1}{2}\,\mathrm{tr}(\frac{R}{2\pi})^2$.

String one-loop perturbation theory automatically generates the B-field one point function

$$2\pi i N \int B$$

where

$$N = \frac{1}{8\pi} \int_{\text{fund.reg.}} Z_{\text{ell}}(q) d\mu = -a + \frac{b}{24}.$$

[Vafa-Witten hep-th/9505053] [Sen hep-th/9604070]

Note also that

$$H = dB + CS(\omega)$$

therefore $\int B$ has the same anomalous variation as $I_4 = \frac{p_1}{2}$.

Summarizing, when the elliptic genus of the internal CFT is

$$Z_{\text{ell}}(q) = aq^{-1} + b + O(q^1),$$

the fermion anomaly is

$$I_4 = (-24a + b)(-\frac{p_1}{48})$$

while the B-field coupling is

$$2\pi i(-a + \frac{b}{24}) \int B.$$

Since $\int B$ has the same anomalous variation as $I_4 = \frac{p_1}{2}$, the two contributions to the perturbative anomaly cancel out.

This is the Green-Schwarz mechanism in 2d.

The B-field coupling

$$2\pi i(-a + \frac{b}{24}) \int B$$

still poses a problem when $b/24 \notin \mathbb{Z}$,
since it transforms nontrivially under the large gauge transformation

$$B \to B + 1.$$

As b is naturally an integer, **this is a $\mathbb{Z}_{24}$ global anomaly.**

The central question can now be formulated as follows:

Take an $\mathcal{N}=(0,1)$ SCFT with $(c_L, c_R) = (24, 12)$.

Let's say its elliptic genus is $Z_{\text{ell}} = aq^{-1} + b + O(q^1)$.

Is b always divisible by 24?

This is not a string theory / quantum gravity question.

It is rather a 2d CFT question, a quite peculiar one.

You can go over various 2d compactifications studied in

[Sen hep-th/9604070],

[Font, López hep-th/0405151],

[Florakis, Garcia-Etxebarria, Lust, Regalado 1712.04318].

In all cases, b is divisible by $\mathbf{24}$.

How do we show this in general?

Let us first use the theory of **modular forms**.

Modular forms

Take a 2d $\mathcal{N}=(0,1)$ theory T. Its elliptic genus is

$$Z_{\text{ell}}(T;q) = \mathrm{tr}_R(-1)^{F_R} q^{L_0 - c/24},$$

which is the partition function of the theory
on T^2 with the periodic spin structure.

Therefore, $Z_{\text{ell}}(T;q)$ is modular invariant,
up to subtle phases given by 24-th roots of unity
dictated by the gravitational anomaly $c_R - c_L \neq 0$.

Given a 2d $\mathcal{N}=(0,1)$ theory T with $2(c_R - c_L) = \nu$,
let us **add ν left-moving fermions**, to cancel this worldsheet anomaly.

What we have is then

$$\phi_W(T;q) := \eta(q)^\nu Z_{\text{ell}}(T;q)$$

where

$$\eta(q) = q^{1/24}\prod_n (1-q^n).$$

is the fermion contribution. We now have

$$\phi_W(T;\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{\nu/2}\phi_W(T;\tau).$$

This transformation law defines a modular form of weight $\nu/2$.

(Mathematicians call ϕ_W the Witten genus.)

Let us come back to heterotic compactifications to 2d.

Since $\nu = 2(c_R - c_L) = -24$, we consider

$$\phi_W(T;q) = \eta(q)^{-24} Z_{\text{ell}}(T;q) = \eta(q)^{-24}(aq^{-1} + b + \cdots).$$

This is a modular form
of weight -12
with integer coefficients
and poles of order at most 2.

From this we conclude

$$\phi_W(T;q) = aE_4^3\Delta^{-2} + (-744a + b)\Delta^{-1}$$

where

$$E_4 = \frac{45}{\pi^4}\sum_{(n,m)\neq 0}\frac{1}{(n\tau+m)^4} = 1 + 240q + \cdots$$

and

$$\Delta = \eta(q)^{24}.$$

The theory of modular forms allow us to determine the entire elliptic genus from a and b, but **it does not tell that b is divisible by 24.**

Topological modular forms come to the rescue.

Topological modular forms

The ring of **topological modular forms** generalizes and refines the ring of modular forms.

It was mathematically constructed by Hopkins et al. around 2000, using an amalgam of algebraic topology and algebraic geometry.

[Hopkins math.AT/0212397]

The Segal-Stolz-Teichner conjecture says

$$\mathbf{TMF}_{\nu} = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(\mathbf{0},\mathbf{1}) \text{ supersymmetric theory} \\ \text{with } \boldsymbol{\nu} = \mathbf{2}(\boldsymbol{c_R} - \boldsymbol{c_L}) \end{array} \right\}}{\text{continuous deformation}}$$

[Segal 1988] [Stolz-Teichner 2002] [Stolz-Teichner 1108.0189]

A $\mathcal{N}{=}(\mathbf{0},\mathbf{1})$ SCFT $\boldsymbol{T}$ with $\mathbf{2}(\boldsymbol{c_R} - \boldsymbol{c_L}) = \boldsymbol{\nu}$ should then determine an element

$$[\boldsymbol{T}] \in \mathbf{TMF}_{\nu}.$$

Why is this conjecture plausible?

Mathematicians constructed a map ϕ_W,
which for us extracts the elliptic genus

$$\begin{array}{ccccc} \phi_W: & \mathbf{TMF}_\nu & \to & \left\{ \begin{array}{c} \text{modular forms of} \\ \text{weight } \frac{\nu}{2} \text{ with} \\ \text{integer coeff.s and poles} \end{array} \right\} \\ & \Downarrow & & \Downarrow \\ & \text{a theory } T & \mapsto & \phi_W(T) = \eta(q)^\nu Z_{\text{ell}}(T; q) \end{array}$$

Mathematicians also know how to describe the sigma models:

$$\begin{pmatrix} M: & \nu\text{-dim'l manifold} \\ B: & B\text{-field on } M \end{pmatrix} \mapsto [\sigma(M_\nu, B)] \in \mathbf{TMF}_\nu$$

Mathematicians showed that

$$\phi_W([\sigma(M_\nu, B)]) = \eta(q)^\nu \underbrace{Z_{\text{ell}}(\sigma(M_\nu, B); q)}_{\text{physicists know this part!}}$$

The image of the map

$$\phi_W : \mathbf{TMF}_\nu \to \left\{ \begin{array}{c} \text{modular forms of} \\ \text{weight } \frac{\nu}{2} \text{ with} \\ \text{integer coeff.s and poles} \end{array} \right\}$$

has been mathematically determined.

In particular,

$d\Delta^k$ *is in the image of* ϕ_W *iff* d *is a multiple of* $\dfrac{24}{\gcd(24, k)}$.

We can finally come back to our question

Take an $\mathcal{N}=(0,1)$ *SCFT with* $(c_L, c_R) = (24, 12)$.

Let's say its elliptic genus is $Z_{\text{ell}} = aq^{-1} + b + O(q^1)$.

Is b *always divisible by* 24*?*

Recall that otherwise heterotic compactifications to 2d has a $\mathbb{Z}_{24}$ global anomaly.

We already argued that

$$\phi_W(T;q) = aE_4^3\Delta^{-2} + (-744a + b)\Delta^{-1}$$

and recall

$d\Delta^k$ *is in the image of* ϕ_W *iff* d *is a multiple of* $\dfrac{24}{\gcd(24,k)}$.

Taking $k = -1$, we find that $-744a + b$ is a multiple of

$$\frac{24}{\gcd(24,-1)} = 24.$$

So b is divisible by 24. **Done.**

How about Witten's $SU(2)$ anomaly?

More generally, the question of global anomaly of
heterotic compactifications down to d dimensions
with gauge symmetry G can be translated to the study of

$$\mathrm{TMF}_{-22-d}(BG)_k$$

where k is the level of the current algebra.

They are very hard to compute.

But **there is a way** to show that global anomaly always vanishes,
by **considering all cases at once**,
without doing any case-by-case analyses.

[Work in progress with Yamashita, a mathematician in Kyoto]

Comments

Clearly we are not really done.

We simply transferred

the question of global anomalies of heterotic strings

to

the validity of the Segal-Stolz-Teichner conjecture

$$\mathbf{TMF}_\nu = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(\mathbf{0},\mathbf{1}) \text{ supersymmetric theory} \\ \text{with } \boldsymbol{\nu} = \mathbf{2}(\boldsymbol{c_R} - \boldsymbol{c_L}) \end{array} \right\}}{\text{continuous deformation}}$$

It is important to test the conjecture in our own way.

There are a couple of works:

[Gaiotto, Johnson-Freyd 1811.00589]
[Gaiotto, Johnson-Freyd, Witten 1902.10249]
[Gaiotto, Johnson-Freyd 1904.05788]

Let me add one implication of the conjecture.

As I said, the image of the map

$$\phi_W : \mathbf{TMF}_\nu \to \left\{ \begin{array}{c} \text{modular forms of} \\ \text{weight } \frac{\nu}{2} \text{ with} \\ \text{integer coeff.s and poles} \end{array} \right\}$$

has been mathematically determined.

In particular,

$d\Delta^k$ is in the image of ϕ_W iff d is a multiple of $\dfrac{24}{\gcd(24, k)}$.

One consequence is this:

If the elliptic genus of 2d $\mathcal{N}=(0,1)$ theory is simply $\mathbf{1}$, then $c_L - c_R$ is divisible by $\mathbf{288}$.

Conversely, there should be a 2d $\mathcal{N}=(0,1)$ theory whose elliptic genus is $\mathbf{1}$ and $c_L - c_R = \pm\mathbf{288}$.

That particular theory would be quite a marvelous one.

Summary

Today, I considered **global anomalies in heterotic string theories.**

Such questions can be answered using the **mathematical theory of TMF**, using the **Segal-Stolz-Teichner conjecture**:

$$\mathbf{TMF}_{\nu} = \frac{\left\{ \begin{array}{c} \text{2d } \mathcal{N}{=}(0,1) \text{ supersymmetric theory} \\ \text{with } \nu = 2(c_R - c_L) \end{array} \right\}}{\text{continuous deformation}}$$

This conjecture predicts **many unexplored properties of 2d theories**, which I think are worth pursuing.

The list of hep-th papers on **TMF** is not very long.

The exhaustive list is

Gaiotto, Johnson-Freyd	1811.00589
Gukov, Pei, Putrov, Vafa	1811.07884
Gaiotto, Johnson-Freyd, Witten	1902.10249
Gaiotto, Johnson-Freyd	1904.05788
Johnson-Freyd	2006.02922
YT	2103.12211

It's a young field and new comers are welcomed...