Generalized symmetries, algebras and entropies

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Based on recent works with Marina Huerta, Javier Magan, Diego Pontello

e-Print: 2008.11748 [hep-th]

e-Print: 1905.10487 [hep-th]

Plan of the talk:

A simple and unified perspective on generalized symmetries and completeness.

Structure of the net of operator algebras attached to regions.

Failure in simple properties (additivity/duality) of the algebra-region relations
$\leftrightarrow$ multiple algebras for the same region $\leftrightarrow$ generalized symmetries

Symmetries always come in dual pairs.

Entropic order parameters arise from this non uniqueness.

Certainty relation: relates statistics of dual entropic order parameters.

Expected behaviour of order parameters and some heuristics.

Operator algebras and causal regions

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Algebras:

$\mathcal{A} \longrightarrow \mathcal{A}'$ commutant $\mathcal{A}' = \{b;\ [b,a] = 0, \forall a \in \mathcal{A}\}$

Von Neumann theorem: $\mathcal{A}$ is a (v.n.) algebra $\longleftrightarrow$ $\boxed{\mathcal{A} = \mathcal{A}''}$

Further structure: $\mathcal{A}_1 \cap \mathcal{A}_2$ $\mathcal{A}_1 \vee \mathcal{A}_2 = (\mathcal{A}_1 \cup \mathcal{A}_2)''$

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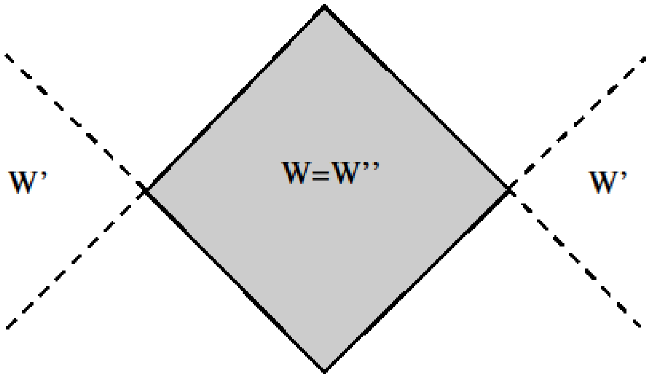
Causal regions:

$W \longrightarrow W'$ causal complement

causal region $\boxed{W = W''}$



Further structure: $R_1 \cap R_2$ $R_1 \vee R_2 = (R_1 \cup R_2)''$

QFT $\longrightarrow$ $\mathcal{A}(R)$

Relations between algebras and regions: minimal requirements for a QFT

$\mathcal{A}(R_1) \subseteq \mathcal{A}(R_2)\,,\qquad R_1 \subseteq R_2$ isotony

$\mathcal{A}(R) \subseteq (\mathcal{A}(R'))'$ causality

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Maximal harmony between algebras and regions

$\mathcal{A}(R) = (\mathcal{A}(R'))'$ Haag duality

$\mathcal{A}(R_1 \vee R_2) = \mathcal{A}(R_1) \vee \mathcal{A}(R_2)$ Additivity

$\mathcal{A}(R_1 \cap R_2) = \mathcal{A}(R_1) \cap \mathcal{A}(R_2)$ Intersection property

"Complete theory"

(Haag's "tentative postulate")

Complete theory $\longleftrightarrow$ assignation $\mathcal{A}(R)$ respects all operations $(\subseteq, \vee, \wedge, ')$

Homomorphism of "orthocomplemented lattices"

Non complete theories?

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- The algebra closes, then the non local operators have fusion rules:

$$a\, a' = \sum_a'' n_{aa'}^{a''}\, a'' \qquad\qquad b\, b = \sum_{b''} \tilde{n}_{bb'}^{b''}\, b''.$$

$\longrightarrow$ generalized symmetries Gaiotto, Kapustin, Seiberg, Willett (2015)

- Types of allowed symmetry algebras depend on the topology of regions and dimension

Completeness $\leftrightarrow$ Additivity + duality $\leftrightarrow$ No generalized symmetries

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Generalized symmetries always appear in dual pairs (because of von Neumann theorem)
This is completely independent of the type of symmetry, dimension or topology .
''Magnetic'' and ''electric'' completeness are the same.

$$
\begin{array}{ccc}
\mathcal{A}_{\rm add}(R) & \subset & \mathcal{A}_{\rm max}(R) \\
{}'\updownarrow & & {}'\updownarrow \\
\mathcal{A}_{\rm max}(R') & \supset & \mathcal{A}_{\rm add}(R')
\end{array}
$$

(and dual non local operators cannot commute to each other)

Example: global symmetries, regions with non trivial homotopy groups π_0 or π_{d-2}

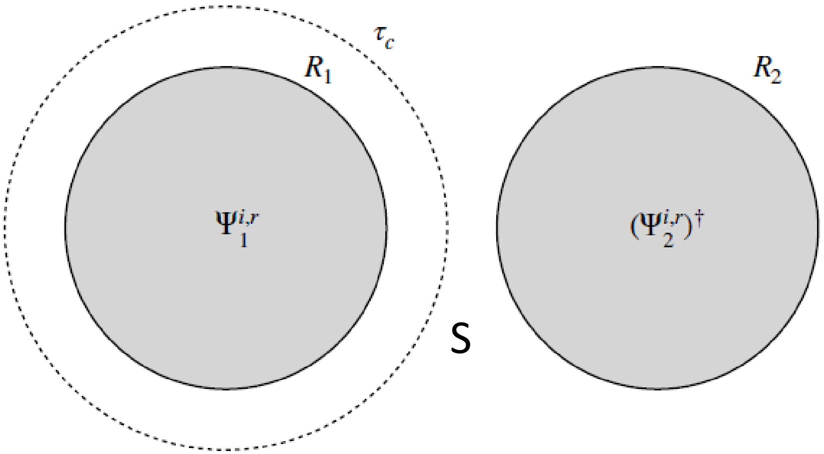
Orbifold of a global symmetry $\mathcal{O} = \mathcal{F}/G$ Take only operators invariant under the group G

(DHR analysis $\mathcal{O} \to \mathcal{F}$)

Non local operators:

Intertwiner (representations) $\mathcal{I}_r = \sum_i \psi_1^{i,r} (\psi_2^{i,r})^\dagger$

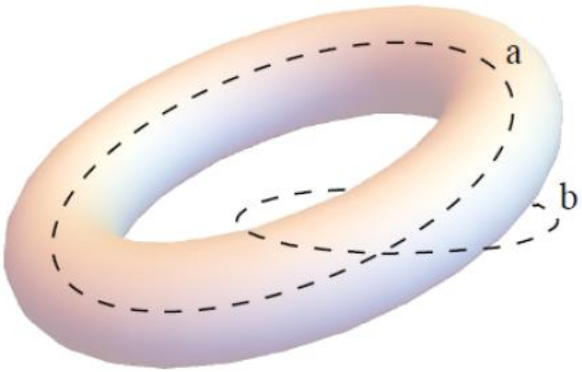
Twists (conjugacy classes) $\tau_c = \sum_{h\in c} \tau_h$



$$(\mathcal{O}_{\rm add}(R_1 R_2))' = \mathcal{O}_{\rm add}(S) \vee \{\tau_c\}\,,$$
$$(\mathcal{O}_{\rm add}(S))' = \mathcal{O}_{\rm add}(R_1 R_2) \vee \{\mathcal{I}_r\}$$

Example: Regions with non trivial homotopy groups π_1 or π_{d-3}

Non local operators form Abelian dual groups (d>3)



Commutation relations are fixed $a\, b = \chi_b(a)\, b\, a$

Gauge theories: Wilson loops $W_r \equiv \mathrm{Tr}_r \mathcal{P}\, e^{i \oint_C dx^\mu A^r_\mu}$

Not all Wilson loops are non locally generated in a ring-like region: for charged representations the loops can be broken in Wilson lines ended in charged operators

$\phi_r(x)\, Pe^{i\int_x^y dx^\sigma A_\sigma}\, \phi_r^\dagger(y)$ $F_{\mu\nu}(x)\, Pe^{i\int_x^y dx^\sigma A_\sigma}\, F_{\alpha\beta(y)}$ → Representations generated by the adjoint are locally generated

Dual operators are the t' Hooft loops

Entropic order parameters:

Two algebras for the same region → two states for the same algebra → relative entropy

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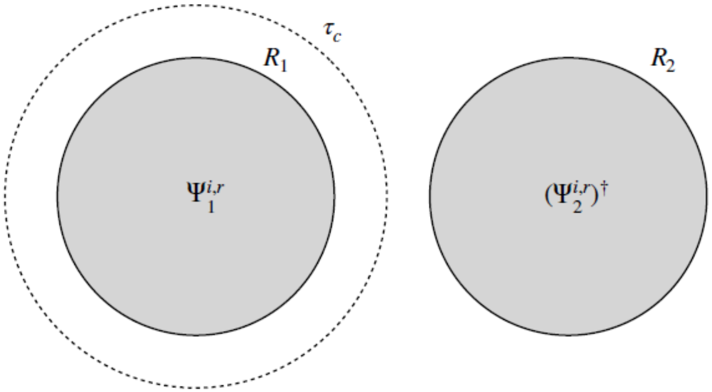
Conditional expectation: linear positive map (chanel) from an algebra to a subalgebra that keeps the subalgebra invariant

$$\varepsilon \,:\, \mathcal{A}_{\max} \to \mathcal{A}_{\text{add}}$$

Uses: lift a state from the subalgebra to the algebra $\omega_{\mathcal{A}_{\text{add}}} \to \omega_{\mathcal{A}_{\text{add}}} \circ \varepsilon$

Conditional expectation produced by the non local operators

$$m \in \mathcal{A}_{\max}\,, \qquad \varepsilon(m) = \frac{1}{|G|}\sum_g \tau_g\, m\, \tau_g^{-1} \in \mathcal{A}_{\text{add}}$$



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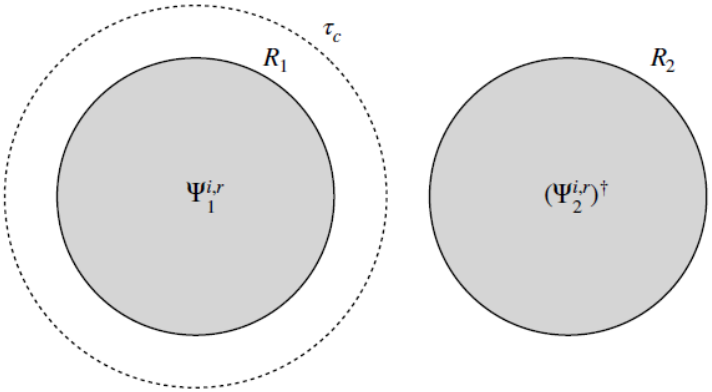

Entropic order parameter

$$S_{\mathcal{A}_{\max}}(\omega|\omega \circ \varepsilon)$$

Relative entropy between the vacuum and the vacuum with the non local operators set to zero expectation value.

It is a measure of the statistics (expectation values) of non local operators.

It is function of the geometry of the region only

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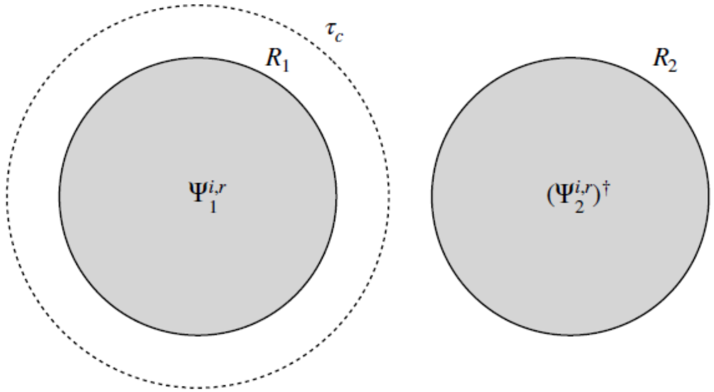

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Standard non local operators determined only by geometry and the vacuum can also be defined

Complementarity diagram

$$
\begin{array}{ccc}
\mathcal{A}_{\text{add}}(R) \vee \{a\} & \xrightarrow{\varepsilon} & \mathcal{A}_{\text{add}}(R) \\
\updownarrow \prime & & \updownarrow \prime \\
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Dual conditional expectations are unique in QFT

For a pure global state we have the entropic certainty relation

$$S_{\mathcal{A}_{\max}(R)}(\omega|\omega \circ \varepsilon) + S_{\mathcal{A}_{\max}(R')}(\omega|\omega \circ \varepsilon') = \log \lambda \qquad (\log |G|)$$

Index of the inclusion $\mathcal{A}_{\text{add}}(R) \subset \mathcal{A}_{\max}(R)$

(R.Longo)

H. Casini, M. Huerta, J. Magan, D. Pontello e-Print: 1905.10487 [hep-th]

J.Magan, D. Pontello e-Print: 2005.01760 [hep-th]

S. Hollands e-Print: 2009.05024 [quant-ph]

Feng Xu e-Print: 1812.01119 [math-ph]

Entropic certainty and uncertainty relations

$$S_{\mathcal{A}_{\max}(R)}(\omega|\omega \circ \varepsilon) + S_{\mathcal{A}_{\max}(R')}(\omega|\omega \circ \varepsilon') = \log|G|$$

$$\longrightarrow \quad S_{\mathcal{A}_{\text{add}}(R)\vee\{a\}}\left(\omega|\omega \circ \varepsilon\right) \leq \log|G|\,, \qquad S_{\mathcal{A}_{\text{add}}(R')\vee\{b\}}\left(\omega|\omega \circ \varepsilon'\right) \leq \log|G|$$

But they cannot saturate at the same time: uncertainty relations for non commuting opertators a,b

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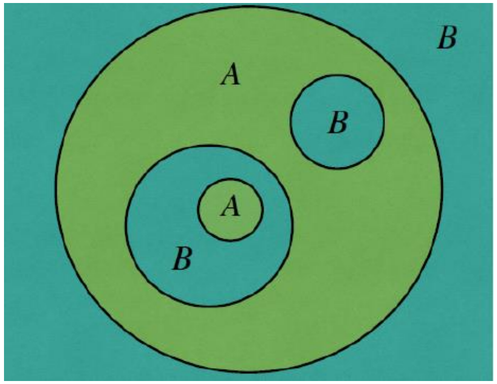
But they cannot saturate at the same time: uncertainty relations for non commuting opertators a,b

It gives a handle in computations

$$S_{\{a\}}\left(\omega|\omega\circ\varepsilon\right)\leq S_{\mathcal{A}_{\mathrm{add}}(R)\vee\{a\}}\left(\omega|\omega\circ\varepsilon\right)\leq\log|G|-S_{\{b\}}\left(\omega|\omega\circ\varepsilon'\right)$$

Uses



Topological contributions to the entropy

$$S_{\mathcal{A}_{\max}(AB)} = I_{\mathcal{F}}(A,B) - I_{\mathcal{O}}(A,B) = n_\partial \, \log|G|$$

Entropic order parameters for gauge fields

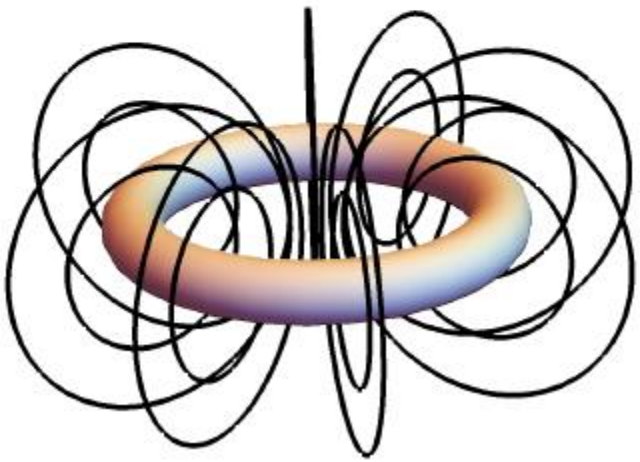
(infrared, thin loops L/R>>1)

Constant law – area law

- confinement $S_W \sim e^{-c\,A}\,, \quad S_T \sim \text{cons}$
- Higgs $S_W \sim \text{cons}\,, \quad S_T \sim e^{-c\,A}$

Conformal $\longrightarrow$ $S_W\,, S_T \sim e^{-c\,L/R}$

- Perimeter law for the conformal case cannot be renormalized to a constant
- Constant law for Higgs/ confinement phases is only for R larger than the gap. (and does not subsist for exponentially large L)



Heuristics of area law:

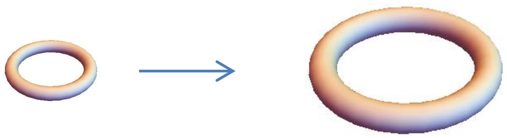
an area worth of decoupled loops with approx constant expectation value

→ the relative entropy for the exterior of the orange ring approaches Log|G| exponentially in the number of loops (then exponentially in the area)

→ The relative entropy in the ring has an area law because of the certainty relation

Some final remarks:

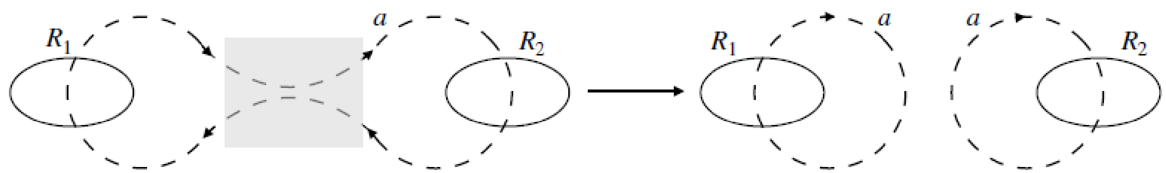
Importance of wide loops
(as opposed to line operators which are partially UV and partially IR)
a) Dual to thin loops
b) For RG fate of symmetries



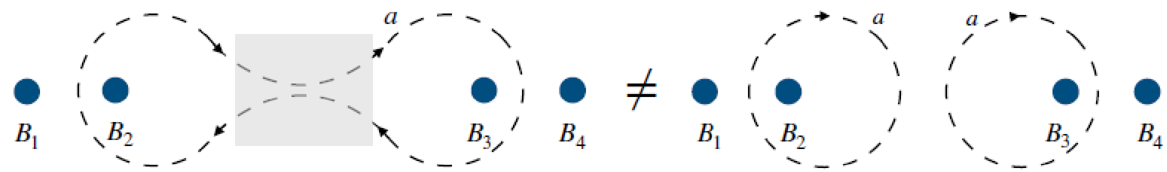
How to relate the UV and IR? Important for confinement for asymtotically free theories

Classification of posible fussion algebras for topologies of regions

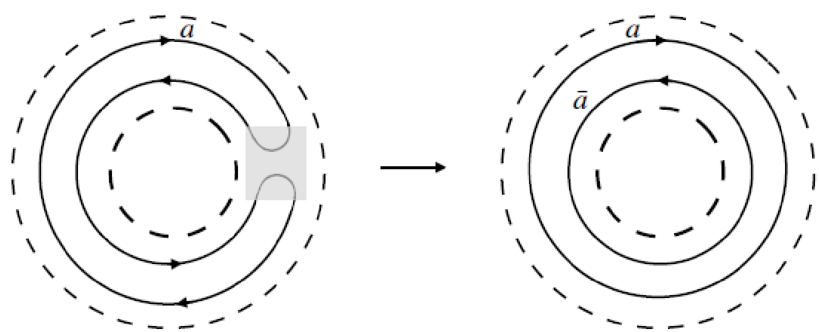
Compute order parameters for non Abelian theories and holography.



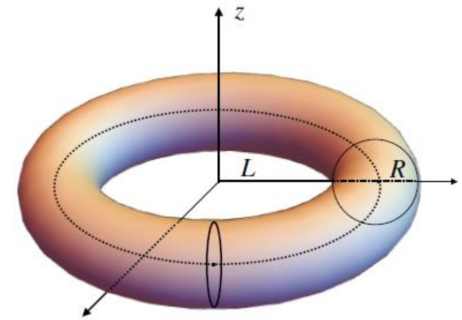
R_1
a
R_2
R_1
a
a
R_2



a
B_1
B_2
B_3
B_4
$\neq$
a
a
B_1
B_2
B_3
B_4



a
a
$\bar{a}$



In the conformal case the relative entropies are fixed for the symmetric cross ratio 1/2 (both the region and its complement have the same cross ratio)

$$S_{WT,0}(1/2) = \log|Z|$$

$$S_{W,0}(1/2) + S_{WT,W}(1/2) = S_{T,0}(1/2) + S_{WT,T}(1/2) = \log|Z|$$

Simplified terminology

confinement	$S_W \to 0\,, \quad S_T \to \log\|G\|$	Saturated, independent of shape
Higgs	$S_W \to \log\|G\|\,, \quad S_T \to 0$	
conformal	$S_W\,, S_T$	depend on conformal shape (with values intermediate between 0 and log|G|)

Symmetry breaking seems related to gap or at least no correlations between non local operators (because their expectation values saturate to 1 or 0).