Higher Central Charges

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In this talk only bosonic theories would be considered.

Consider a 1+1 dimensional system. Alvarez-Gaume & Witten have shown that it must be gapless (massless) if

$$c_- \equiv c_L - c_R \neq 0 \ .$$

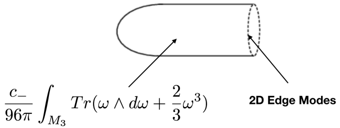
From a modern perspective it is important to distinguish two cases:

$c_- \in 8\mathbb{Z}$

Then we are dealing with an almost genuine 1+1 dimensional system. It requires a contact term in a 2+1 dimensional bulk to be fully diffeomorphism invariant

$$\frac{c_-}{96\pi}\int_{M_3} Tr\left(\omega\wedge d\omega+\frac{2}{3}\omega^3\right) \ .$$

In this case, $c_- \neq 0$ is a 't Hooft gravitational anomaly.



$c_- \notin 8\mathbb{Z}$

We cannot add a contact term in 2+1 dimensions to render the theory diffeomorphism invariant. Such theories are called "relative."

To render them fully diffeomorphism invariant we need a nontrivial theory in the 2+1 dimensional bulk. The 2+1 dimensional theory could be gapless or a topological theory.

Examples:

- A chiral boson has $c_- = 1$ and it is famously the boundary of $U(1)_k$ Chern-Simons theory in the bulk. This is the QHE and the boundary is a relative theory.

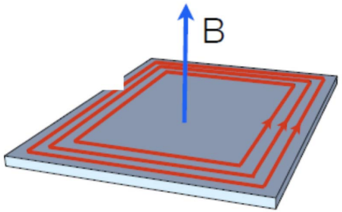


figure from Pascal Simon

- 8 chiral bosons moving on the E_8 lattice have $c_- = 8$ and they do not require a bulk topological theory. They can be the boundary of the contact term

$$\frac{1}{12\pi}\int_{M_3}\left(\omega\wedge d\omega+\frac{2}{3}\omega^3\right)\ .$$

This system has a standard gravitational 't Hooft anomaly.

- Consider

$$U(1)_k \times U(1)_{-k'}$$

($k, k' \in 2\mathbb{N}$) Chern-Simons theory in 2+1 dimensions:

$$S = \frac{k}{4\pi}\int ada - \frac{k'}{4\pi}\int bdb \ .$$

It has many different boundary conditions with $c_- = 0$. Though there is no gravitational anomaly, no gapped boundary condition exists **unless**

$$kk' = m^2 \ , \quad m \in \mathbb{Z} \ . \quad \longrightarrow a = \pm\sqrt{\frac{k'}{k}}b \ .$$

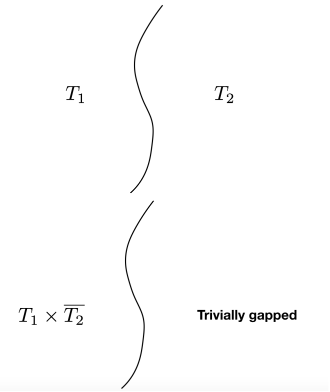
Therefore, protected edge modes do not always require $c_- \neq 0$.

A non-Abelian Example: The Fibonacci anyons, i.e. $(G_2)_1$ Chern-Simons theory.

It has $c_- = 14/5$ and leads to a relative theory on the boundary. But $[(G_2)_1]^{20}$ has $c_- = 56 \in 8\mathbb{Z}$ and hence admits boundary conditions with $c_- = 0$.

Yet one cannot make the boundary gapped. In fact no power of $(G_2)_1$ Chern-Simons theory has a gapped boundary.

Folding Trick:



The interface could be gapless or gapped. The corresponding boundary condition is gapless or gapped, respectively.

An Abelian group structure:

Suppose $T_1 \times \overline{T_2}$ has a gapped boundary. Then we declare that $T_1 \simeq T_2$. It is an equivalence relation. This defines the Witt group of topological field theories [Davydov-Müger-Nikshych-Ostrik], [Fuchs-Schweigert-Valentino], [Freed-Teleman].

This leads us to our main topic: Are there additional "central charges" that could perhaps diagnose the absence of a gapped boundary even when $c_- = 0$?

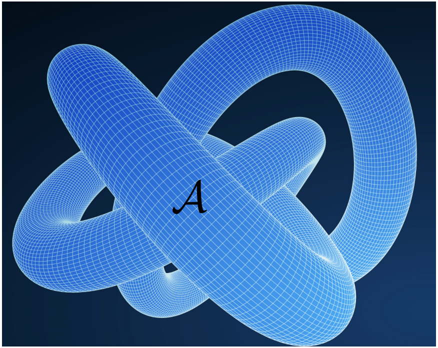
In other words we are asking if there are additional obstructions to having a gap on the boundary other than the famous gravitational anomaly.

The subject of gapped boundaries is of relevance to mathematicians, condensed matter physicists, and high-energy physicists. We will not be able to cover all the points of view in this talk.

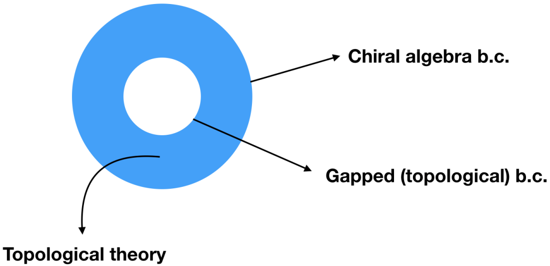
A gapped boundary is a topological boundary. By carving out an empty tube with empty boundary and squeezing it, we see that an empty boundary can be represented by a non-negative integer sum over the anyons!

$$\mathcal{A} = \sum_a Z_{0a} a \ .$$

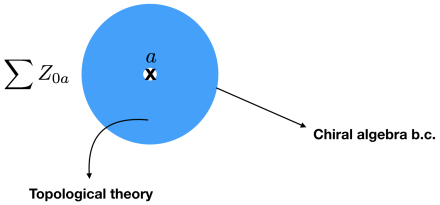
There are many restrictions on $\mathcal{A}$.



An illuminating restriction on $\mathcal{A}$ comes from the annulus partition function with the chiral algebra boundary [Elitzur-Moore-Schwimmer-Seiberg] on one end and the gapped boundary on the other end.



Shrinking/Squeezing the interior:



Therefore the annulus partition function is

$$Z = \sum_a Z_{0a} \chi_a(\tau)$$

It is a holomorphic modular invariant.

$$S_{ab}Z_{0b} = Z_{0a}$$

follows from the modular invariance of $\sum Z_{0a}\chi_a(\tau)$. It also follows that all the anyons a with $Z_{0a} \neq 0$ have zero spin, i.e.

$$T_{ab}Z_{0b} = Z_{0a} \ .$$

In particular, the quantum dimensions satisfy $d_{\mathcal{A}} = \sqrt{\sum_a d_a^2}$. This is a useful way to set up the problem of gapped boundary from the RCFT point of view. It can be implemented algorithmically.

For theories of the type $T \times \overline{T}$ the anyon $\mathcal{A}$ always exists. Before folding it is just the empty and transparent interface. The explicit expression is

$$\mathcal{A} = \sum_a a \otimes \bar{\tilde{a}}$$

For Abelian theories, $d_{\mathcal{A}} = \sqrt{\sum_a d_a^2}$ means that the number of anyons, $|G|$, is a square. This explains why $kk' = m^2$. More generally, the anyons with $Z_{0a} \neq 0$ are the elements of an anomaly free one-form symmetry subgroup of dimension $\sqrt{|G|}$.

Such a one-form symmetry subgroup is also called a "Lagrangian subgroup." It has been known for some time [Kapustin-Saulina, Fuchs-Schweigert-Valentino, Levin, Barkeshli-Jian-Qi...] that indeed for Abelian theories the existence of a Lagrangian subgroup is equivalent to a gapped boundary.

In general there could be more than one Lagrangian subgroup. For instance, $\mathbb{Z}_2$ gauge theory has two such subgroups.

The gapped boundary is related to “condensing” $\mathcal{A}$. The theory with $\mathcal{A}$ condensed is trivial. The gapped boundary separates the original theory from this trivial phase.

For Abelian theories condensing $\mathcal{A}$ is the same as gauging a non-anomalous one-form symmetry. We understand what this means in a clear QFT language in terms of coupling to background fields and then summing over them.

For non-Abelian theories, in general, condensing $\mathcal{A}$ is a mysterious process from the continuum point of view. We can imagine inserting $\mathcal{A}$ on a very fine mesh but there is no clean path integral interpretation, yet. Analogs of this mysterious process also appear in 2d [Frohlich-Fuchs-Runkel-Schweigert] [Carqueville-Runkel] [Bhardwaj-Tachikawa].

Recall that the chiral central charge appears in the partition function of the topological theory on S^3 (*in some particular scheme*) [Witten...]:

$$Z[S^3] = \frac{1}{\sum_a d_a^2} \sum_a d_a^2 \theta_a = \frac{1}{\sqrt{\sum_a d_a^2}} e^{2\pi i \frac{c_-}{8}}$$

The **phase** of $Z[S^3]$ is therefore an obstruction to a gapped boundary. ($\theta_a = e^{2\pi i s_a}$)

If there is a gapped boundary, $Z[S^3] = \frac{1}{d_{\mathcal{A}}} > 0$. This follows from $c_- = 0$ and $d_{\mathcal{A}} = \sqrt{\sum_a d_a^2}$, or from a nice geometric argument of nucleating a tube of nothing.

Many more obstructions in the form of certain partition functions can be found using nice properties of $\mathcal{A}$!

For any topological theory we can define the partition functions on Lens spaces:

$$Z[L(n,1)] = \frac{1}{\sum_a d_a^2} \sum_a d_a^2 \theta_a^n.$$

It is useful to define N_{FS} as the smallest integer for which $\theta_a^{N_{FS}} = 1$. Then, [Ng-Schopieray-Wang, Ng-Rowell-Wang-Zhang]

Gapped boundary $\longrightarrow$ $Z[L(n,1)]$ is positive for $gcd(n, N_{FS}) = 1$

The phases of $Z[L(n,1)]$ are higher central charges. They give nontrivial obstructions to gapped boundaries even when the boundary chiral central charge vanishes.

Additional constraints on theories with a gapped boundary for Abelian theories:

- $Z[M_3]$ for any three manifold M_3 with $gcd(|H_1(M_3)|, |G|) = 1$ must be positive. $|G|$ is the number of anyons. One nice proof is using gauging of one-form symmetry on such three manifolds.
- Lens spaces $L(n, 1)$ with

$$gcd(n, 2|G|/gcd(n, 2|G|)) = 1 \ .$$

The corresponding partition functions must be positive. It is a *necessary and sufficient* condition for having a gapped boundary. The proof invokes a decomposition of Abelian theories into "prime theories."

Some fun facts:

- Every Abelian theory T^8 has a gapped boundary.
- Some non-Abelian theories, such as the Fibonacci anyons, never have a gapped boundary no matter to what power they are raised.
- In Abelian theories with a K-matrix, there is a curious symmetry relating $Z_K[L]$ and $\overline{Z_L[K]}$ for $gcd(|\det K|, |\det L|) = 1$. (3d-3d correspondence?). Here L is the surgery linking matrix.
- Every Abelian theory that has a gapped boundary is equivalent to a discrete Abelian gauge theory (Dijkgraaf-Witten theory)! All discrete Abelian gauge theories have (at least one) gapped boundary.

Conclusion: There are certain higher central charges protecting the 1+1 dimensional boundary from developing a gap. They can be computed via partition functions on three-manifolds, analogously to c_- mod 8.

Q1: There is not yet a complete set of higher central charges for the general case but we have found a complete set for Abelian theories. Can one find more general higher central charges in the non-Abelian case?

Q2: Oftentimes, a given theory may admit more than one gapped boundary condition. (e.g. $\mathbb{Z}_2$ gauge theory). Can we count them?

Q3: There may not be a gapped boundary condition preserving some symmetry (e.g. $e \leftrightarrow m$ symmetry in $\mathbb{Z}_2$ gauge theory). Can this be characterized via some central charges?

Q4: What does non-Abelian anyon condensation mean?

Q5: This talk was about non-spin theories – what happens if we relax this condition?

slide for experts

A relation between conformal embeddings and gapped boundaries:

The conformal embeddings in the talk of Jaume Gomis could be understood through special topological interfaces of a theory T to itself. They are equivalent to nontrivial gapped boundary conditions for $T \times \overline{T}$. Such interfaces of T to itself are sometimes called Kapustin-Saulina surfaces. For conformal embeddings, they can be understood from fusion of more "elementary" surfaces from the original theory to another theory (the one related by conformal embedding).