Infrared Phases of $2d$ QCD

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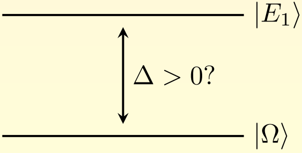
Introduction

- a central theme in physics is unraveling the low energy phenomena that emerges from a physical system defined by a collection of microscopic degrees of freedom and interactions

$$H_{\mathrm{UV}} \rightsquigarrow H_{\mathrm{IR}}$$

  - gives physics its richness and beauty

- the spectrum can be either gapped or gapless, but determining which phase is realized is often a nonperturbative problem



- Yang-Mills theory is a remarkable example of a theory whose nonperturbative dynamics gaps the Hamiltonian of massless gluons

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- QCD_n with dynamical massless quarks in a representation R of G
    - which QCD_n theories are gapped and which are gapless?
    - what is the low energy description?

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  - which QCD_n theories are gapped and which are gapless?
  - what is the low energy description?
    - asymptotic freedom bounds which QCD_4 theories can be gapped
      * adjoint QCD_4 is gapped — Witten, Affleck, Dine, Seiberg
    - QCD_3 flows to a CFT in the large R expansion — Appelquist, Nash
      * gapped QCD_3 theories argued to exist — J.G., Komargodski, Seiberg

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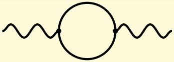
What emerges in the infrared?

2d QCD

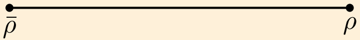
- QCD with gauge group G and representations (R_ℓ, R_r) for left/right quarks

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2g^2}\text{tr}(F_{\mu\nu}F^{\mu\nu}) + i\psi_\ell^\dagger(\partial_- - iA_-^a t_\ell^a)\psi_\ell + i\psi_r^\dagger(\partial_+ - iA_+^a t_r^a)\psi_r + \mathcal{L}_\theta$$

- symmetries:
  - flavor symmetries
  - one-form symmetry $\Gamma \subset Z(G)$
  - non-invertible symmetries

- QCD obtained by gauging $G \subset$ flavor symmetry of quarks in UV:
  - anomaly cancellation: a  $b \Longrightarrow \text{tr}(t_\ell^a t_\ell^b) = \text{tr}(t_r^a t_r^b)$
  - g triggers an RG flow. What happens in the infrared?

- QCD Hilbert space splits into distinct topological sectors
  - background with a flux tube created by probe quarks at ∞ Coleman,Witten



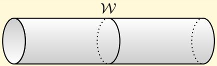
$$\rho \in \Gamma^*$$

  - QCD is gapped iff the Hamiltonian is gapped in the $\rho = 0$ sector
    - QCD has topological lines $\mathcal{W}$ charged under Γ

Komargodski, Ohmori, Roumpedakis, Seifnashri

    - $\mathcal{W}$ interpolates between flux sectors and cannot lower the energy

$$\mathcal{W}H = H_\rho \mathcal{W}$$



  - suffices to study QCD with simply connected gauge group since

$$G_{\rm sc}/\Gamma + (R_\ell, R_r) + \mathcal{L}_{\theta=\rho} \; = G_{\rm sc} + (R_\ell, R_r) + \rho - \text{flux tube}$$

’t Hooft anomalies and infrared phases

- ’t Hooft anomalies have a topological classification, making them invariant under symmetric deformations, including RG transformations: $\alpha_{\mathrm{UV}} = \alpha_{\mathrm{IR}}$

- ’t Hooft anomalies constrain the infrared dynamics of a physical system

  - symmetry is continuous ($\alpha \in \mathbb{Z}$):

    1. symmetry preserving gapless phase (CFT)
    2. symmetry breaking gapless phase (Goldstone bosons)

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    3. symmetry preserving, gapped and topologically ordered phase (TQFT)

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Towards gapped QCD

- a necessary condition for a QCD theory to be gapped is that it has no 't Hooft anomalies for continuous symmetries preserved under RG flow:
  - gapped QCD theory cannot have continuous chiral flavor symmetries

$$\partial_- j_+ = 0 \Longrightarrow \langle \partial_- j_+ \rangle_B \neq 0$$

  - absence of gravitational anomalies: $c_\ell - c_r = 0$
- these constrain the quark content of QCD_2 theories that are gapped

QCD Lightcone Hamiltonian

- x^+ (or x^-) as time in lightcone quantization
- mass spectrum by diagonalizing P^+ and P^- since $M^2 = P^+P^-$ 't Hooft,Pauli, Brodsky,Hornsbostel,Klebanov,Demeterfi,Kutasov,Schwimmer,Gross,Hashimoto,Pufu,Dempsey,...

$$P^-_{\text{QCD}} \propto -g^2 \int \mathrm{d}x^- :J^a \frac{1}{\partial_-^2} J^a: \propto g^2 \sum_{n=1}^{\infty} :J^a_{-n} J^a_n: \geq 0$$

where $J^a = :\psi_r^\dagger t_r^a \psi_r:$ generates a G_k current algebra, where $k = I(R_r)$

- P^-_{QCD} acts on the quark Hilbert space $\mathcal{H}$

$$|\Psi^{i_1 i_2 \dots i_L}\rangle \equiv a^\dagger_{i_1}(k_1) a^\dagger_{i_2}(k_2) \dots a^\dagger_{i_L}(k_L) |\Omega\rangle$$

- necessary and sufficient conditions for $|\Psi^{i_1 i_2 \dots i_L}\rangle \in \mathcal{H}$ to have $P^- = 0$:
  1. $|\Psi^{i_1 i_2 \dots i_L}\rangle$ is a primary state of the current algebra $G_{I(R_r)}$
  2. $|\Psi^{i_1 i_2 \dots i_L}\rangle$ transforms in a trivial representation of G

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QCD is gapped iff $|\Omega\rangle$ is the unique $G_{I(R)}$ primary, singlet state in $\mathcal{H}$

Hilbert Space

- $\mathcal{H}$ decomposes into modules of $SO(\dim(R))_1$ current algebra Witten, Ji,Shao,Wen
  - gauging $G \subset SO(\dim(R))$ induces an embedding of current algebras:

$$G_{I(R)} \subset SO(\dim(R))_1$$

  - embedding of modules encoded in the decomposition of affine characters

$$\chi_\Lambda(q) = \sum_\lambda b_{\Lambda\lambda}(q)\chi_\lambda(q)$$

    - Λ and λ label integrable representations of $SO(\dim(R))_1$ and $G_{I(R)}$
    - $b_{\Lambda\lambda}(q)$ counts the primary states of $G_{I(R)}$ in the module of $SO(\dim(R))_1$
    - $b_{\Lambda\lambda}(q)$ is a character of the commutant chiral algebra:
      - $T_{SO(\dim(R))_1} - T_{G_{I(R)}}$
      - current algebra $H_{k'}$ generated by currents J^α if $H \times G \subset SO(\dim(R))$
      - ...

- QCD is gapped iff $|\Omega\rangle$ is the only $G_{I(R)}$ primary, singlet state in $\mathcal{H}$

$$\implies \text{QCD is gapped iff } b_{\Lambda 0}(q) \text{ is independent of } q$$

- consider first $b_{00}(q) = q^{c(G_{I(R)})/24 - \dim(R)/48}\left(1 + a_1 q + a_2 q^2 + \dots\right)$
  - $a_1 =$ dimension of flavor symmetry group
    - gapped spectrum requires $a_1 = 0 \iff$ no continuous global symmetries
  - the q^2 term corresponds to the following state in quark Hilbert space $\mathcal{H}$

$$(T_{SO(\dim(R))_1} - T_{G_{I(R)}})|\Omega\rangle$$

    - $T_{SO(\dim(R))_1} = -\frac{1}{2}{:}\psi^i \partial \psi^i{:}$
    - $T_{G_{I(R)}} = \frac{1}{2(I(R)+h)}{:}J^a J^a{:}$ where $J^a = {:}\psi^i t^a_{ij} \psi^j{:}$
  - gapped spectrum requires that $T_{SO(\dim(R))_1} = T_{G_{I(R)}} \implies b_{\Lambda 0}(q) = \delta_{\Lambda 0}$

$\iff$ necessary and sufficient condition for a QCD theory to be gapped is that

$$\boxed{T_{SO(\dim(R))_1} = T_{G_{I(R)}}}$$

Gapped QCD

- this yields a Jacobi-like identity for the generators t^a in representation R of G under which quarks transform

$$\sum_{a=1}^{\dim G} t^a_{ij} t^a_{kl} + t^a_{ik} t^a_{lj} + t^a_{il} t^a_{jk} = 0$$

  - solutions are in one-to-one correspondence with symmetric spaces

Goddard, Nahm, Olive

  - describe conformal embeddings of G_k into $SO(\dim(R))_1$

$\Longrightarrow$

  - complete list of QCD theories that are gapped $\Longrightarrow (G, R_\ell, R_r)$
  - any other theory is gapless

Gapped QCD Theories

$\mathfrak{g}$	R	$\mathfrak{g}$	R
$\mathfrak{g}$	adj	$\mathfrak{su}(2)$	$\mathbf{5}$
$\mathfrak{so}(N)$	$\square$	$\mathfrak{so}(9)$	$\mathbf{16}$
$\mathfrak{u}(N)$	$\square_q$	F_4	$\mathbf{26}$
$\mathfrak{so}(N)$	$\square\!\square$	$\mathfrak{sp}(4)$	$\mathbf{42}$
$\mathfrak{sp}(N)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathfrak{su}(8)$	$\mathbf{70}$
$\mathfrak{u}(N)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}_q$	$\mathfrak{so}(16)$	$\mathbf{128}$
$\mathfrak{u}(N)$	$\square\!\square_q$	$\mathfrak{so}(10)+\mathfrak{u}(1)$	$\mathbf{16}_q$
$\mathfrak{su}(M)+\mathfrak{su}(N)+\mathfrak{u}(1)$	$(\square,\square)_q$	$E_6+\mathfrak{u}(1)$	$\mathbf{27}_q$
$\mathfrak{so}(M)+\mathfrak{so}(N)$	$(\square,\square)$	$\mathfrak{su}(2)+\mathfrak{su}(2)$	$(\mathbf{2},\mathbf{4})$
$\mathfrak{sp}(M)+\mathfrak{sp}(N)$	$(\square,\square)$	$\mathfrak{su}(2)+\mathfrak{sp}(3)$	$(\mathbf{2},\mathbf{14})$
$\oplus_i\mathfrak{u}(n_i)$	$\oplus_i(1,\ldots,\square_i,\ldots,1)_{\boldsymbol{q}_i,\tilde{\boldsymbol{q}}_i}$	$\mathfrak{su}(2)+\mathfrak{su}(6)$	$(\mathbf{2},\mathbf{20})$
		$\mathfrak{su}(2)+\mathfrak{so}(12)$	$(\mathbf{2},\mathbf{32})$
		$\mathfrak{su}(2)+E_7$	$(\mathbf{2},\mathbf{56})$

Infrared Dynamics of 2d QCD

- what description emerges in the deep infrared for gapped and gapless QCD?
- conjecture is that infrared description is given by $g^2 \to \infty$ limit of $\mathcal{L}_{\text{QCD}}$
  - gauged WZW description of coset $SO(\dim(R))_1/G_{I(R)}$
    - TQFT when QCD is gapped
    - CFT when QCD theory is gapless
- examples:
  - $SU(N) + N_F \,\square \xrightarrow{\text{IR}} U(N_F)_N$ WZW
  - $SU(2) + \mathbf{7} \xrightarrow{\text{IR}} \mathcal{N} = 1$ minimal model (tricritical Ising)

Conclusions

- QCD_2 theories exhibit interesting phenomena:
  - supersymmetric spectrum by virtue of ' t Hooft anomalies
  - nonperturbative quark condensates
  - quark deconfinement
  - ...
- problem of determining the gapped QCD_2 theories can be solved
- study QCD_2 with light quarks using the proposed infrared description
- Hamiltonian methods very fruitful in tackling the problem. Tackle QCD_3?