Thermodynamics applied to game theory: study of the risk associated with cooperation in the Public Goods Games

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Cooperation risk
Our lab. in Ribeirão Preto

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Our lab. in Ribeirão Preto
Authors

Gilberto and Guilherme.
Cooperation risk and Nash equilibrium: Quantitative description for realistic players

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HIGHLIGHTS

- Mathematical description of cooperation risk to restore Nash equilibrium in potential games.
- First- and second-order phase transitions between cooperative and other behaviors.
- Punishments and cooperation risk in public goods games with algebraic operators.

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ABSTRACT

The emergence of cooperation figures among the main goal of game theory in competitive-cooperative environments. Potential games have long been hinted as viable alternatives to study realistic player behavior. Here, we expand the potential games approach by taking into account the inherent risks of cooperation. We show the Public Goods game reduce to a Hamiltonian with one-body operators, with the correct Nash Equilibrium as the ground state. The inclusion of punishments to the Public Goods game reduces cooperation risks, creating two-body interactions with a rich phase diagram, in which phase transitions segregates cooperative from competitive regimes.

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Yes, we are using all information we have in game theory?
Outline

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- Ideal gas: state equations, fundamental equation (thermodynamics potential)
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Extend this comparison to $N$-player games
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Stress this evidence in evolutive potential games
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- Ideal gas: state equations, fundamental equation (thermodynamics potential)
- Compare two-player and two-strategy games to Ising model
- Extend this comparison to $N$-player games
- Consideration of something else (cooperation risk)
- Stress this evidence in evolutive potential games
- Set up a natural connection with the thermodynamics, only with cooperation risk.
Ideal Gas Law

\[ PV = nRT \]

- \( P \) is the pressure of the gas
- \( V \) is the volume of the gas
- \( T \) is the temperature of the gas
- \( n \) is the number of moles
- \( R \) is the gas constant

\( R = 0.0820573 \ \text{L atm K}^{-1} \ \text{mol}^{-1} \)
\( R = 8.3144598 \ \text{J K}^{-1} \ \text{mol}^{-1} \)

State equation: partial information.
Ideal gas

Equipartition law

\[ \frac{1}{2} kT \text{ per molecule} \]
\[ \frac{1}{2} RT \text{ per mole} \]

\[ k = \text{Boltzmann's constant} \]
\[ R = \text{gas constant} \]

\[ \frac{3}{2} kT \]
\[ \frac{3}{2} RT \]

For three translational degrees of freedom, such as in an ideal monoatomic gas.

State equation: partial information.
Ideal gas

Entropy

Entropy of an Ideal Gas

\[ S(N, V, U) = N k_B \ln \left( \frac{V}{N} \left( \frac{4 \pi m U}{3 \hbar^2} \right)^{3/2} \right) + \frac{5}{2} N k_B + \ln(2\Delta p) \]

Monatomic ideal gas:

\[ S(N, V, U) = N k_B \ln \left( \frac{V}{N} \left( \frac{4 \pi m U}{3 \hbar^2} \right)^{3/2} \right) + \frac{5}{2} \]

In general, for a gas of polyatomic molecules:

\[ S(N, V, T) = N k_B \ln \frac{V}{N} + \frac{f}{2} N k_B \ln T + \phi(N, m) \]

\( f \Rightarrow 3 \) (monatomic), \( 5 \) (diatomic), \( 6 \) (polyatomic)

Fundamental equation: all information, but not handy.
The most handy and optimized way to keep all the system information.
**Card players, Paul Cézanne.**

**Conflict**: global minimum is better (cooperation), but player chose local ones (competition).
Two-player game

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**Values** : $T$: temptation, $P$: punishment, $R$: reward and $P$: punishment.

Card players, Paul Cézanne.
Two-player game

Conflict: global minimum is better (cooperation), but player chose local ones (competition)


What game to play?

$R > T > P > S$ Stag hunt

$T > R > P > S$ Prisoner’s dilemma

$T > R > S > P$ Chicken (hawk-dove)
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**Two-player game**

What game to play?

\[
\begin{align*}
R & > T > P > S & \text{Stag hunt} \\
T & > R > P > S & \text{Prisoner’s dilemma} \\
T & > R > S > P & \text{Chicken (hawk-dove)}
\end{align*}
\]

Payoff matrix:

\[
\begin{pmatrix}
C & D \\
C & (R, P) & (S, T) \\
D & (T, S) & (P, P)
\end{pmatrix}
\]

Solution to these dilemmas: *Nash equilibrium (local minima)!!!!*
Player 1 payoff:

\[ G_1 = R \ b_1 b_2 + T \ (1-b_1) b_2 + S \ b_1 (1-b_2) + P \ (1-b_1)(1-b_2) \]

with \( b_i = \{0, 1\} \).

Calling: \( t = (T - P)/R \), \( s = (S - P)/R \) and \( t = (T - P)/R \)

\[ g_1 = \frac{G_1 - P}{R} = (1 - t - s) \ b_1 b_2 + s \ b_1 + t \ b_2 \]
\[ g_2 = \frac{G_2 - P}{P} = (1 - t - s) \ b_1 b_2 + t \ b_1 + s \ b_2 . \]
Two-player game

Examples of two-player games.

Player 1 payoff:

\[ G_1 = R b_1 b_2 + T (1-b_1) b_2 + S b_1 (1-b_2) + P (1-b_1)(1-b_2) \]

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\[ g_2 = \frac{G_2 - P}{P} = (1 - t - s) b_1 b_2 + t b_1 + s b_2 . \]

Total payoff: \( g = g_1 + g_2 = 2(1 - t - s)b_1 b_2 + (t + s)(b_1 + b_2) \).
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Total payoff: \( g = g_1 + g_2 = 2(1 - t - s) b_1 b_2 + (t + s)(b_1 + b_2) \).

Changing to **Ising variables**, \( s_i = \{-1, 1\} \): \( b_i = (s_i + 1)/2 \):

\[ 2g - 1 - t - s = \underbrace{(1 - t - s) s_1 s_2 + s_1 + s_2}_{\text{-}E_2} \]
Two-player game and the Ising model

Two-player game Hamiltonian:

\[ E_2 = -J s_1 s_2 - (s_1 + s_2). \]
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- \( J > 0 \) Stag hunt
- \( J < 0 \) Chicken (hawk-dove)

Prisoner’s dilemma can be both!

To compare to the Ising model

\[ E_2^{(I)} = -J s_1 s_2 + H (s_1 + s_2). \]

with \( J \) and \( H \) being independent parameters.
$N$ player on a graph

$$E_N = -J \frac{1}{\langle k \rangle} \sum_{i>j=1}^N A_{i,j} s_i s_j - \sum_{i=1}^N s_i,$$

mean degree: $\langle k \rangle = \sum_{i=1}^N \langle k_i / N \rangle$ with $k_i = \sum_{j=1}^N A_{i,j}$, $i$-th node degree and no self-interation: $A_{i,i} = 0$. 

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To compare with Ising model:

\[ E_N^{(I)} = -\frac{J}{\langle k \rangle} \sum_{i>j=1}^{N} A_{i,j}s_is_j + H \sum_{i=1}^{N} s_i . \]

Symmetric case: \( A_{i,j} = A_{j,i} \):

\[ E_N = \sum_{i=1}^{N} s_i \left( -1 - \frac{J}{2\langle k \rangle} \sum_{j=1}^{n} A_{i,j}s_j \right) \]
\[ E_N^{(I)} = \sum_{i=1}^{N} s_i \left( H - \frac{J}{2\langle k \rangle} \sum_{j=1}^{n} A_{i,j}s_j \right) . \]
Game Hamiltonian: $E_N = \sum_{i=1}^{N} s_i \left( -1 - \frac{J}{2\langle k \rangle} \sum_{j=1}^{n} A_{i,j} s_j \right)$. No way to have vanishing external field (**only one parameter**)!
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- A simple way to fix thing up and make the two systems equivalent is to add, by hand, something that depends on $N$ to the game hamiltonian.
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- A simple way to fix thing up and make the two systems equivalent is to add, by hand, something that depends on $N$ to the game hamiltonian.

- But what is that stuff that brings us additional information of game theory? THE COOPERATION RISK.
The risk of one agent to cooperate and the other does not.
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The risk of being a sucker.
Cooperation risk

The risk of one agent to cooperate and the other does not.

It has been introduced by Nash, in a qualitative way; remained only been intuitive, for decades; shows up in evolutive potential games.
All players in evolutive games choose the best strategy;

\[ p(s) = e^{-\beta U(s)} \]

The potential \( U(s) \) is taken to be the total payoff. Something is missing to make Nash configurations to correspond to potential minima. The payoff may be the equivalent of a state equation and another one is missing.

Adding a chemical potential \( \mu \) to \( U(s) \) solves the problem.
Evolutive potential games

All players in evolutive games choose the best strategy; but in potential games, they choose \textit{randomly} a strategy.
Evolutive potential games

All players in evolutive games choose the best strategy; but in potential games, they choose randomly a strategy. The best one is the most probable.

\[ p(s) = \frac{e^{-\beta U(s)}}{Z} \quad \text{with} \quad Z = \sum_{s'} e^{-\beta U(s')} . \]
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**WELL KNOWN ISSUE**

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Player favor gains over risks: (in thermo: work over heat)
Information about the cooperation risk is missing.
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For each player $k$: $\mu_k \sim \langle g_k \rangle$. 
Cooperation risk

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Information about the cooperation risk is missing.

For each player $k$: $\mu_k \sim \langle g_k \rangle$.
Strategies are not correlated: $\langle b_k b_{k+1} \rangle \equiv \langle b_k \rangle \langle b_{k+1} \rangle$.
We define the cooperation risk as:

$$\mu \equiv - \frac{\partial \langle \mathcal{H}_k \rangle}{\partial \langle b_k \rangle}$$
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We define the cooperation risk as:

$$
\mu \equiv - \frac{\partial \langle \mathcal{H}_k \rangle}{\partial \langle b_k \rangle}
$$

This is the constant we have to add up to make the game Hamiltonian equivalent to the Ising one.
Conclusions

- Working with evolutive potential games without cooperation risk is equivalent to work with Clayperon equation without equipartition theorem in ideal gases;
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Conclusions

Working with evolutive potential games without cooperation risk is equivalent to work with Clayperon equation without equipartition theorem in ideal gases;

in evolutive potential public game with punishment a cooperative transition occurs;

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we would like to understand its role in other games!!!