Scaling Laws in Urban Systems:
The Physics of Cities

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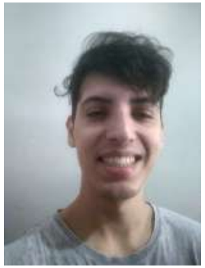
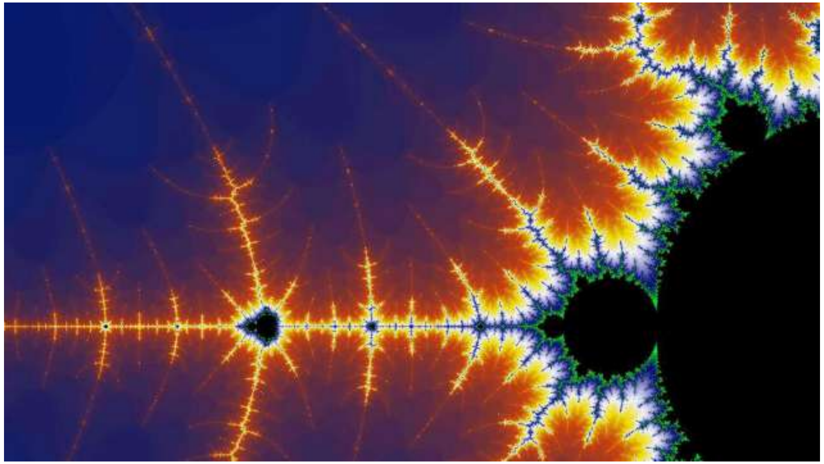


Lucas Marques
(Post-grad. Student - UFLA)

Sumary

- Fractal Cities;
- Urban Scaling;
- Scaling in Biology;
- Gravity Models to explain urban scaling

Fractal Cities



Mandelbrot Set

Shenyang and Fushun, China - population: 10M

Fractal Cities

Box counting method:

Ribeiro, Rev. Morf. Urb. (2020)

Fractal Cities

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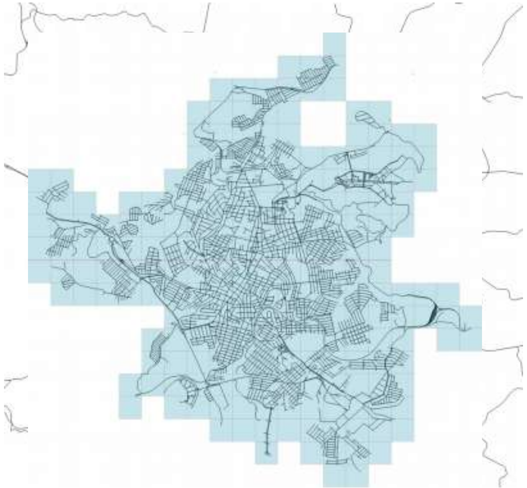

Ribeiro, Rev. Morf. Urb. (2020)

Fractal Cities

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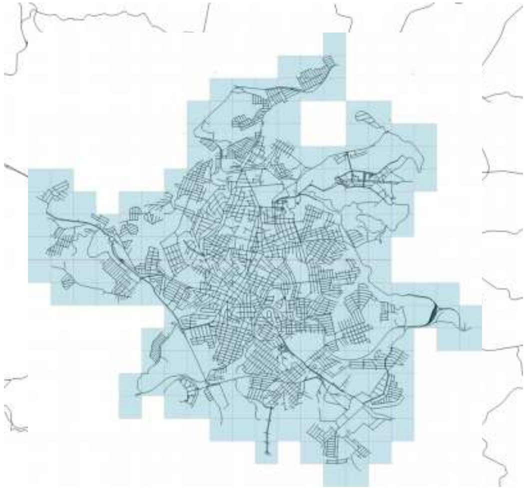

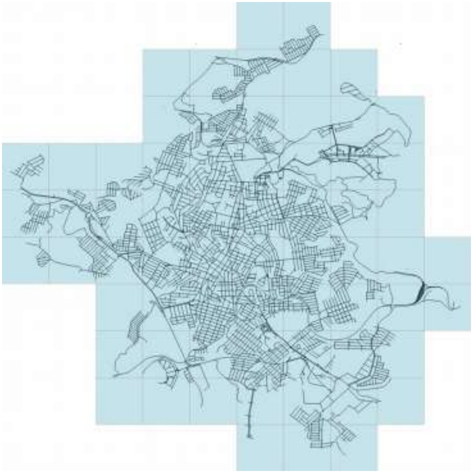

Ribeiro, Rev. Morf. Urb. (2020)

Fractal Cities

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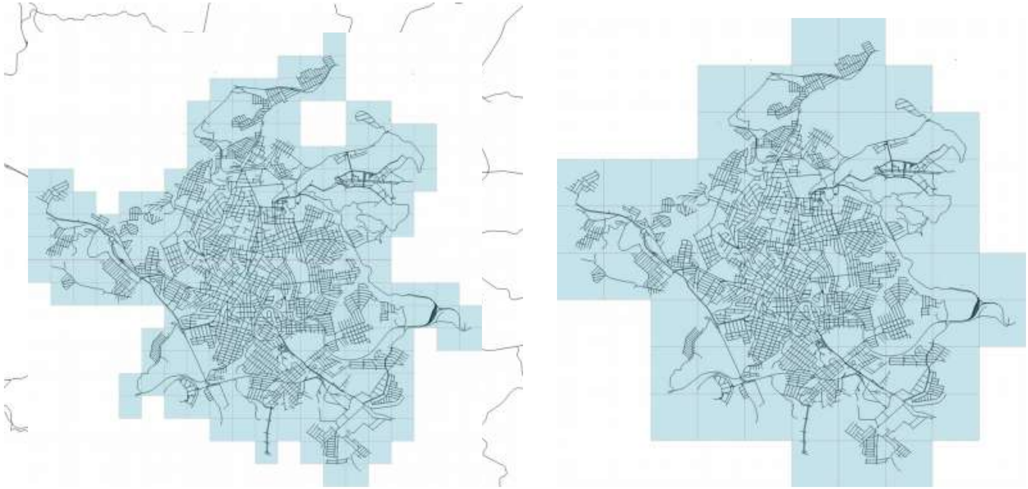

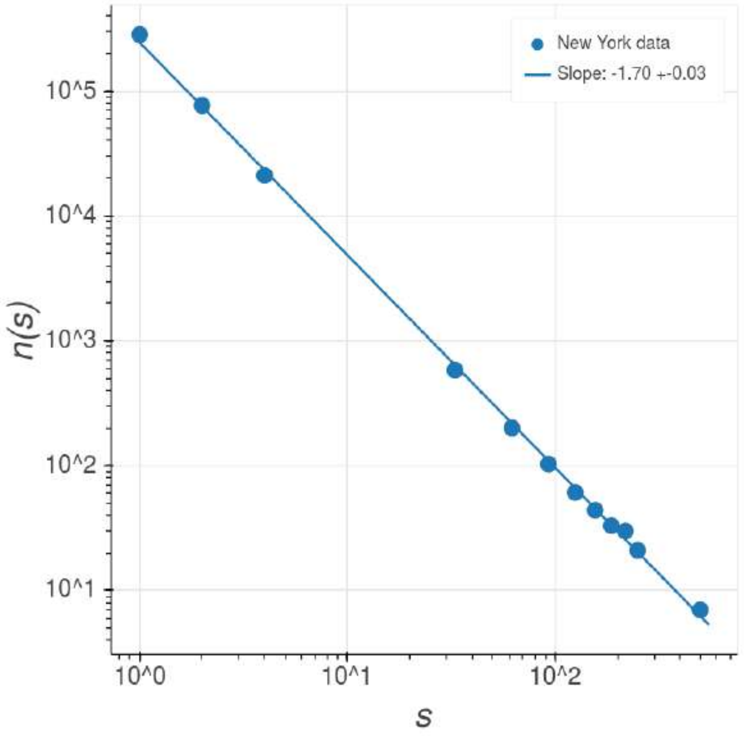

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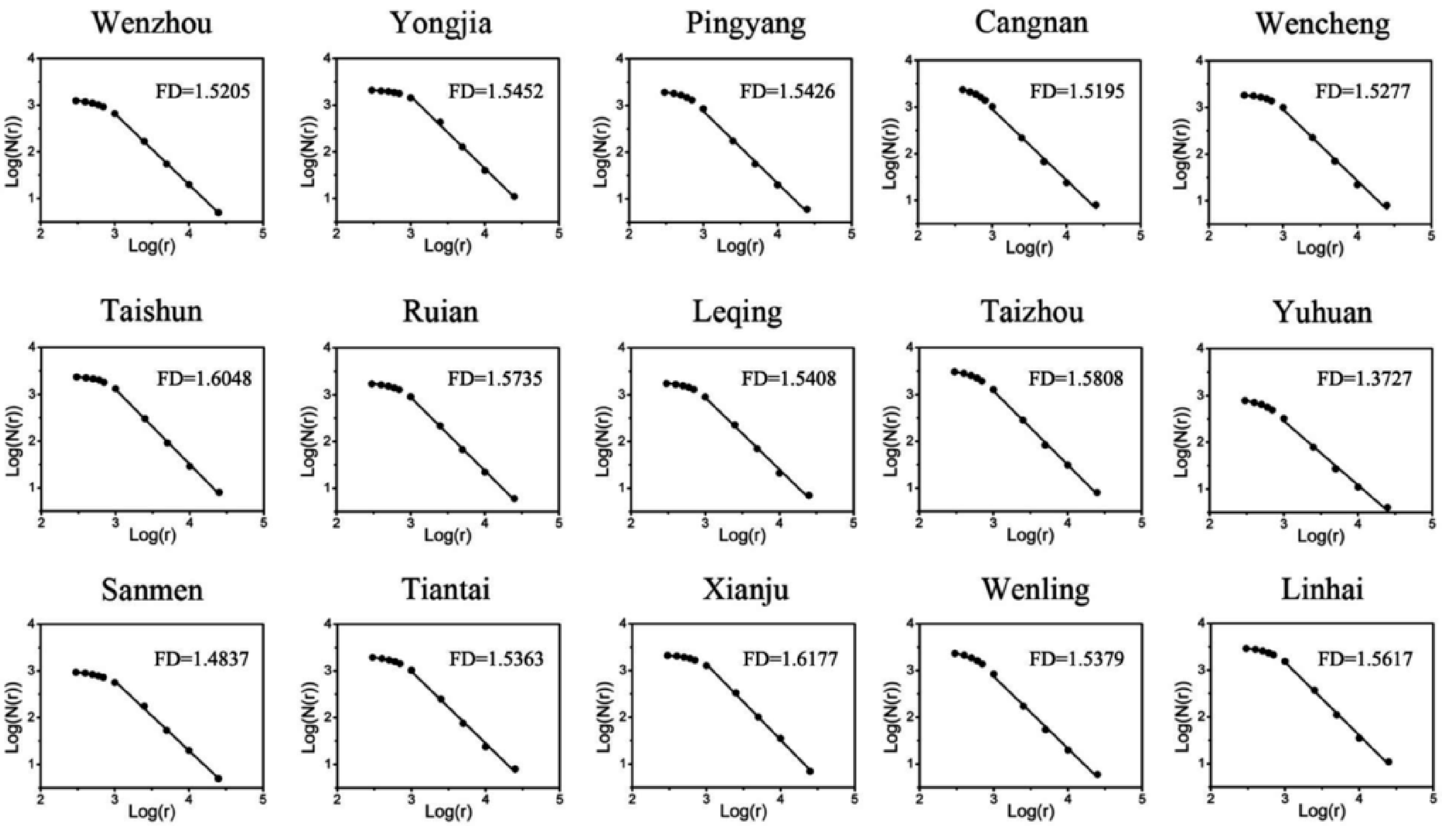
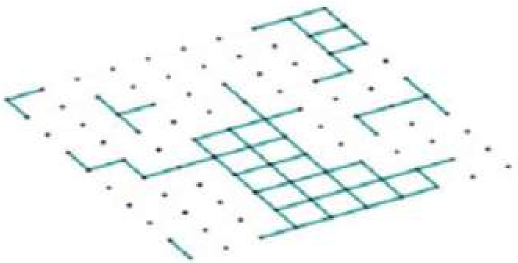
Fractal Cities

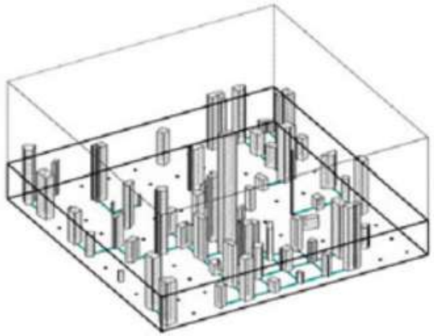


Z. Zhang et al, ISPRS (2015).

Fractal Cities



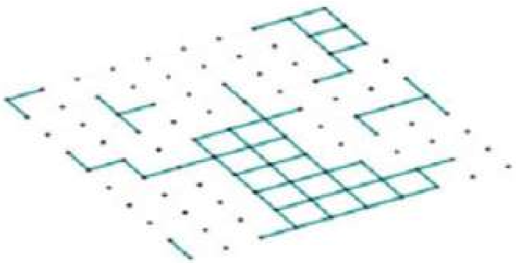
$$L \sim r^{D_{infra}}$$

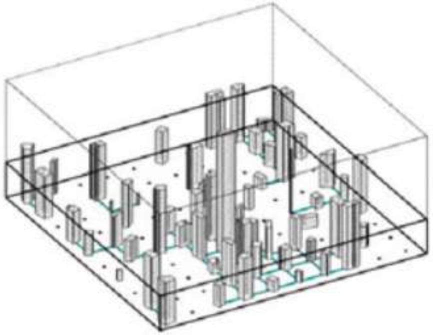


$$N \sim r^{D_p}$$

Molinero & Thurner, Interface (2021).

Fractal Cities





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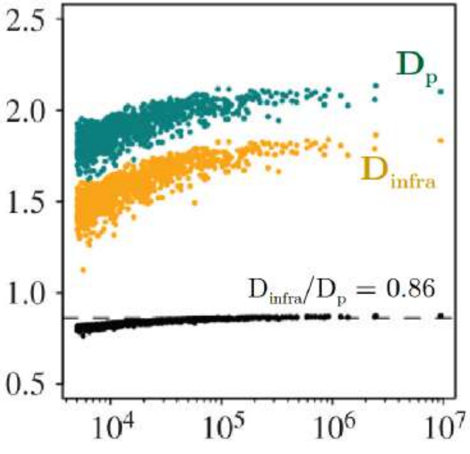


N

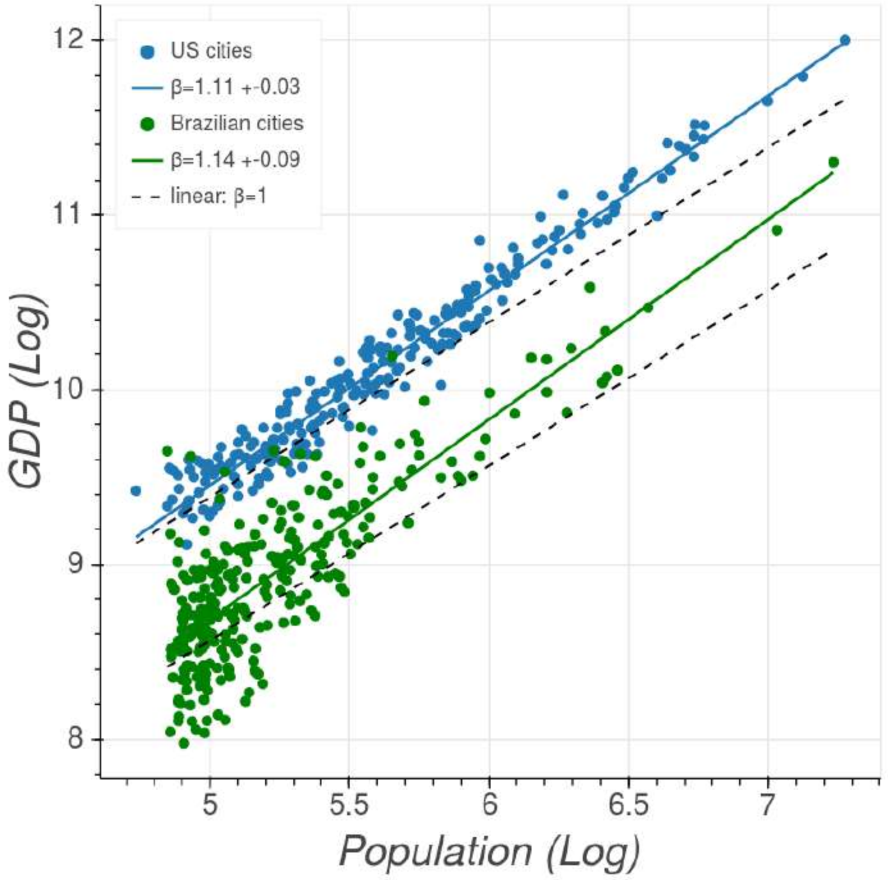
$$N \sim r^{D_p}$$

Molinero & Thurner, Interface (2021).

Growth Domestic Product (GDP)

$$Y = Y_0 N^{\beta}$$

$$\beta_{se} \approx 1.15$$



Bettencourt et al , Pnas (2007).
Ribeiro et al., R. Soc. open sci. (2017).
Ribeiro, Rev. Morf. Urb. (2020)

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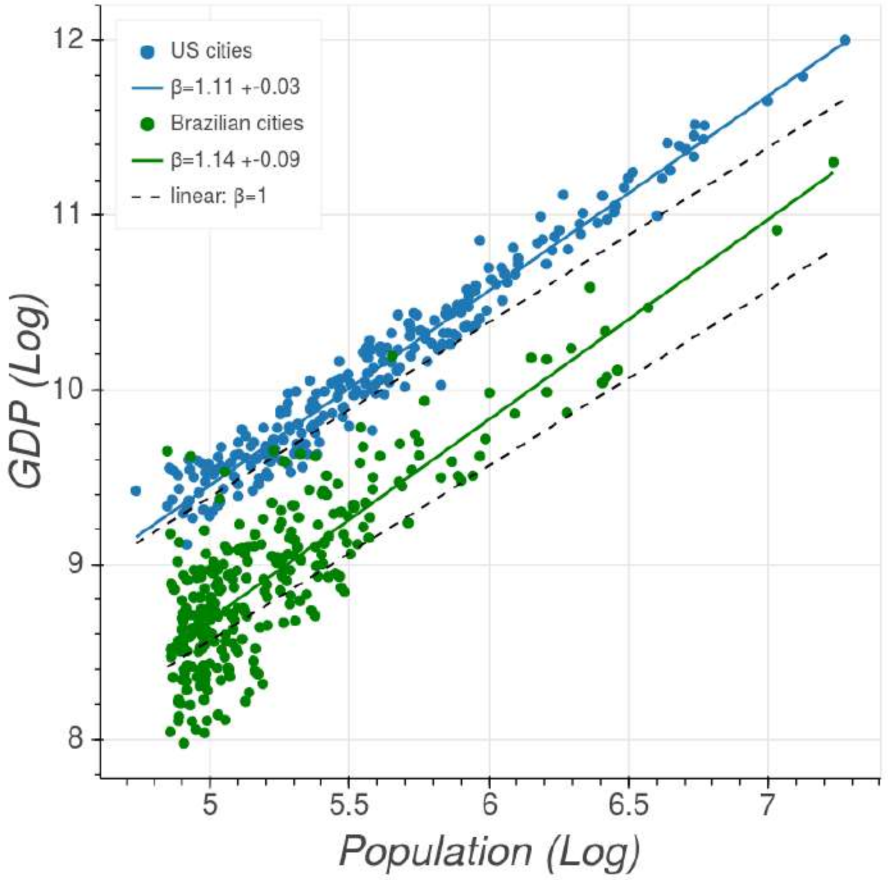

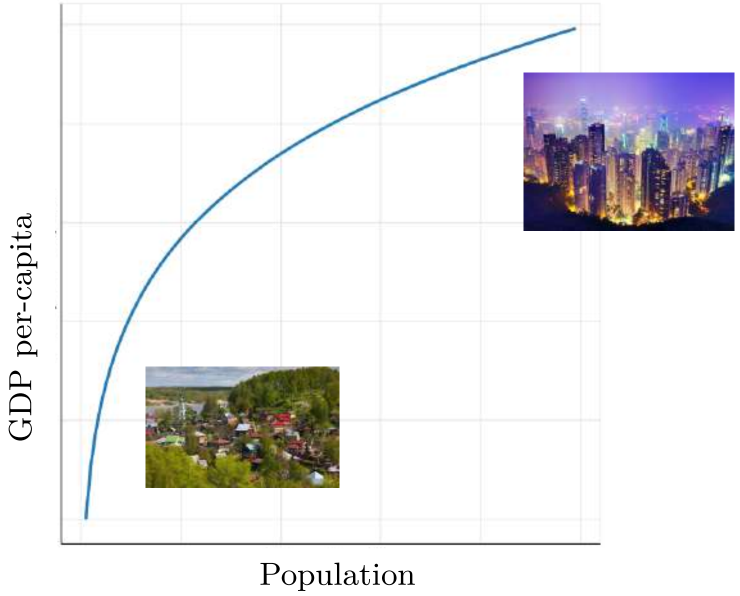

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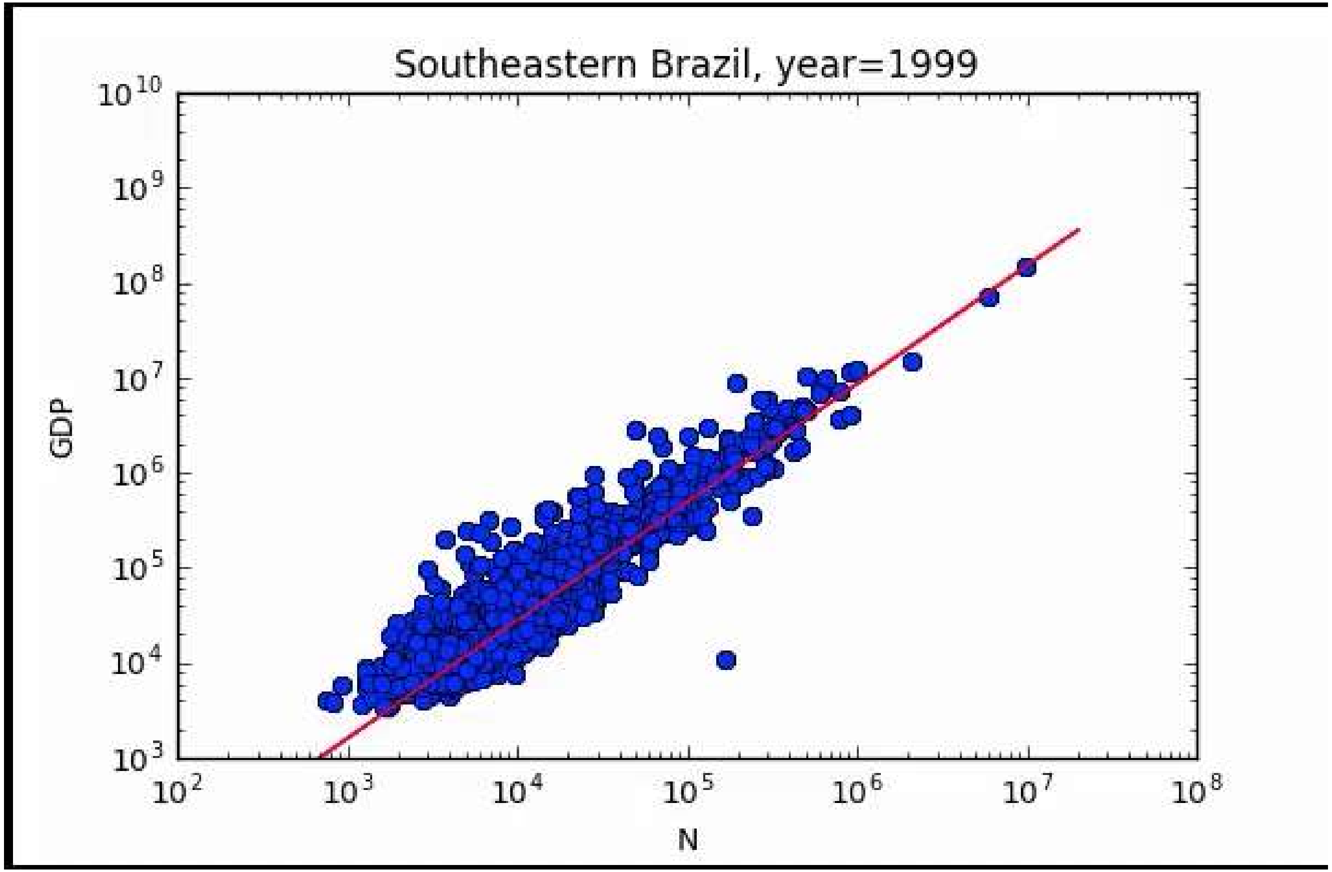
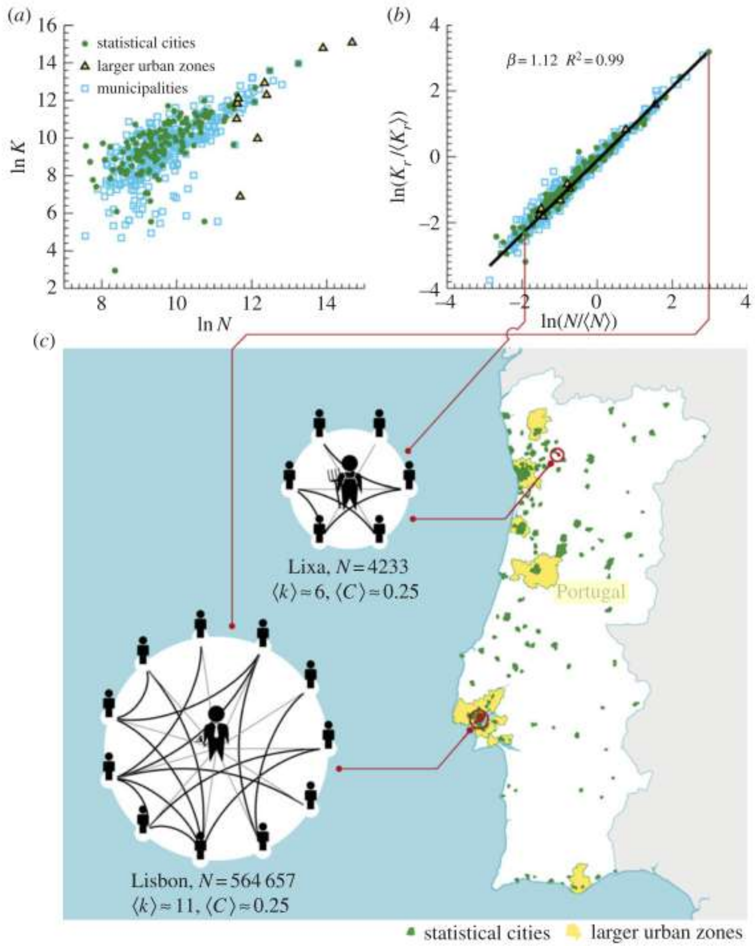
Growth Domestic Product (GDP)



Ribeiro et al, Plos One (2020)

Mobile Phones Contacts



rsif.royalsocietypublishing.org J. R. Soc. Interface 11: 20130789

Number of Petrol Stations

$\beta_{infra} \approx 0.85$

C. Kühnert et al. / Physica A 363 (2006) 96–103 101

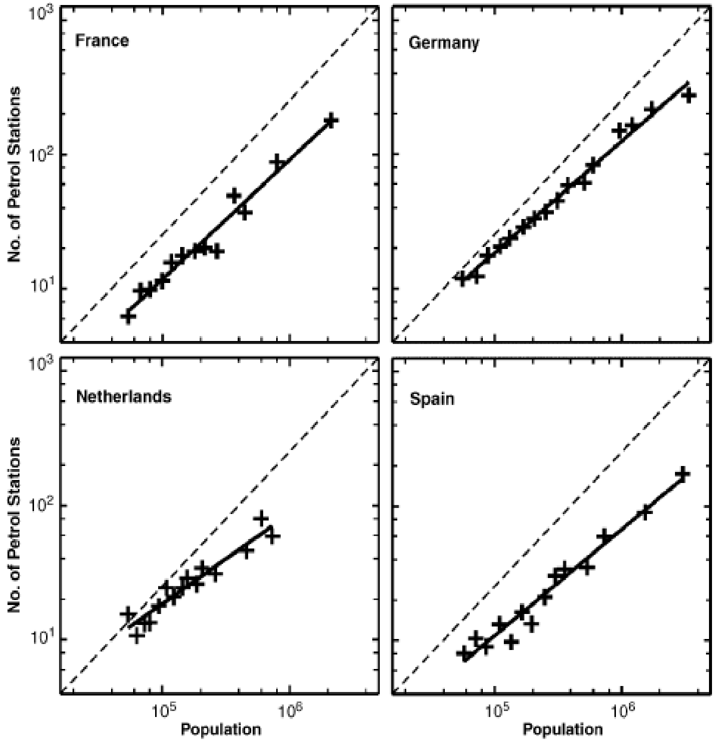


Fig. 4. Double-logarithmic representation of the number of petrol stations as a function of the population size of cities of France, Germany, Netherlands and Spain, after a logarithmic binning method has been applied. The solid lines correspond to the respective linear regression and the dashed lines indicate the slope 1.

Z. Zhang et al, ISPRS (2015).

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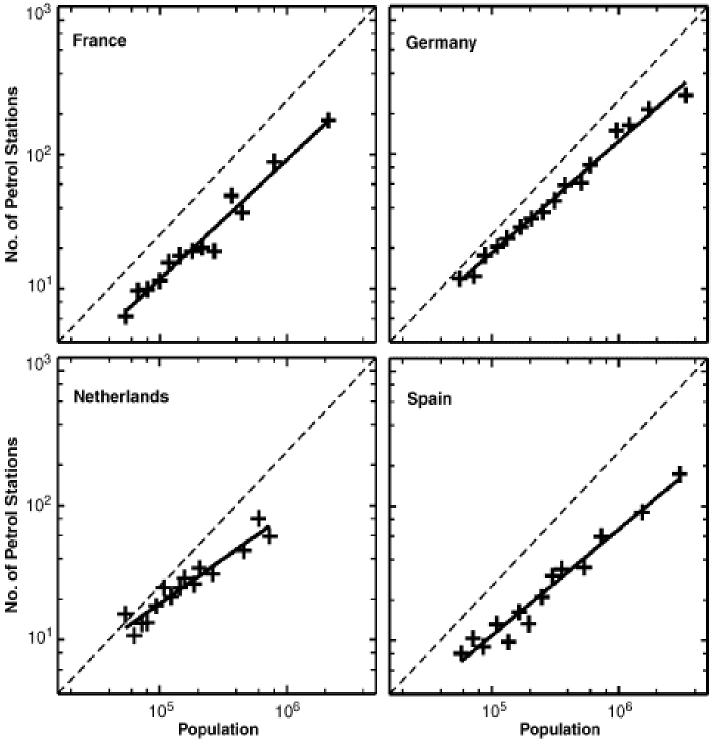

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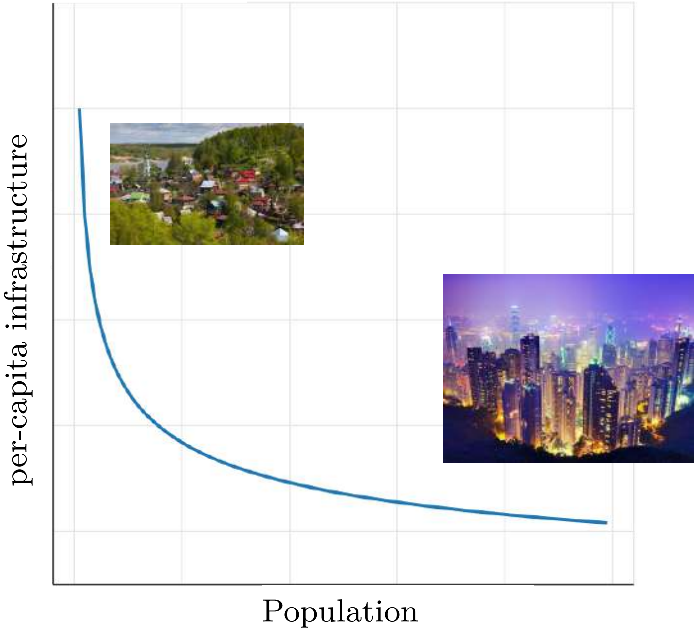


Z. Zhang et al, ISPRS (2015).

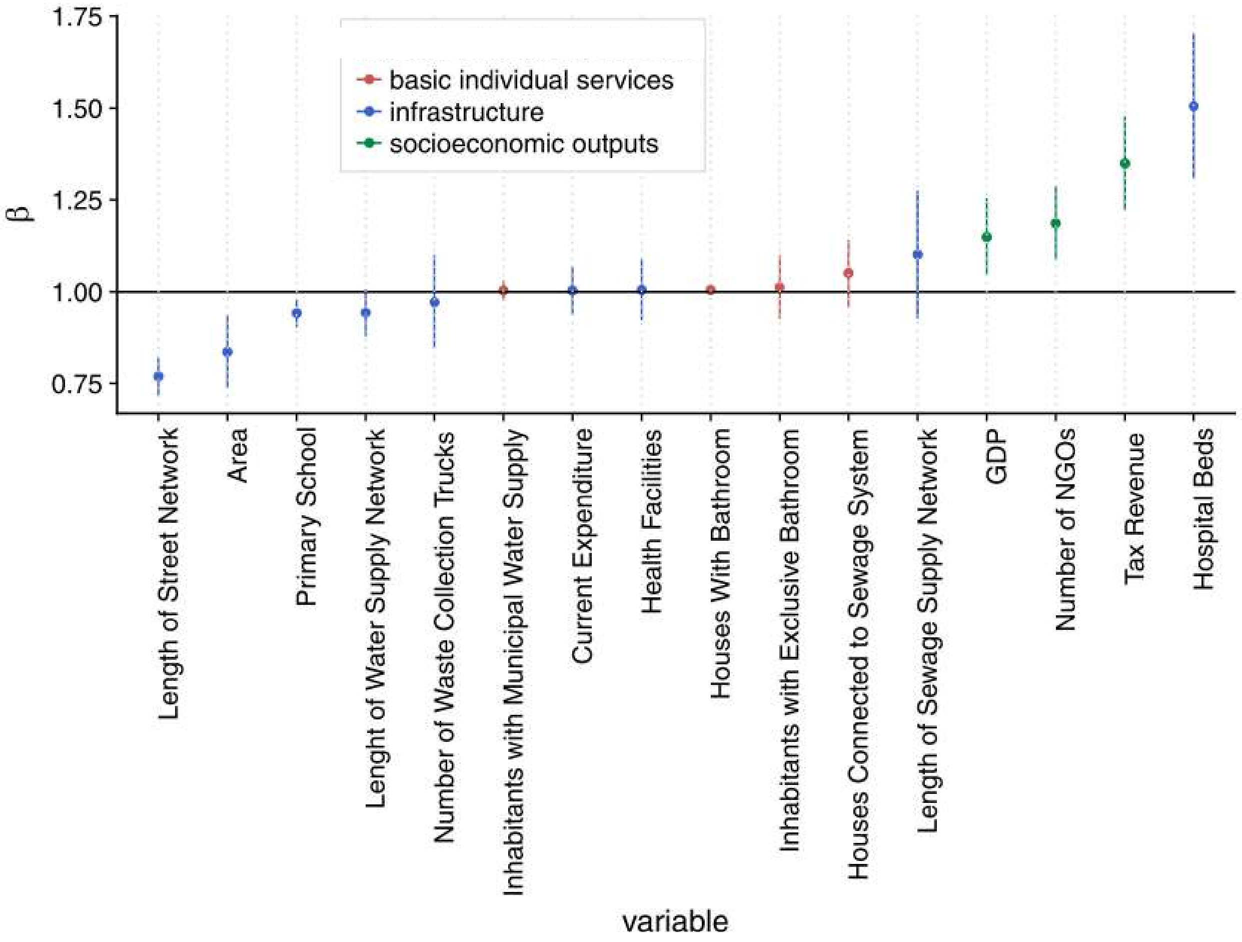
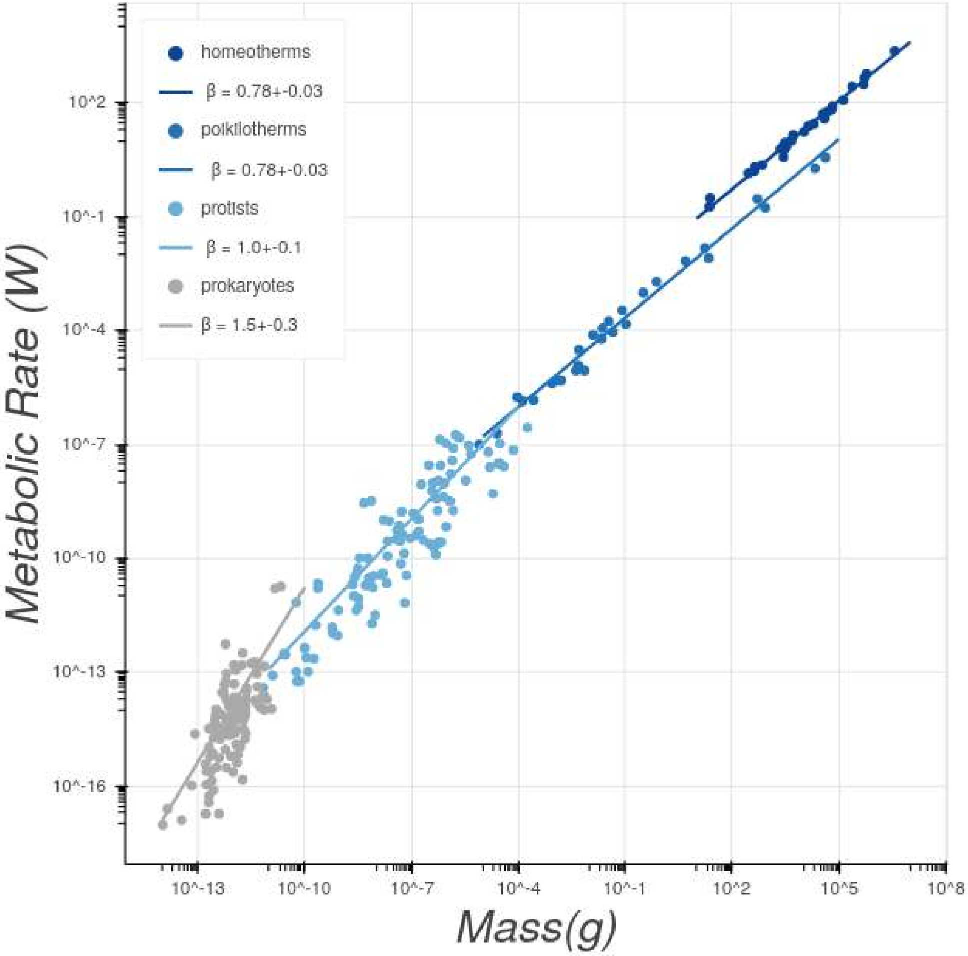


Fig 4. Exponents values for different urban indicators in the Brazilian urban system. Each dot represents the scaling exponent related to the best-fit line from the OLS regression of the population against the studied variable; vertical line segments represent 95% confidence interval (CI) of those regressions; colors are based on Bettencourt's classification; the horizontal black line indicates linear relationship.

https://doi.org/10.1371/journal.pone.0204574.g004

Meirelles et al., Plos One (2018)

Scaling in Biology

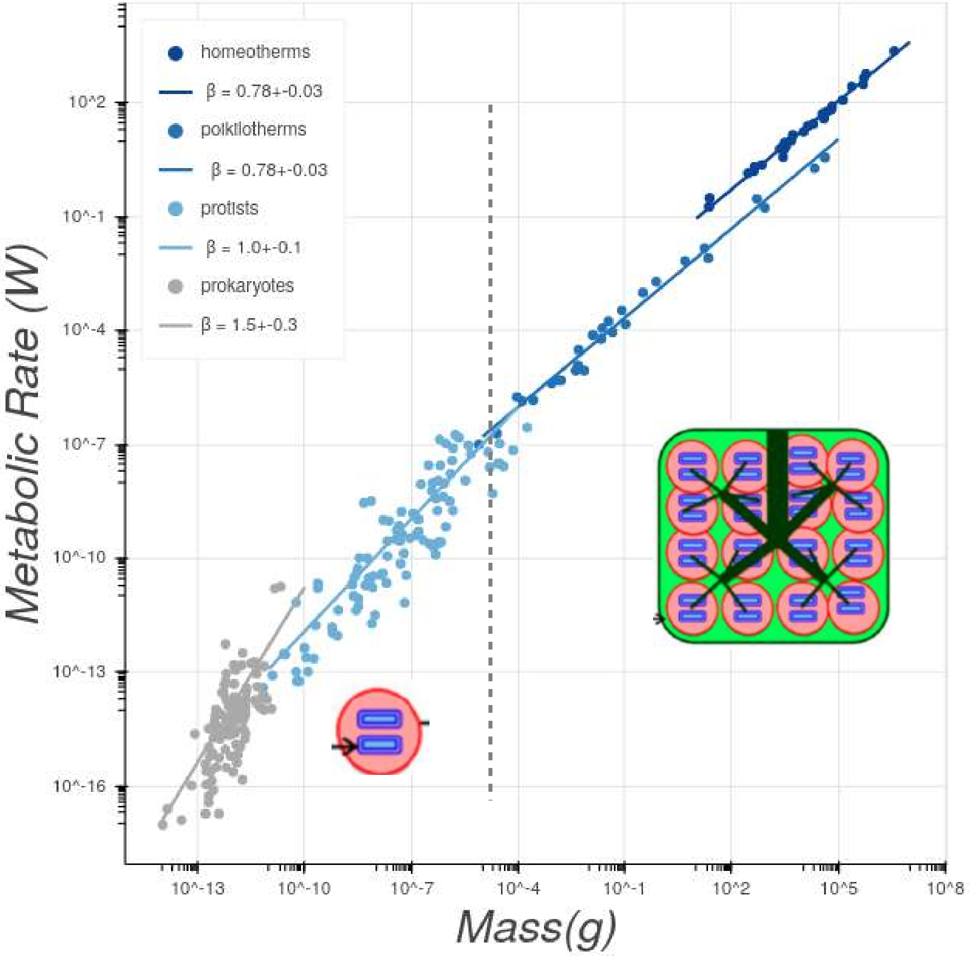


$$B = B_0 M^{\beta}$$

DeLong et al. PNAS (2010)

Ribeiro et al.. Submitted to RevsBrasEnsPhys.

Scaling in Biology



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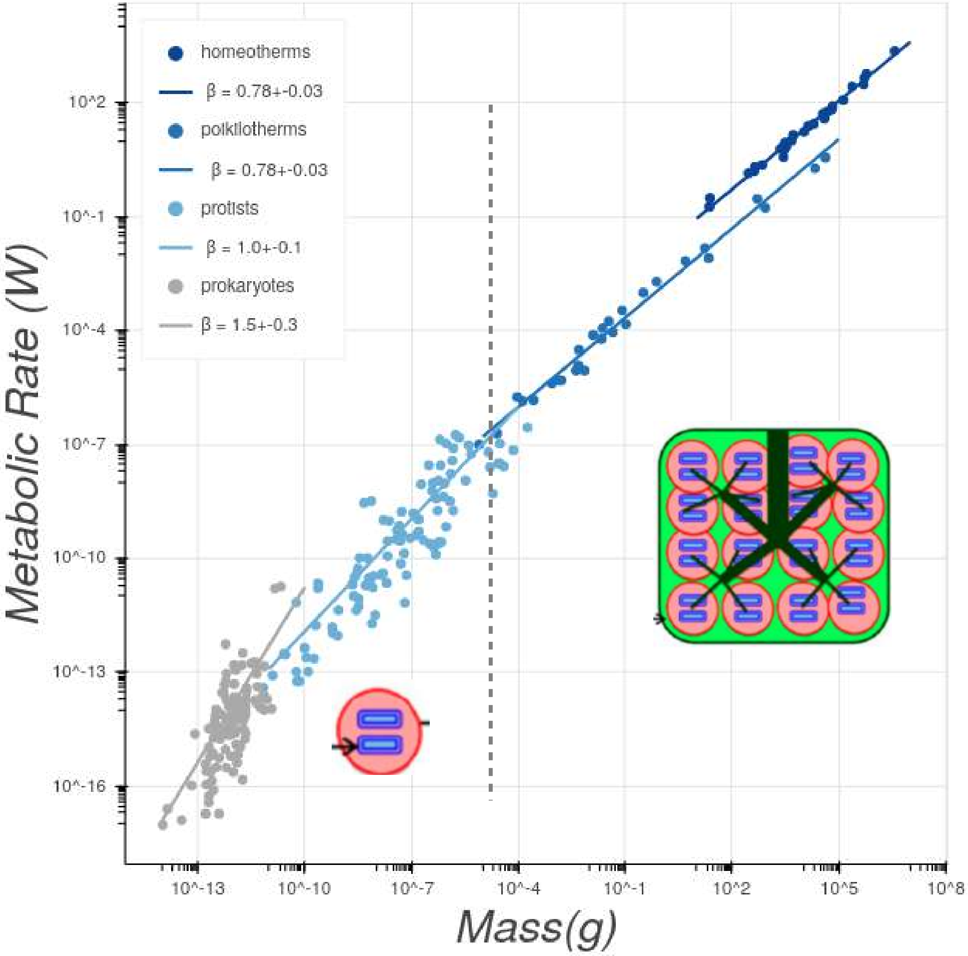
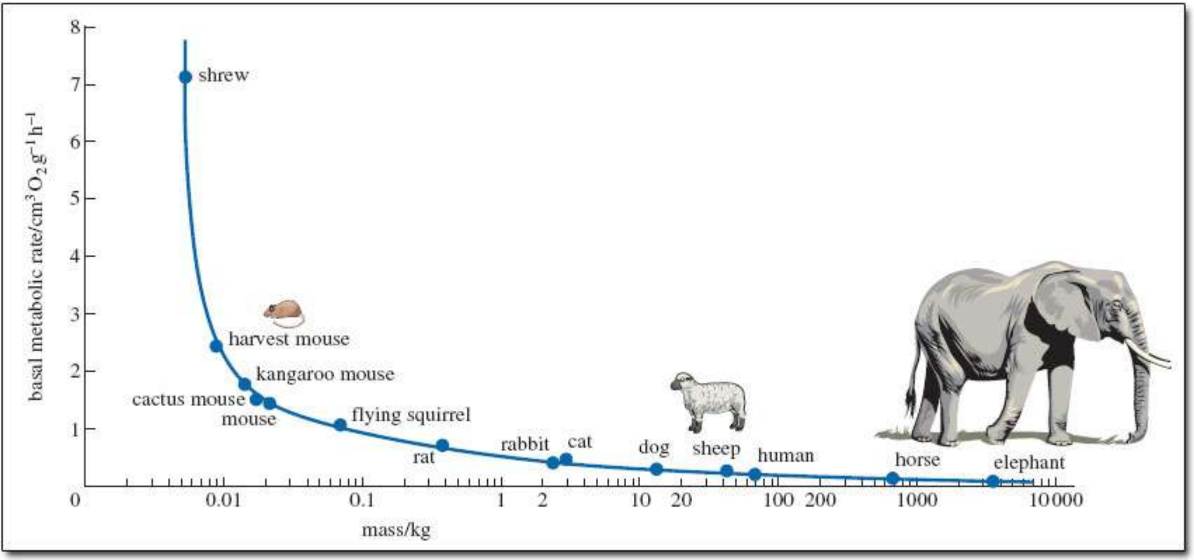


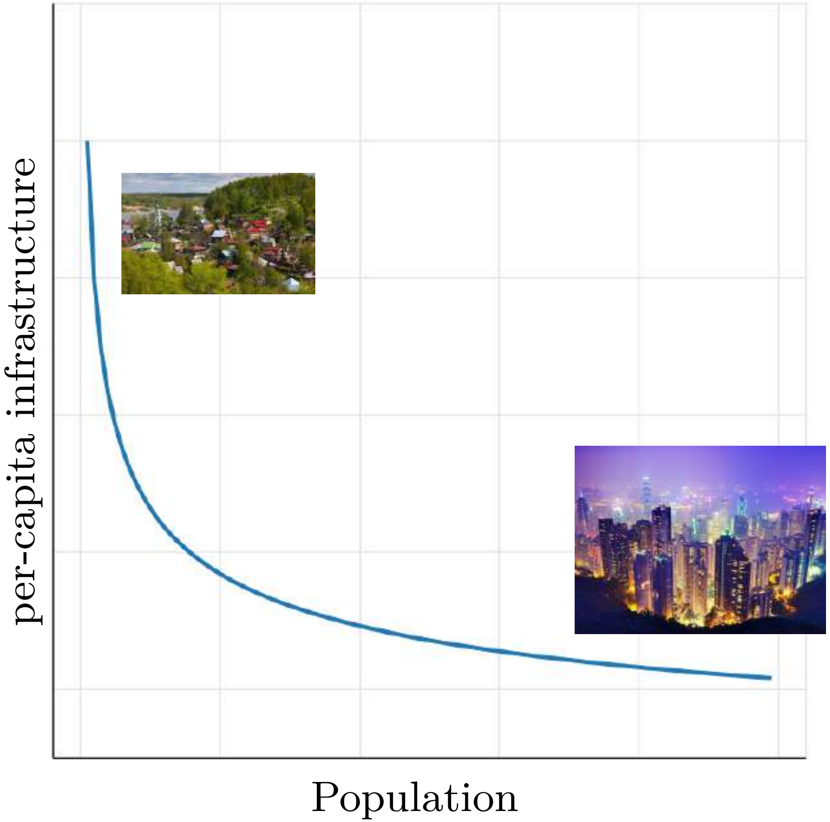
$$B = B_0 M^{\beta}$$

In vascular organisms: $$\beta \approx \frac{3}{4}$$

DeLong et al. PNAS (2010)
Ribeiro et al.. Submitted to RevsBrasEnsPhys.

Scaling Economy

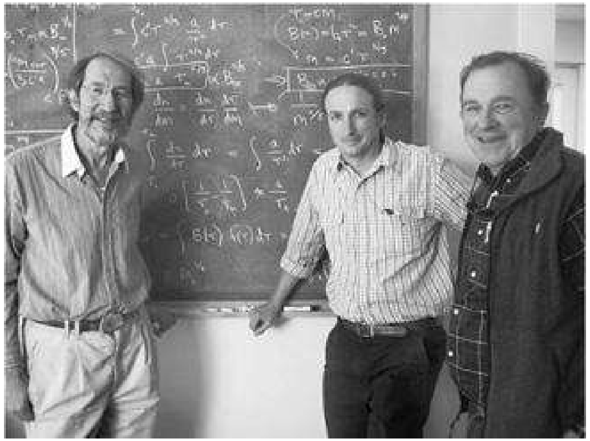




West, Brown & Enquist Theory

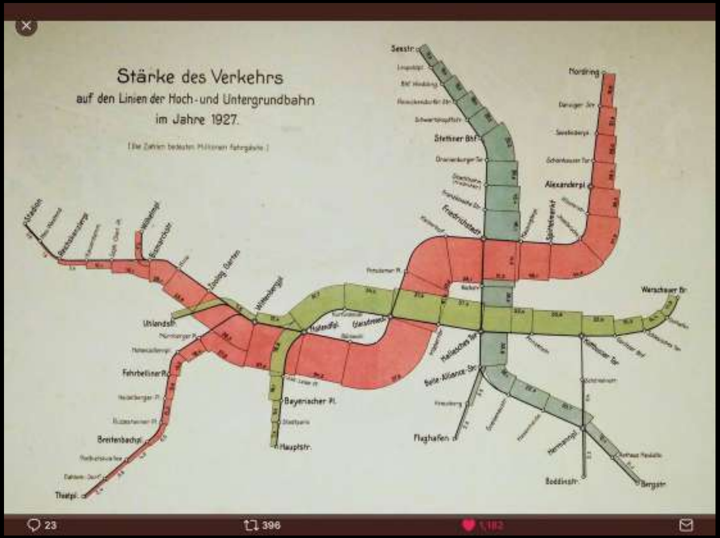
- fractal network distribution;
- terminal units (e.g. cells and capillaries)

  do not vary with the size of organisms;
- natural selection and energy minimization.



$$\Rightarrow \quad \beta = \frac{3}{4}$$

West,Brown & Enquist, Science (1997).



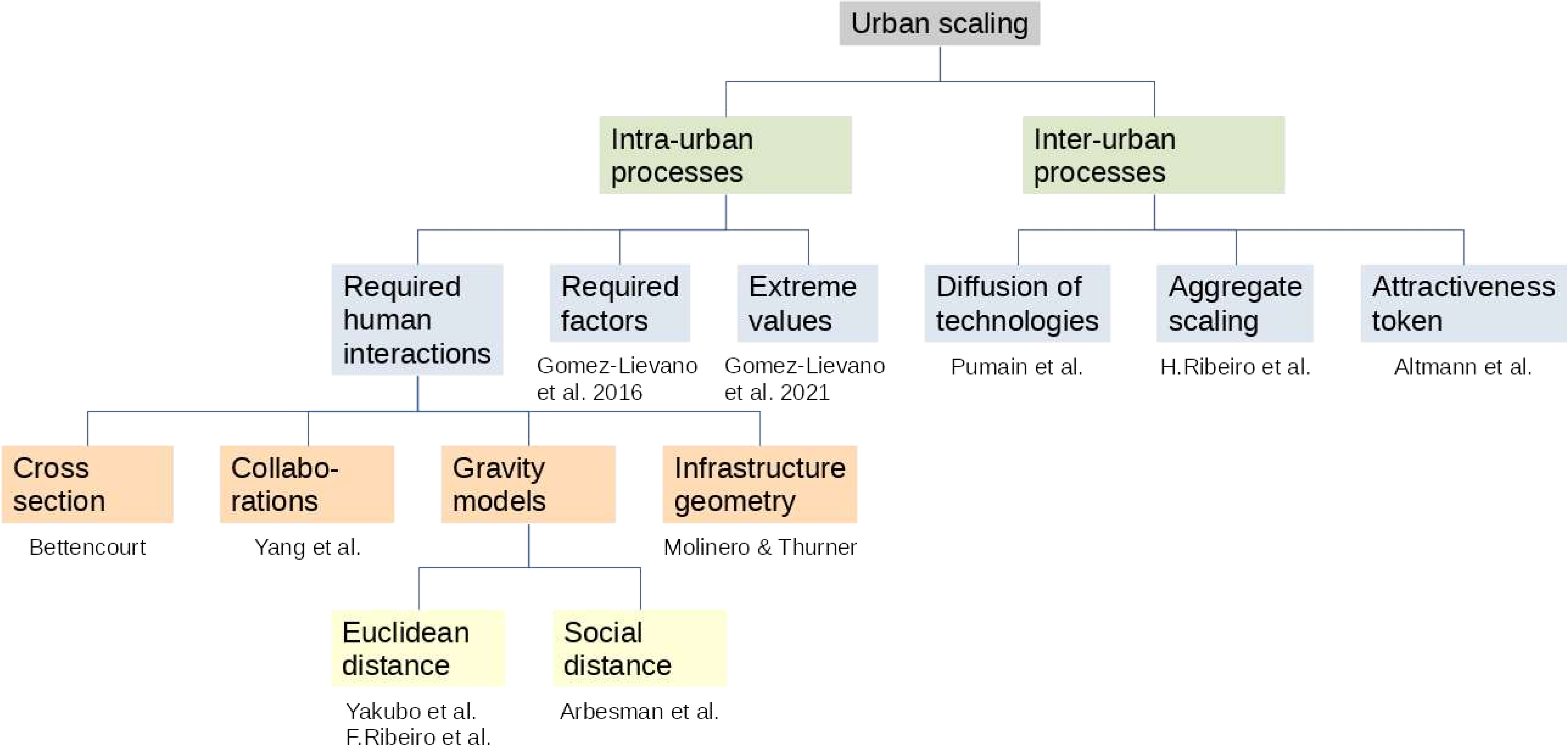
Stärke des Verkehrs
auf den Linien der Hoch- und Untergrundbahn
im Jahre 1927.
Seestr.
Nordring
Stettiner Bhf.
Alexanderpl.
Warschauer Br.
Uhlandstr.
Fehrbelliner Pl.
Breitenbachpl.
Bayerischer Pl.
Hauptstr.
Flughafen
Boddinstr.
Bergstr.
23
396

Scaling Economy

- Cities : $\sim$ 15% (*) $Y = Y_0 N^{\frac{5}{6}}$

- Biology: $\sim$ 25% (*) $B = B_0 M^{\frac{3}{4}}$

(*) for a small increase in size



Ribeiro & Rybsky, in preparation (2021).

Gravity Models to explain urban scaling

- r_{ij} : euclidean distance between two individuals: i and j
- $p(r_{ij})$: propability they are conected;
- Conection implies socio-economic production;
- Hiphotesis:

$$\boxed{p(r) \propto \frac{1}{r^{\gamma}}}$$

Ribeiro et al, Royal Society Open science (2017).

Empirical Evidences

Figure 2: The Distribution of Email Distances

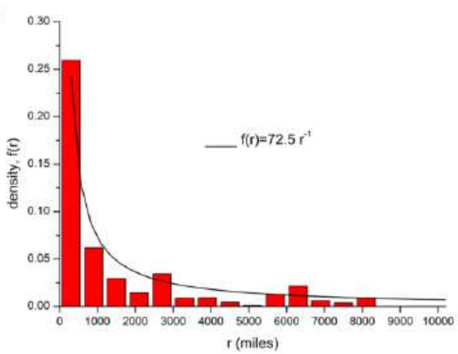
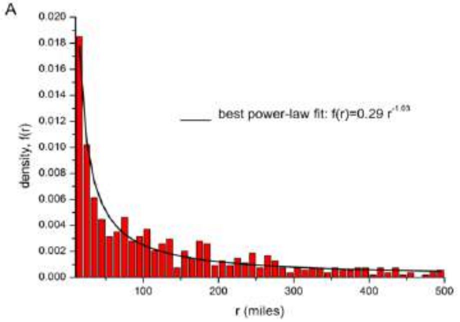
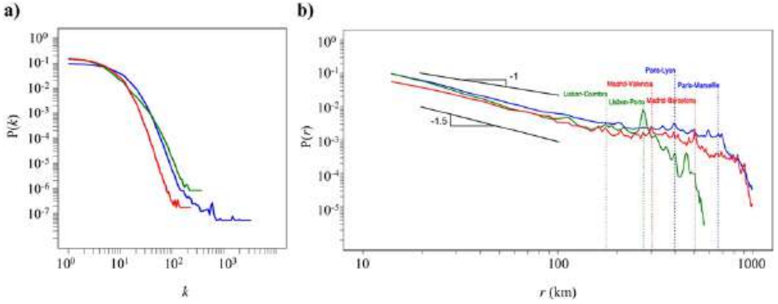


Figure 1: The Distribution of Physical Distances of Facebook Contacts





Country-wide social networks structure. (**a**) Degree distribution for each of the country level networks. (**b**) Probability of a link to have distance r in each of the networks. Distances are grouped in 7 km bins. In all three countries, distribution present a power law decay (exponents between −1 and −1.5) up to 100 km. A large fraction of links lie within the same tower ($r = 0$), averaging 40% in Spain (red), 18% in France (blue) and 21% in Portugal (green).

$$p(r) \propto \frac{1}{r^{\gamma}}$$

$$1 \leq \gamma \leq 1.5$$

J. Goldenberg and M. Levy (2009), C. Herrera-Yague et al. (2015)

Empirical Evidences

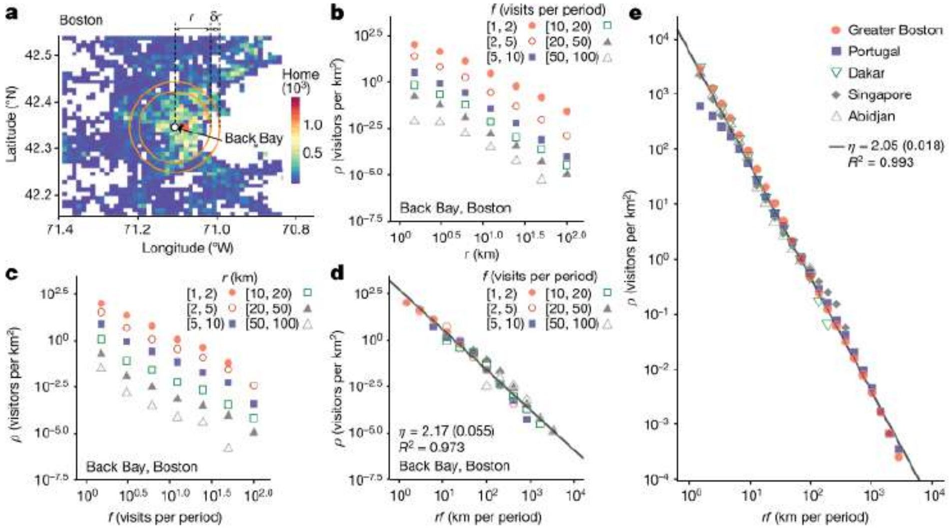
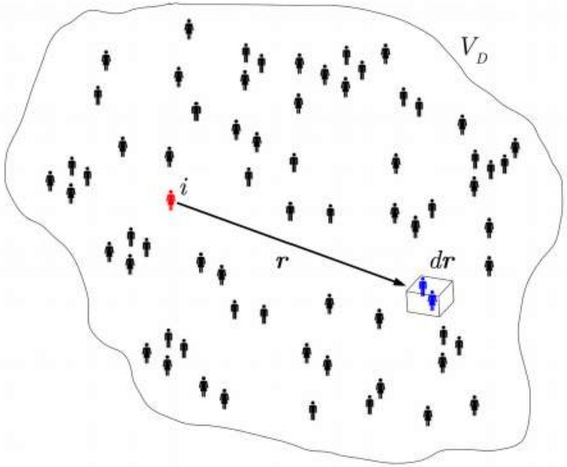


Fig. 1 | The universal distance–frequency distribution of population flows. **a**, For each location, we count the number of visitors who are living at a distance $[r, r+\delta r)$ away and are visiting with frequency f. The map colours indicate the population density derived from the mobile phone data (users per grid cell). **b**, For a fixed frequency f, the visitor flow to a specific location, $\rho_i(r,f)$, decreases with increasing distance r. **c**, When keeping the distance r fixed, the flow decreases similarly with increasing frequency f. **d**, Rescaled values collapse onto a single curve, making the flows dependent only on the single variable rf. The entire distance–frequency distribution is very well described by a power law of the form $\rho_i(r,f) = \mu_i/(rf)^\eta$, with exponent $\eta \approx 2$ (η is the slope of the best-fit line by the least squares method; standard error in parentheses). **e**, Rescaled flows across all studied regions, demonstrating that the same scaling relation holds for radically different urban regions worldwide. Symbols are average values across all locations in each region. To visually compare the different world regions, the shown curves were superimposed by normalizing the distance–frequency distribution of each individual location.

$$\rho \sim \frac{1}{(rf)^2}$$

Schläpfer et al., Nature (2021)

Socio-economic production



$$Y \sim N \int p(\vec{r})\rho(\vec{r})\, d\vec{r}$$

$$\frac{\gamma}{D_p} > 1 \longrightarrow \begin{cases} \bullet\ \textbf{Short range interaction;} \\ \bullet\ Y \sim N \end{cases}$$

$$\frac{\gamma}{D_p} < 1 \longrightarrow \begin{cases} \bullet\ \textbf{Long range interaction;} \\ \bullet\ Y \sim N^{\beta_{se}} \end{cases}$$

$$\boxed{\beta_{se} = 2 - \frac{\gamma}{D_p}}$$

Ribeiro et al, Royal Society Open science (2017).

Infrastructure: Number of Amenities

r_{ik} : distance between individual i and the amenity k

$f(r_{ik})$: amount of a product consumed by i in k

$$\boxed{f(r_{ik}) \propto \frac{1}{r_{ik}^{\gamma}}}$$

$$u_i = \sum_{k=1}^{P} f(r_{ik})$$

$$U = \sum_{i=1}^{N} u_i = \sum_{i=1}^{N}\sum_{k=1}^{P} f(r_{ik})$$

$$U = \sum_{k=1}^{P}\left(\sum_{i=1}^{N} f(r_{ik})\right)$$

- $\sum_{i=1}^{N} f(r_{ik}) \sim \int f(\mathbf{r})\rho(\mathbf{r})d\mathbf{r} \sim N^{1-\frac{\gamma}{D_p}}$
- $U \sim N$

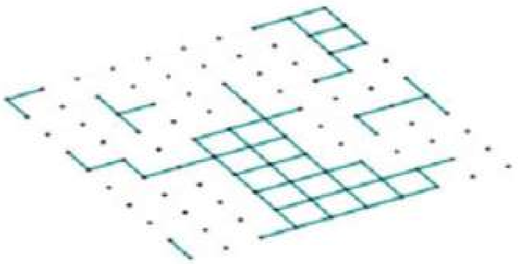
U: The total quantity of an individual need product.

$$U \sim PN^{1-\frac{\gamma}{D_p}}$$

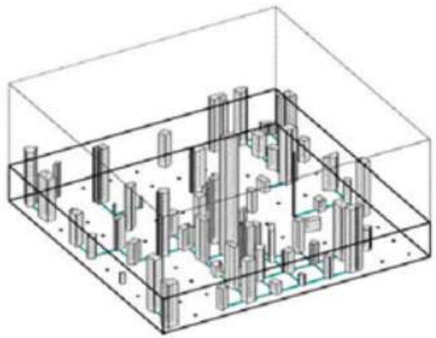
$$P \sim N^{\frac{\gamma}{D_p}} \quad \Rightarrow \quad \boxed{\beta_{infra} = \frac{\gamma}{D_P}}$$

Ribeiro et al., Royal Society Open science (2017).

Molinero & Thurner Model



$$L \sim r^{D_{infra}}$$



$$N \sim r^{D_p}$$

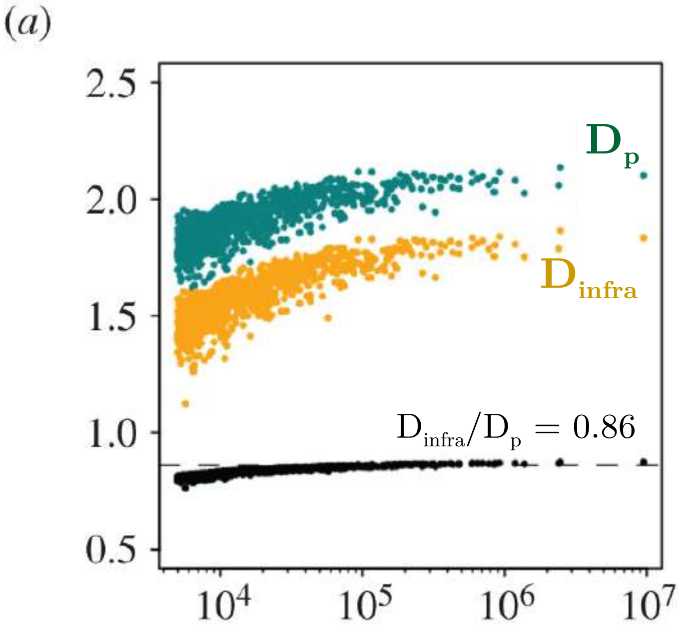
$$\Rightarrow \quad L \sim N^{\frac{D_{infra}}{D_p}}$$

Amenities are along (attached) to the streets

$$\boxed{P \sim N^{\frac{D_{infra}}{D_p}}}$$

Molinero & Thurner, Interface (2021).

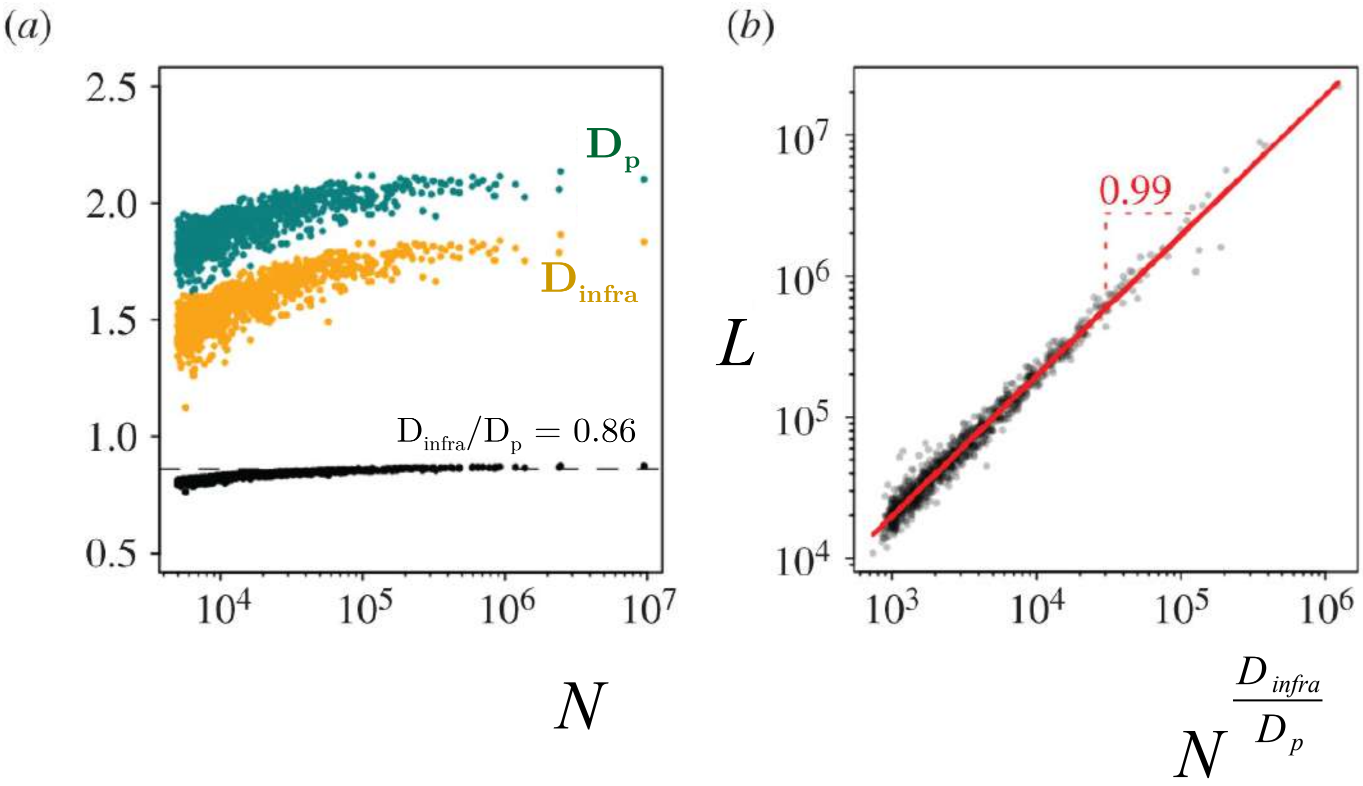
Molinero & Thurner Model



N

Molinero & Thurner, Interface (2021).

Molinero & Thurner Model



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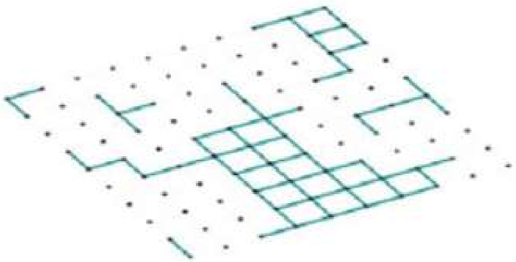
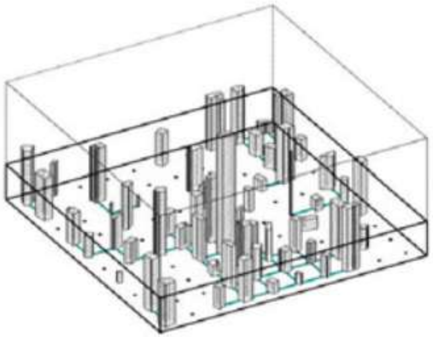


$$L \sim r^{D_{infra}}$$



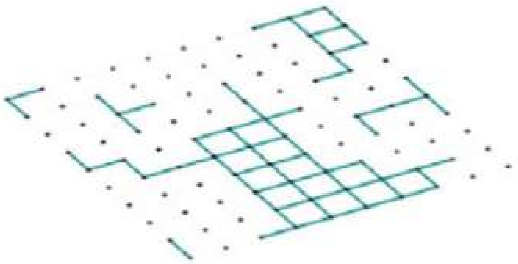
$$N \sim r^{D_p}$$

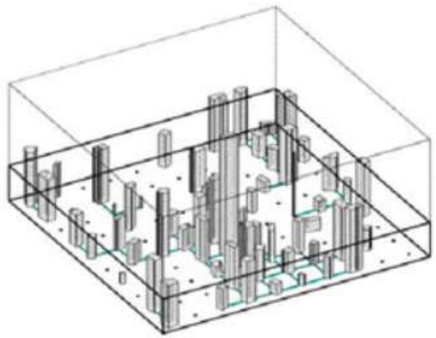
$$\Rightarrow \quad L \sim N^{\frac{D_{infra}}{D_p}} \xrightarrow{\text{Amenities are along (attached) to the streets}} \boxed{P \sim N^{\frac{D_{infra}}{D_p}}}$$

Molinero & Thurner, Interface (2021).

Ribeiro & Rybski , in preparation (2021)

Molinero & Thurner Model





$$L \sim r^{D_{infra}}$$

$$N \sim r^{D_p}$$

Ribeiro et al.
Model:

$$\Rightarrow \quad L \sim N^{\frac{D_{infra}}{D_p}}$$

Amenities are along (attached) to the streets

$$\xrightarrow{}$$

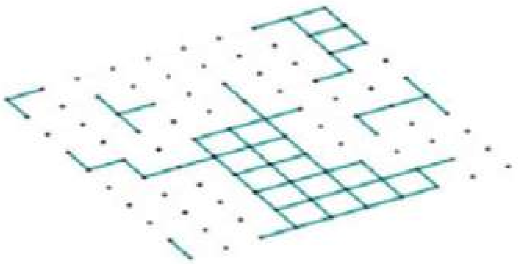
$$\boxed{P \sim N^{\frac{D_{infra}}{D_p}}}$$

$$\boxed{P \sim N^{\frac{\gamma}{D_p}}}$$

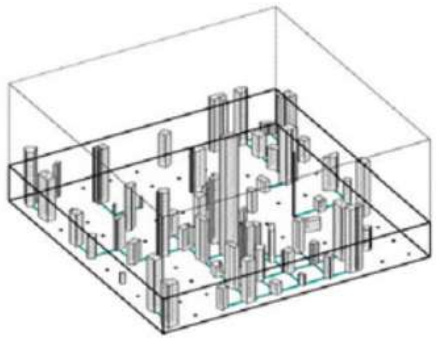
Molinero & Thurner, Interface (2021).

Ribeiro & Rybski , in preparation (2021)

Molinero & Thurner Model



$$L \sim r^{D_{infra}}$$



$$N \sim r^{D_p}$$

$$\Rightarrow \quad L \sim N^{\frac{D_{infra}}{D_p}}$$

Amenities are along (attached) to the streets

$$\longrightarrow \quad \boxed{P \sim N^{\frac{D_{infra}}{D_p}}}$$

Ribeiro et al. Model:

$$\boxed{P \sim N^{\frac{\gamma}{D_p}}}$$

Equivalence between the models implies:

$$\boxed{p_{ij}(r) = \frac{1}{r^{D_{infra}}}}$$

Molinero & Thurner, Interface (2021).

Ribeiro & Rybski , in preparation (2021)

PLOS ONE

RESEARCH ARTICLE

On the relation between transversal and longitudinal scaling in cities

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Abstract

Does the scaling relationship between population sizes of cities with urban metrics like economic output and infrastructure (transversal scaling) mirror the evolution of individual cities in time (longitudinal scaling)? The answer to this question has important policy implications, but the lack of suitable data has so far hindered rigorous empirical tests. In this paper, we advance the debate by looking at the evolution of two urban variables, GDP and water network length, for over 5500 cities in Brazil. We find that longitudinal scaling exponents are city-specific. However, they are distributed around an average value that approaches the transversal scaling exponent provided that the data is decomposed to eliminate external factors, and only for cities with a sufficiently high growth rate. We also introduce a mathematical framework that connects the microscopic level to global behaviour, finding good agreement between theoretical predictions and empirical evidence in all analyzed cases. Our results add complexity to the idea that the longitudinal dynamics is a micro-scaling version of the transversal dynamics of the entire urban system. The longitudinal analysis can reveal differences in scaling behavior related to population size and nature of urban variables. Our approach also makes room for the role of external factors such as public policies and development, and opens up new possibilities in the research of the effects of scaling and contextual factors.

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A model of urban scaling laws based on distance-dependent interactions

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Socio-economic related properties of a city grow faster than a linear relationship with the population, in a log–log plot, the so-called *superlinear scaling*. Conversely, the larger a city, the more efficient it is in the use of its infrastructure, leading to a *sublinear scaling* on these variables. In this work, we addressed a simple explanation for those scaling laws in cities based on the interaction range between the citizens and on the fractal properties of the cities. To this purpose, we introduced a measure of social potential which captured the influence of social interaction on the economic performance and the benefits of amenities in the case of infrastructure offered by the city. We assumed that the population density depends on the fractal dimension and on the distance-dependent interactions between individuals. The model suggests that when the city interacts as a whole, and not just as a set of isolated parts, there is improvement of the socio-economic indicators. Moreover, the bigger the interaction range between citizens and amenities, the bigger the improvement of the socio-economic indicators and the lower the infrastructure costs of the city. We addressed how public policies could take advantage of these properties to improve cities development, minimizing negative effects. Furthermore, the model predicts that the sum of the scaling exponents of social-economic and infrastructure variables are 2, as observed in the literature. Simulations with an agent-based model are confronted with the theoretical approach and they are compatible with the empirical evidences.

Scaling Laws in Urban Systems:

The Physics of Cities

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