Mixing innovative and imitative dynamics in evolutionary games: the role of update rules

Marco Antonio Amaral

Universidade Federal do Sul da Bahia

20 de Outubro de 2021
Classical Game Theory

- “Interdependent decision making strategies”, Von Neumann (1928)
- Mainly regarding economics and war decisions.
- Which is the best strategy?
Maynard Smith (1970) applies game theory to evolution. Best strategies leave more descendants. Natural selection based on birth-death process (imitation). Which is the **evolutionary stable** strategy?
- Game round, payoff accumulation.
- Strategy update round.
- Parametrization $R=1, P=0$.
- Coordination, anti-coordination, dilemma.
Usual imitative update.
● Innovative update.

![Diagram showing network dynamics with a node labeled D appearing in the transition from left to right.]
Logit Model (Ising)

- Player consider its future payoff, if it had another strategy.
- Cognitive analysis of the future situation.
- Similar to Glauber dynamics from Ising Models.

\[ p(\Delta u_i) = \frac{1}{1 + e^{-(u_{i*} - u_i)/k}} \]  

\[ u=4R \quad \text{to} \quad u^*=4T \]
Dynamic Win-Stay-Lose-Shift

- Player compares its payoff with the neighbors average payoff, $\langle u \rangle$.
- In times of crises we lower our expectations.

$$p(\Delta u_i) = \frac{1}{1 + e^{-(\langle u \rangle - u_i)/k}}$$  (2)

![Diagram showing a grid with payoffs 4R, 3R, and 4T]
- Cooperation ($\rho$) versus Temptation to betray ($T$).
- Weak prisoners dilemma ($S = 0$), square lattice.
- Imitative dynamics, square lattice.
- Minimum cooperation level for innovative models.
- Nevertheless, WSLS and Logit are quite different.


- Scale-Free networks have small effects on innovative dynamics.
- Such behavior was seen in human experiments (Gracia-Lázaro et al, *PNAS* 2012).
- $T - S$ parameter space.
- There is always coexistence of strategies in the WSLS model.
- Logit dynamics is more similar to Imitation models.
- Curious homogeneous region for SD game.
Continuous change from imitative to innovative is possible

\( \alpha \% \) of the lattice is innovative

Non-linear behavior: few innovators can greatly help cooperation

---

- Logit and WSLS innovative models.
- Both maintain a minimum cooperation level.
- Still, they are strikingly different in the $T - S$ plane.
- Reduced topology dependence.
- Coexistence of strategies.
- A small change in update rule can lead to very different dynamics.
Thanks

We would like to thank CNPq and CAPES for the financial support.
Appendix
• Lattice sizes $N = [100, 16000]$
• Samples $= [10, 500]$
• Connectivity coefficient for SF networks $= 2.7$
• MC steps until thermalization $MCS = [1000, 60000]$
• MC steps averaged over $= [1000, 5000]$

![Diagram of lattice and networks](image.png)
Level of Cooperation

A) experiment

B) control

C) lattice

D) heterogeneous

(Gracia-Lázaro et al, PNAS 2012)
- Average population payoff ($\langle u \rangle$) increases with $T$.
- Related to spatial organization.
Monte Carlo and mean-field pair approximation, Logit.
Monte Carlo and mean-field pair approximation, WSLS.
- Fixed $T$, varying $S$ vertically.
- $\rho$ is independent from $T$ for the Logit model.
- Cooperation even for $S = -0.2$. 
- Typical snapshot from the lattice.
- No cooperators island!

![Imitation](image1.png)  ![WSLS](image2.png)
- Typical WSLS process.
- Cooperation spreads through second neighbors.
- Average frustration ($\phi$) for $T - S$ plane.
- Logit model drives the system to the lower frustration.
- Which coincides with highest payoff.
Subtle differences with spin models.

Energy is conserved, payoff is not. (Non-zero sum games).

\[
\begin{pmatrix}
C & D \\
C & R & S \\
D & T & P
\end{pmatrix},
\uparrow \begin{pmatrix}
-J - B & J - B \\
J + B & -J + B
\end{pmatrix}.
\]

\(J + B > 0, \quad 0 > J \quad \text{e} \quad J > B\),

(3)
Different thermalization times.
- Payoff and aspirations grows with $T$. 

![Graph showing the payoffs and aspirations with $T$.]
Mapeamento em processo Markoviano ⇒ Equação Mestra:

$$\frac{d}{dt} P({s}, t) = \sum_{\{s\}'} P({s}', t) W({s}' \rightarrow {s})$$

$$- P({s}, t) W({s}, \rightarrow {s}')$$

Uso de campo médio e aproximações gera uma eq. dinâmica:

$$\dot{\rho} = \rho W_+ - (1 - \rho) W_-$$

Vizinhança com n cooperadores

$$W_\pm = \sum_{n}^N \binom{N}{n} \rho^n (1 - \rho)^{(N-n)} P(u_i)$$
Sítio $i$ atualiza sua estratégia com probabilidade:

$$p(\Delta u_i) = \frac{1}{1 + e^{-(\bar{u} - u_i)/k}}.$$ (8)
Equação de campo médio:

\[ \dot{\rho} = (1 - \rho) W_{+(D \rightarrow C)} - \rho W_{-(C \rightarrow D)} , \]  

(9)

com taxas de transição (grafo totalmente conectado):

\[ W_+ = \frac{1}{1 + e^{-(\bar{u} - \bar{u}_d)/k}} , \]  

(10)

\[ W_- = \frac{1}{1 + e^{-(\bar{u} - \bar{u}_c)/k}} . \]  

(11)
Para o dilema do prisioneiro fraco, \( R = 1 \) e \( S = P = 0 \):

\[
\bar{u} = \rho^2 R + (1 - \rho)^2 P + \rho(1 - \rho)(T + S) = \\
= \rho^2 + \rho(1 - \rho)T,
\]
\( \bar{u}_c = \rho R + (1 - \rho)S = \rho, \) \hspace{1cm} (12)
\[
\bar{u}_d = \rho T + (1 - \rho)P = \rho T.
\] \hspace{1cm} (13)
Substituindo tais expressões na equação de taxas:

\[
\frac{d\rho}{dt} = \frac{1 - \rho}{1 + e^{-\left(\rho^2(1-T)/k\right)}} - \frac{\rho}{1 + e^{-\left(\rho(\rho-1)(1-T)/k\right)}}.
\] (15)
A aproximação de primeiros vizinhos
Vizinhança com \( n \) cooperadores
Payoff do sítio focal: \( u_{iD} = nT \) e \( u_{iC} = n \).
Taxa com que um sítio se torna \( C \) ou \( D \):

\[
W_\pm = \sum_{n=0}^{4} \binom{4}{n} \rho^n (1 - \rho)^{(4-n)} P_\pm (u_i, \bar{u}),
\]  

(16)
Payoff médio dos vizinhos:

$$
\bar{u} = \frac{1}{4} \left[ n3\rho + (4 - n)3T\rho \right] = \frac{3}{4}(n\rho + 4T\rho - nT\rho) . \quad (17)
$$

Assim

$$
P_+ = \frac{1}{1 + e^{-[T(12\rho - 3n\rho - 4n) + 3n\rho]/4k}} , \quad (18)
$$

e

$$
P_- = \frac{1}{1 + e^{[T(3n\rho + n - 4 - 12\rho) - 3n\rho + 3n]/4k}} . \quad (19)
$$
\[
W_\pm = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1 - \rho)^{N-n} P_\pm ( u_i, u_\Omega ) . \tag{20}
\]

A equação dinâmica para o modelo de Logit se torna:

\[
\dot{\rho} = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1 - \rho)^{N-n} \left( \frac{1}{1 + e^{-n(1-\tau)/k}} - \rho \right) . \tag{21}
\]
Aproximação de campo médio de primeiros vizinhos para esse modelo.

\[
\dot{\rho} = (1 - \rho)W_{+(C \rightarrow D)} - \rho W_{-(D \rightarrow C)}, \quad (22)
\]

\[
W_{\pm} = \sum_{n=0}^{N} \binom{N}{n} \rho^n(1 - \rho)^{(N-n)} P_{\pm}(u_i, u_\Omega). \quad (23)
\]
Para o modelo de Logit usamos:

\[ P_{\pm}(u_i, u_\Omega) = \frac{1}{1 + e^{- (u^* - u_i)/k}}. \]  

(24)

Probabilidades dependem somente do payoff de \( i \).

Diferença de payoff atual para o futuro (\( n \) vizinhos \( C \)):

\( (u^* - u_i)_{D \rightarrow C} = n(1 - T) \),  

(25)

\( (u^* - u_i)_{C \rightarrow D} = n(T - 1) \).  

(26)
Usando $A = (1 - T)/k$ para simplificar, obtemos:

$$P_{\pm}(u_i, u_\Omega) = \frac{1}{1 + e^{\mp nA}}.$$  \hspace{1cm} (27)

Substituindo esse termo nas taxas de transição,

$$W_{\pm} = \sum_{n=0}^{N} \binom{N}{n} \rho^n(1 - \rho)^{(N-n)} \frac{1}{1 + e^{\mp nA}}.$$  \hspace{1cm} (28)

O que leva a eq. dinâmica:

$$\dot{\rho} = (1 - \rho) \sum_{n=0}^{N} \binom{N}{n} \rho^n(1 - \rho)^{(N-n)} \frac{1}{1 + e^{-nA}}$$

$$-\rho \sum_{n=0}^{N} \binom{N}{n} \rho^n(1 - \rho)^{(N-n)} \frac{1}{1 + e^{+nA}}.$$  \hspace{1cm} (29)
A expressão acima pode ser simplificada:

\[
\dot{\rho} = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1 - \rho)^{(N-n)} \times \\
\left[ (1 - \rho) \frac{1}{1 + e^{-nA}} - \rho \frac{1}{1 + e^{nA}} \right],
\]

(30)

\[
\dot{\rho} = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1 - \rho)^{(N-n)} \times \\
\left[ \frac{1}{1 + e^{-nA}} - \rho \left( \frac{1}{1 + e^{-nA}} + \frac{1}{1 + e^{nA}} \right) \right].
\]

(31)
A equação dinâmica para o modelo de Logit se torna:

$$\dot{\rho} = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1 - \rho)^{(N-n)} \left( \frac{1}{1 + e^{-\frac{-n(1-T)}{k}}} - \rho \right). \quad (32)$$

$$\dot{\rho} = 0$$ nos dá pontos fixos da eq.

Polinômio de 7 ordem.

Pelo menos uma raiz no intervalo de $0 < \rho^* < 1$.

Independente de $T$. 
Aproximação de Pares:

- Estrutura espacial e solução analítica (para primeiros vizinhos)
- Aproximação de Campo-Médio para segundos vizinhos

\[
\begin{align*}
    u_c &= R \rho_c + S \rho_d \\
    u_d &= T \rho_c + P \rho_d
\end{align*}
\] (33)
\[ P(\{k\}; \alpha \beta) W_{\alpha \beta \rightarrow \gamma \beta} = A_{\alpha \beta} + B_{\alpha \beta} + C_{\alpha \beta} \]
Equações de taxa para ligações (ao invés de sítios):

\[
\dot{\Gamma}_{cc} = \sum_{\{k\}} P(\{k\}; cd) W_{cd\rightarrow cc}\{k\} - P(\{k\}; cc) W_{cc\rightarrow cd}\{k\} \tag{35}
\]

\[
\dot{\Gamma}_{cd} = \sum_{\{k\}} P(\{k\}; cc) W_{cc\rightarrow cd}\{k\} + P(\{k\}; dd) W_{dd\rightarrow cd}\{k\} - P(\{k\}; cd) W_{cd\rightarrow cc}\{k\} - P(\{k\}; cd) W_{cd\rightarrow dd}\{k\} \tag{36}
\]

\[
\dot{\Gamma}_{dd} = \sum_{\{k\}} P(\{k\}; cd) W_{cd\rightarrow dd}\{k\} - P(\{k\}; dd) W_{dd\rightarrow cd}\{k\} \tag{37}
\]
Condições de contorno:

\[ \Gamma_{cc} + \Gamma_{cd} + \Gamma_{dd} = 1 \]

\[ \rho_c = \frac{\Gamma_{cc} + \Gamma_{cd}}{2} \]

\[ \rho_c + \rho_d = 1 \]
\[ A_{(i=\alpha,j=\beta)} = \sum_{x,y,z,v,w,u} \eta_{\beta} \left[ \frac{\Gamma_{ix} \Gamma_{iy} \Gamma_{iz}}{\Gamma_{i}^3} \right] \left[ \frac{\Gamma_{ij} \Gamma_{jv} \Gamma_{jw}}{\Gamma_{j}^3} \right] p(\Delta u_{ij}) \text{, } \tag{B7} \]

\[ B_{(i=\alpha,x=\beta)} = 3 \sum_{y,z,v,w,u} \sum_{j} \eta_{\beta} \left[ \frac{\Gamma_{ix} \Gamma_{iy} \Gamma_{iz}}{\Gamma_{i}^3} \right] \left[ \frac{\Gamma_{ij} \Gamma_{jv} \Gamma_{jw}}{\Gamma_{j}^3} \right] p(\Delta u_{ix}) \text{, } \tag{B8} \]

\[ C_{(i=\beta,x=\alpha)} = 3 \sum_{y,z,v,w,u} \sum_{j} \left[ \frac{\Gamma_{ix} \Gamma_{iy} \Gamma_{iz}}{\Gamma_{i}^3} \right] \left[ \frac{\Gamma_{ij} \Gamma_{jv} \Gamma_{jw}}{\Gamma_{j}^3} \right] p(\Delta u_{xi}) \text{. } \tag{B9} \]