Mixing innovative and imitative dynamics in evolutionary games: the role of update rules

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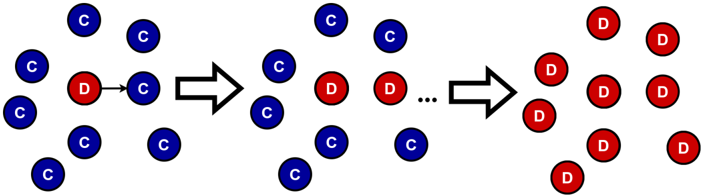
Classical Game Theory

- "Interdependent decision making strategies", Von Neumann (1928)
- Mainly regarding economics and war decisions.
- Which is the **best** strategy?

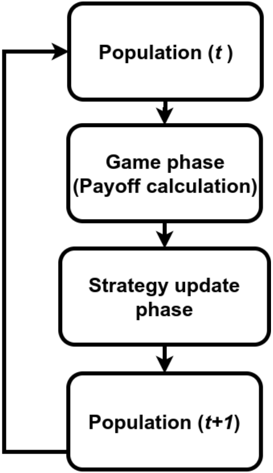
	C	**D**
C	R	S
D	T	P

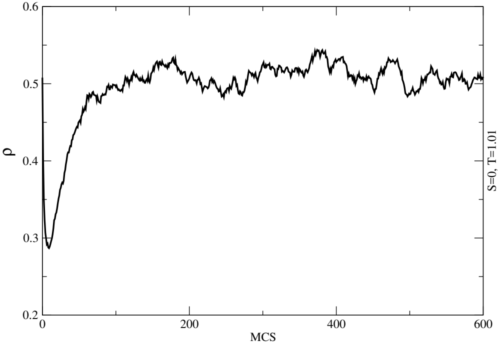
Evolutionary Game Theory

- Maynard Smith (1970) applies game theory to evolution.
- Best strategies leave more descendants.
- Natural selection based on birth-death process (imitation).
- Which is the **evolutionary stable** strategy?



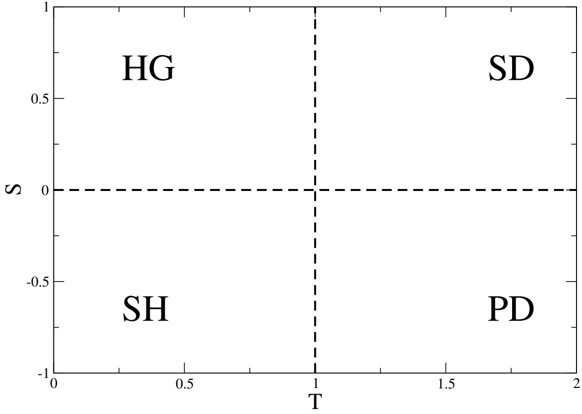
- Game round, payoff accumulation.
- Strategy update round.



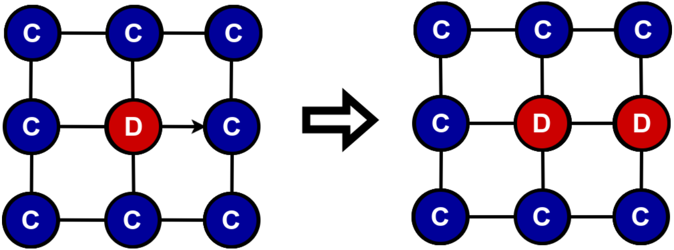


- Parametrization R=1,P=0.
- Harmony Game, Stag-Hunt, Snow-Drift, Prisoner Dilemma.
- Coordination, anti-coordination, dilemma.

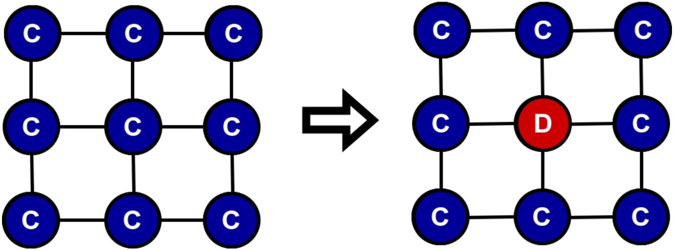
	C	**D**
C	1	S
D	T	0



- Usual imitative update.



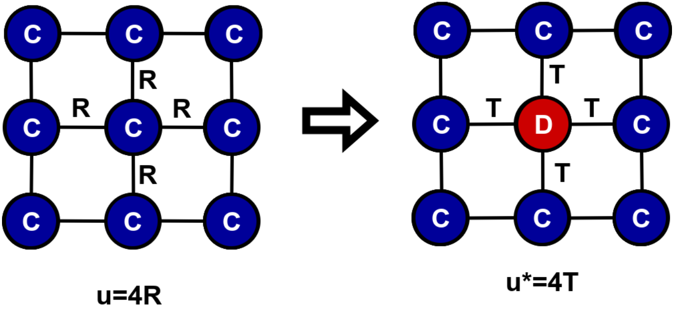
- Innovative update.



Logit Model (Ising)

- Player consider its future payoff, if it had another strategy.
- Cognitive analysis of the future situation.
- Similar to Glauber dynamics from Ising Models.

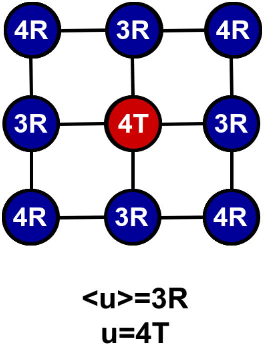
$$p(\Delta u_i) = \frac{1}{1 + e^{-(u_{i*} - u_i)/k}} \tag{1}$$



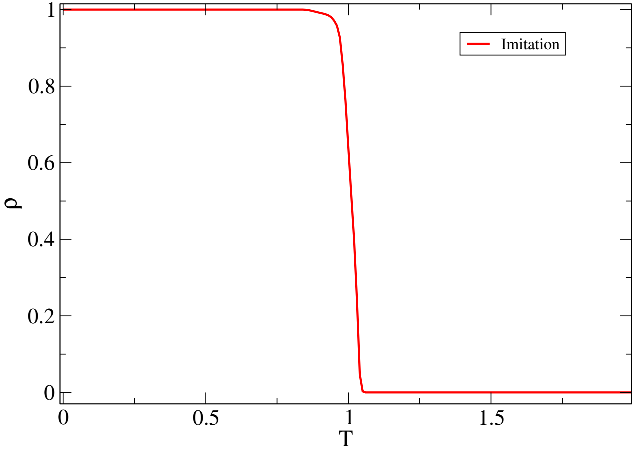
Dynamic Win-Stay-Lose-Shift

- Player compares its payoff with the neighbors average payoff, $\langle u \rangle$.
- In times of crises we lower our expectations.

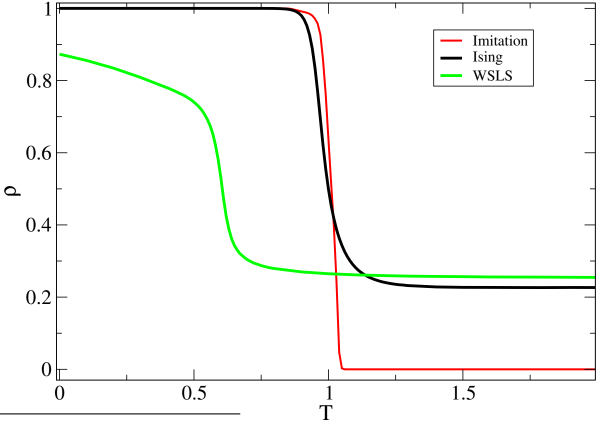
$$p(\Delta u_i) = \frac{1}{1 + e^{-(\langle u \rangle - u_i)/k}} \tag{2}$$



- Cooperation (ρ) versus Temptation to betray (T).
- Weak prisoners dilemma ($S = 0$), square lattice.
- Imitative dynamics, square lattice.

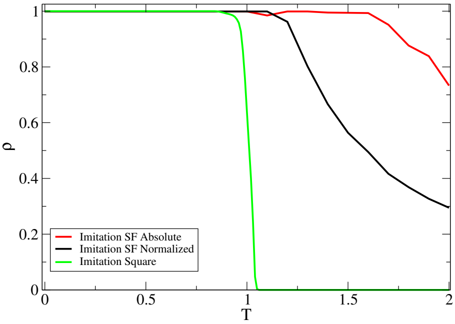


- Minimum cooperation level for innovative models.
- Nevertheless, WSLS and Logit are quite different.

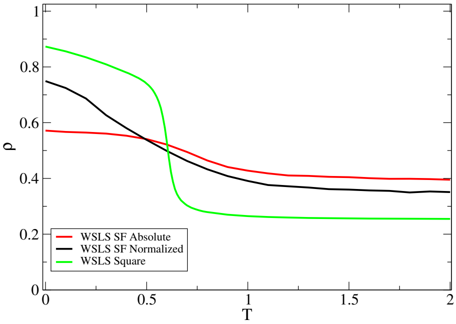


Amaral, M. A., Wardil, L., Perc, M. and da Silva, J. K. L. (2016). Stochastic win-stay-lose-shift strategy with dynamic aspirations in evolutionary social dilemmas. Physical Review E, 94(3), 032317.
Amaral, M. A., Perc, M., Wardil, L., Szolnoki, A., da Silva Júnior, E. J. and da Silva, J. K. L. (2017). Role-separating ordering in social dilemmas controlled by topological frustration. Physical Review E, 95(3), 032307.

- Scale-Free networks have small effects on innovative dynamics.
- Such behavior was seen in human experiments (Gracia-Lázaro et al, **PNAS** 2012).

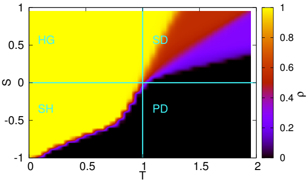


Imitation

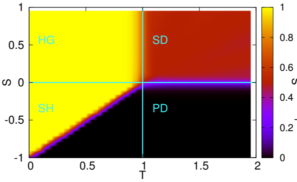


WSLS

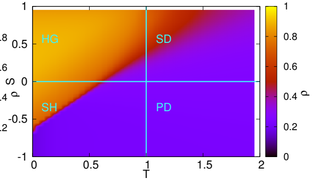
- $T - S$ parameter space.
- There is always coexistence of strategies in the WSLS model.
- Logit dynamics is more similar to Imitation models.
- Curious homogeneous region for SD game.



Imitation

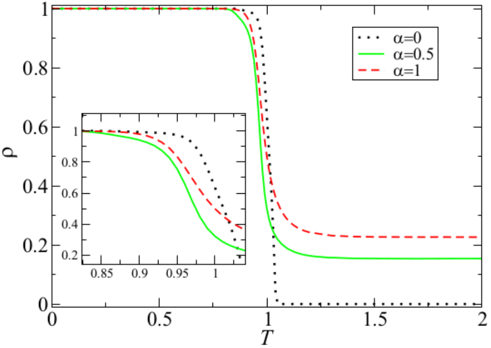


Logit



WSLS

- Continuous change from imitative to innovative is possible
- α % of the lattice is innovative
- Non-linear behavior: few innovators can greatly help cooperation



Amaral, M. A., and Javarone, M. A. (2018). Heterogeneous update mechanisms in evolutionary games: Mixing innovative and imitative dynamics. Physical Review E, 97(4), 042305.

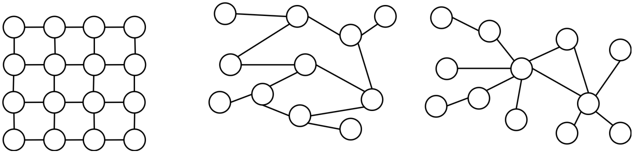
- Logit and WSLS innovative models.
- Both maintain a minimum cooperation level.
- Still, they are strikingly different in the $T - S$ plane.
- Reduced topology dependence.
- Coexistence of strategies.
- A **small** change in update rule can lead to **very** different dynamics.

Thanks

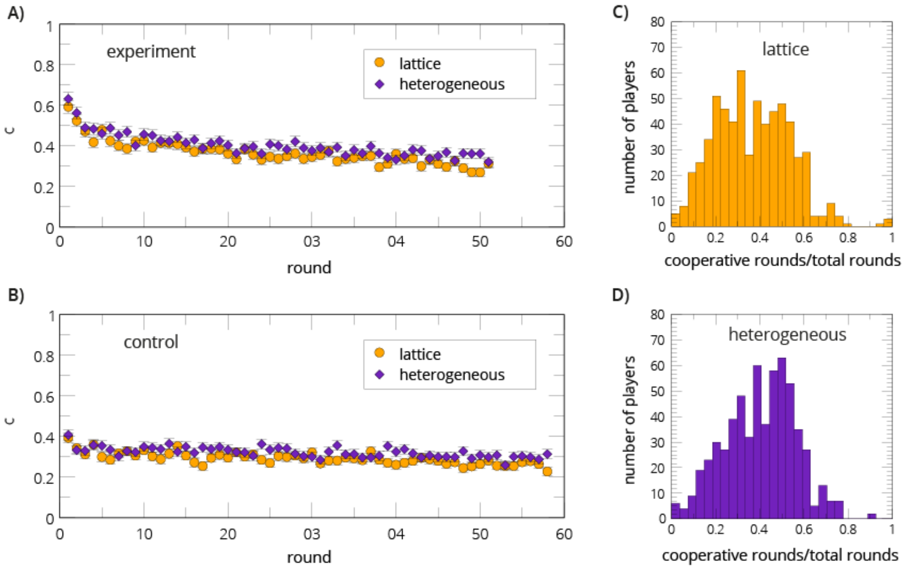
We would like to thank CNPq and CAPES for the financial support.

Appendix

- Lattice sizes N=[100, 16000]
- Samples=[10, 500]
- Connectivity coefficient for SF networks =2.7
- MC steps until termalization MCS=[1000, 60000]
- MC steps averaged over=[1000, 5000]



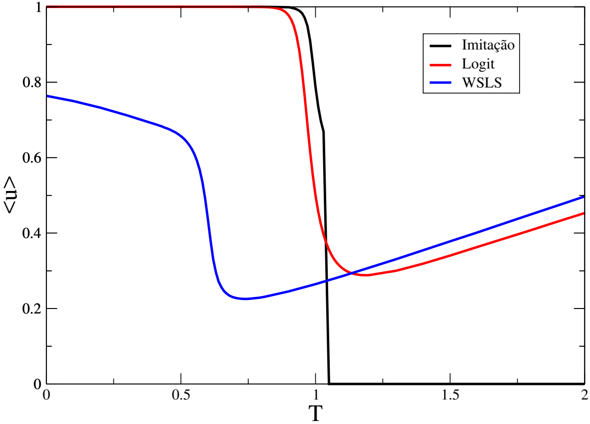
Level of Cooperation



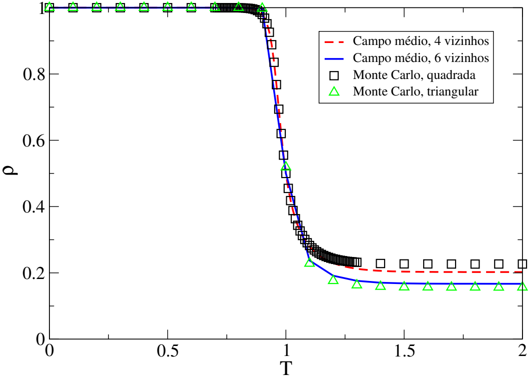
[3]

[3](Gracia-Lázaro et al, **PNAS** 2012)

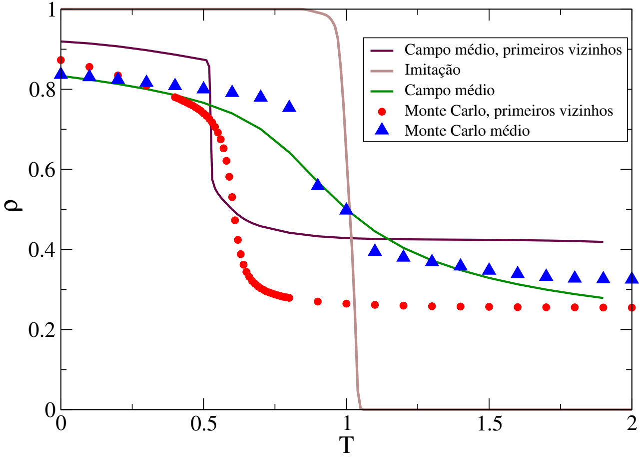
- Average population payoff ($\langle u \rangle$) increases with T.
- Related to spatial organization.



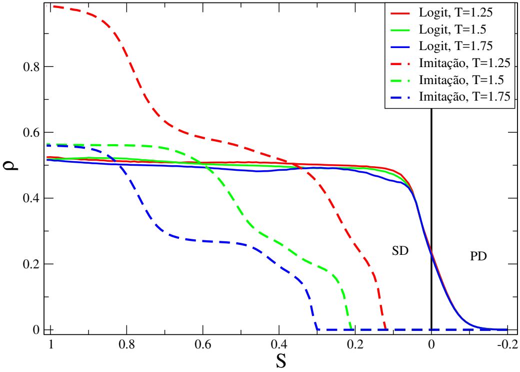
- Monte Carlo and mean-field pair approximation, Logit.

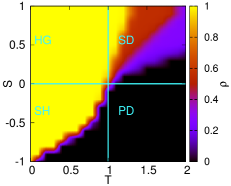


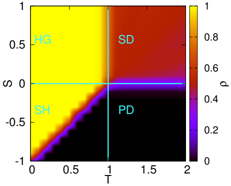
- Monte Carlo and mean-field pair approximation, WSLS.



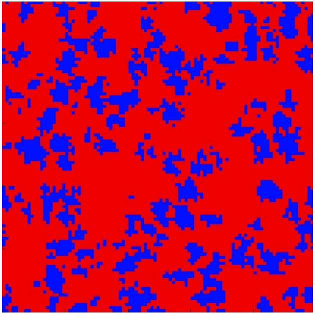
- Fixed T, varying S vertically.
- ρ is independent from T for the Logit model .
- Cooperation even for $S = -0.2$.



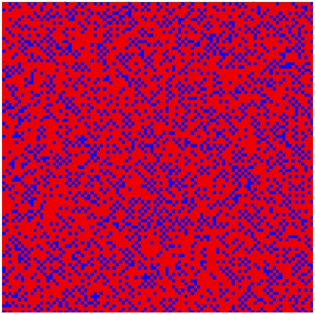




- Typical snapshot from the lattice.
- No cooperators island!

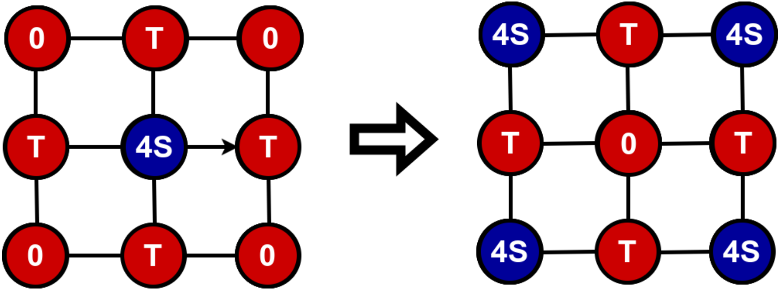


Imitation

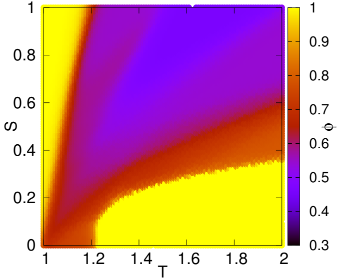


WSLS

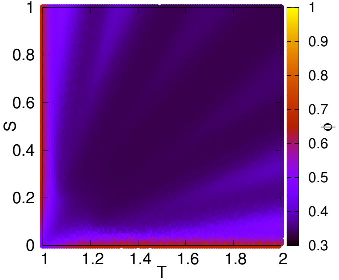
- Typical WSLS process.
- Cooperation spreads through second neighbors.



- Average frustration (ϕ) for $T - S$ plane.
- Logit model drives the system to the lower frustration.
- Which coincides with highest payoff.



Imitation



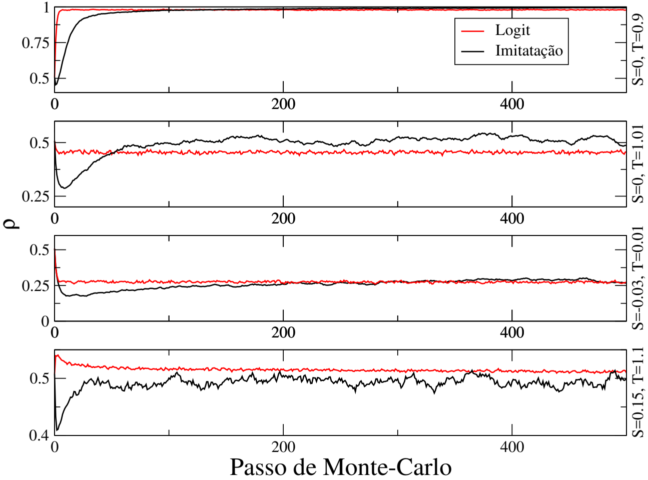
WSLS

- Subtle differences with spin models.
- Energy is conserved, payoff is not. (Non-zero sum games).

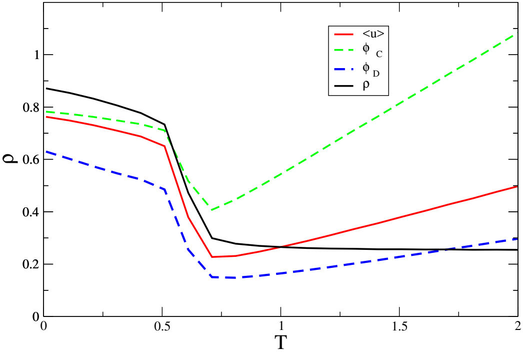
$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array} , \qquad \begin{array}{cc} & \begin{array}{cc} \uparrow & \downarrow \end{array} \\ \begin{array}{c} \uparrow \\ \downarrow \end{array} & \begin{pmatrix} -J-B & J-B \\ J+B & -J+B \end{pmatrix} \end{array} . \tag{3}$$

$$J+B>0, \quad 0>J \quad \mathrm{e} \quad J>B \ , \tag{4}$$

- Different termalization times.



- Payoff and aspirations grows with T.



- Mapeamento em processo Markoviano $\Rightarrow$ Equação Mestra:

$$\frac{d}{dt}P(\{s\},t) = \sum_{\{s\}'} P(\{s\}',t)W(\{s\}' \rightarrow \{s\}) - P(\{s\},t)W(\{s\}, \rightarrow \{s\}') \tag{5}$$

- Uso de campo médio e aproximações gera uma eq. dinâmica:

$$\dot{\rho} = \rho W_+ - (1-\rho)W_- \tag{6}$$

- Vizinhança com n cooperadores

$$W_\pm = \sum_n^N \binom{N}{n} \rho^n (1-\rho)^{(N-n)} P(u_i) \tag{7}$$

- Sítio i atualiza sua estratégia com probabilidade:

$$p(\Delta u_i) = \frac{1}{1 + e^{-(\bar{u} - u_i)/k}} \ . \tag{8}$$

- Equação de campo médio:

$$\dot{\rho} = (1-\rho)W_{+(D\to C)} - \rho W_{-(C\to D)} \ , \tag{9}$$

- com taxas de transição (grafo totalmente conectado):

$$W_{+} = \frac{1}{1+e^{-(\bar{u}-\bar{u}_d)/k}} \ , \tag{10}$$

$$W_{-} = \frac{1}{1+e^{-(\bar{u}-\bar{u}_c)/k}} \ . \tag{11}$$

- Para o dilema do prisioneiro fraco, $R = 1$ e $S = P = 0$:

$$
\begin{aligned}
\bar{u} = & \quad \rho^2 R + (1-\rho)^2 P + \rho(1-\rho)(T+S) = \\
= & \quad \rho^2 + \rho(1-\rho)T \ , \qquad (12) \\
\bar{u}_c = & \quad \rho R + (1-\rho)S = \rho, \qquad (13) \\
\bar{u}_d = & \quad \rho T + (1-\rho)P = \rho T \ . \qquad (14)
\end{aligned}
$$

- Substituindo tais expressões na equação de taxas:

$$\frac{d\rho}{dt} = \frac{1-\rho}{1+e^{-(\rho^2(1-T))/k}} - \frac{\rho}{1+e^{-(\rho(\rho-1)(1-T))/k}}. \tag{15}$$

- A aproximação de primeiros vizinhos
- Vizinhança com n cooperadores
- Payoff do sítio focal: $u_{iD} = nT$ e $u_{iC} = n$.
- Taxa com que um sítio se torna C ou D:

$$W_{\pm} = \sum_{n=0}^{4} \binom{4}{n} \rho^n (1-\rho)^{(4-n)} P_{\pm}(u_i, \bar{u}) \ , \tag{16}$$

- Payoff médio dos vizinhos:

$$\bar{u} = \frac{1}{4}\left[n3\rho + (4-n)3T\rho\right] = \frac{3}{4}(n\rho + 4T\rho - nT\rho)\ . \qquad (17)$$

Assim

$$P_{+} = \frac{1}{1 + e^{-[T(12\rho - 3n\rho - 4n) + 3n\rho]/4k}}\ , \qquad (18)$$

e

$$P_{-} = \frac{1}{1 + e^{[T(3n\rho + n - 4 - 12\rho) - 3n\rho + 3n]/4k}}\ . \qquad (19)$$

$$W_{\pm} = \sum_{n=0}^{N} \binom{N}{n} \rho^{n} (1-\rho)^{(N-n)} P_{\pm}(u_i, u_{\Omega}) \ . \tag{20}$$

- A equação dinâmica para o modelo de Logit se torna:

$$\dot{\rho} = \sum_{n=0}^{N} \binom{N}{n} \rho^{n} (1-\rho)^{(N-n)} \left(\frac{1}{1+e^{\frac{-n(1-T)}{k}}} - \rho \right) \ . \tag{21}$$

- Aproximação de campo médio de primeiros vizinhos para esse modelo.

$$\dot{\rho} = (1-\rho)W_{+(C\to D)} - \rho W_{-(D\to C)} \ , \tag{22}$$

$$W_{\pm} = \sum_{n=0}^{N} \binom{N}{n} \rho^{n}(1-\rho)^{(N-n)} P_{\pm}(u_i, u_\Omega) \ . \tag{23}$$

- Para o modelo de Logit usamos:

$$P_{\pm}(u_i, u_\Omega) = \frac{1}{1 + e^{-(u^* - u_i)/k}} \ . \tag{24}$$

- Probabilidades dependem somente do payoff de i.
- Diferença de payoff atual para o futuro (n vizinhos C):

$$(u^* - u_i)_{D \to C} = n(1 - T) \ , \tag{25}$$
$$(u^* - u_i)_{C \to D} = n(T - 1) \ . \tag{26}$$

- Usando $A = (1 - T)/k$ para simplificar, obtemos:

$$P_{\pm}(u_i, u_\Omega) = \frac{1}{1 + e^{\mp nA}} \; . \tag{27}$$

Substituindo esse termos nas taxas de transição,

$$W_{\pm} = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1-\rho)^{(N-n)} \frac{1}{1 + e^{\mp nA}} \; . \tag{28}$$

O que leva a eq. dinâmica:

$$\begin{aligned} \dot{\rho} = (1-\rho) \sum_{n=0}^{N} \binom{N}{n} \frac{\rho^n (1-\rho)^{(N-n)}}{1 + e^{-nA}} \\ -\rho \sum_{n=0}^{N} \binom{N}{n} \frac{\rho^n (1-\rho)^{(N-n)}}{1 + e^{+nA}} \; . \end{aligned} \tag{29}$$

- A expressão acima pode ser simplificada:

$$\dot{\rho} = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1-\rho)^{(N-n)} \times \left[(1-\rho)\frac{1}{1+e^{-nA}} - \rho\frac{1}{1+e^{+nA}} \right] , \tag{30}$$

$$\dot{\rho} = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1-\rho)^{(N-n)} \times \left[\frac{1}{1+e^{-nA}} - \rho \left(\cancelto{1}{\frac{1}{1+e^{-nA}} + \frac{1}{1+e^{nA}}} \right) \right] . \tag{31}$$

- A equação dinâmica para o modelo de Logit se torna:

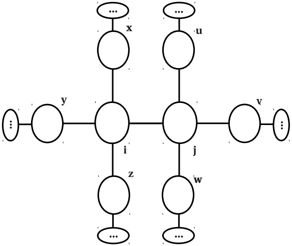
$$\dot{\rho} = \sum_{n=0}^{N} \binom{N}{n} \rho^n (1-\rho)^{(N-n)} \left(\frac{1}{1+e^{\frac{-n(1-T)}{k}}} - \rho \right) \, . \tag{32}$$

- $\dot{\rho} = 0$ nos dá pontos fixos da eq.
- Polinômio de 7 ordem.
- Pelo menos uma raiz no intervalo de $0 < \rho^* < 1$.
- Independente de T.

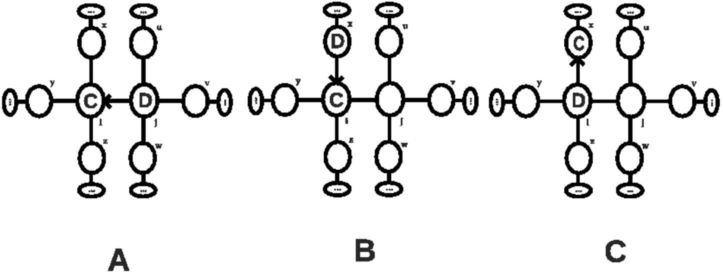
Aproximação de Pares:

- Estrutura espacial e solução analítica (para primeiros vizinhos)
- Aproximação de Campo-Médio para segundos vizinhos

$$
\begin{aligned}
u_c &= R\rho_c + S\rho_d \\
u_d &= T\rho_c + P\rho_d
\end{aligned} \tag{33}
$$



$$P(\{k\};\alpha\beta)W_{\alpha\beta\to\gamma\beta} = A_{\alpha\beta} + B_{\alpha\beta} + C_{\alpha\beta} \tag{34}$$



Equações de taxa para ligações (ao invés de sítios):

$$\dot{\Gamma}_{cc} = \sum_{\{k\}} P(\{k\}; cd) W_{cd \to cc}\{k\} - P(\{k\}; cc) W_{cc \to cd}\{k\} \tag{35}$$

$$\begin{aligned}\dot{\Gamma}_{cd} = &\sum_{\{k\}} P(\{k\}; cc) W_{cc \to cd}\{k\} + P(\{k\}; dd) W_{dd \to cd}\{k\} \\ &- P(\{k\}; cd) W_{cd \to cc}\{k\} - P(\{k\}; cd) W_{cd \to dd}\{k\}\end{aligned} \tag{36}$$

$$\dot{\Gamma}_{dd} = \sum_{\{k\}} P(\{k\}; cd) W_{cd \to dd}\{k\} - P(\{k\}; dd) W_{dd \to cd}\{k\} \tag{37}$$

- Condições de contorno:
- $\Gamma_{cc} + \Gamma_{cd} + \Gamma_{dd} = 1$
- $\rho_c = \Gamma_{cc} + \Gamma_{cd}/2$
- $\rho_c + \rho_d = 1$

$$A_{(i=\alpha,j=\beta)} = \sum_{x,y,z} \sum_{v,w,u} \eta_\beta \left[\frac{\Gamma_{ix}\Gamma_{iy}\Gamma_{iz}}{\Gamma_i^3} \Gamma_{ij} \frac{\Gamma_{ju}\Gamma_{jv}\Gamma_{jw}}{\Gamma_j^3} \right] p(\Delta u_{ij}) \quad , \tag{B7}$$

$$B_{(i=\alpha,x=\beta)} = 3 \sum_{y,z} \sum_{v,w,u} \sum_{j} \eta_\beta \left[\frac{\Gamma_{ix}\Gamma_{iy}\Gamma_{iz}}{\Gamma_i^3} \Gamma_{ij} \frac{\Gamma_{ju}\Gamma_{jv}\Gamma_{jw}}{\Gamma_j^3} \right] p(\Delta u_{ix}) \quad , \tag{B8}$$

$$C_{(i=\beta,x=\alpha)} = 3 \sum_{y,z} \sum_{v,w,u} \sum_{j} \left[\frac{\Gamma_{ix}\Gamma_{iy}\Gamma_{iz}}{\Gamma_i^3} \Gamma_{ij} \frac{\Gamma_{ju}\Gamma_{jv}\Gamma_{jw}}{\Gamma_j^3} \right] p(\Delta u_{xi}) \quad . \tag{B9}$$