Deviations from the Majority: A Local Flip Model

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Summary

- Binary opinion/voting dynamics
- Contrarians do not adopt majority opinion
- Contrarian tendency depends on size of majority
- Phase transitions: number of fixed points and stability change

We study the effect of local distortions to the majority rule on the dynamics of opinions using an extension of the Galam model. At each iteration of the local updates of opinion, the new model accounts for different probabilities of a local flip against the local majority as a function of the ratio of majority / minority within the discussing group. Depending on those probabilities, the model exhibits a wide variety of patterns which include new features in the topology of the landscape driving the dynamics. In particular, we uncover a rich interplay between attractors and tipping points coupled with both monotonic and alternating dynamics. The cases of group sizes 3 and 5 are investigated in detail, and we find regimes that feature competition between three attractors for size 5. Larger groups are also analysed. The local flip model also applies to the study of bottom-top hierarchical voting, where each group elects a representative at the next higher level according to the local majority in the group. The local flip corresponds to a representative who decides to vote against the choice of their electing group, i.e. a 'faithless elector'. The results shed a new light on a series of social phenomena triggered by one single individual who acts against the local majority.

Frames

1. Model of *voting in a hierarchy*. The hierarchy is a homogeneous rooted tree, branching factor $r \in \mathbb{N}$. Each group within the bottom level is comprised of r individuals who vote for one of the options by the majority rule. This decision is carried to the next level of the hierarchy, where r representatives form a group that in turn votes on A versus B. Their decision is passed to the next level, etc.
2. Model of *opinion dynamics*. The entire population randomly meets in groups of fixed size r. A discussion takes place in each group and the majority opinion is adopted by all members of the group. Next, the groups are broken up and the individuals randomly form new groups of size r to resume discussion. This is repeated n times.

Definition of the Model

For group size $r = 3$, the model is summarised by

Configuration	Group vote	Probability
AAA	A	$(1-b)\,p^3$
	B	bp^3
$AAB \cdot 3$	A	$(1-a)\cdot 3p^2(1-p)$
	B	$a \cdot 3p^2(1-p)$
$ABB \cdot 3$	A	$a \cdot 3p(1-p)^2$
	B	$(1-a)\cdot 3p(1-p)^2$
BBB	A	$b(1-p)^3$
	B	$(1-b)(1-p)^3$

For general r, there are flip parameters $a_{(r+1)/2}, \ldots, a_r$ with the subindex being the number of votes for A. Then the update equation is

$$R_{r,\mathbf{a}}(p) = \sum_{i=\frac{r+1}{2}}^{r} \binom{r}{i} \left[(1-a_i)\,p^i(1-p)^{r-i} + a_i p^{r-i}(1-p)^i\right].$$

Results

Group Size $r = 3$

The regimes of this 2-parameter model are classified according to the stability of the universal fixed point $1/2$ and the number of fixed points of the update equation.

As far as the stability of $1/2$ is concerned, we distinguish four regions:

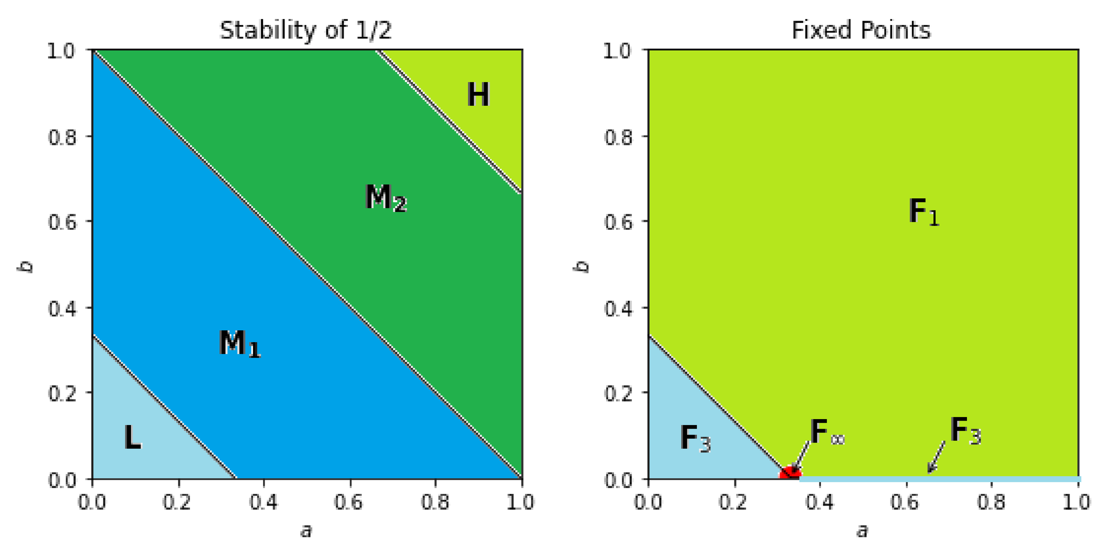
1. The unstable region $\mathbf{L}$ with monotonic dynamics, given by the inequality $b < 1/3 - a$.
2. The stable region M_1 with monotonic dynamics, given by the inequalities $1/3 - a < b < 1 - a$.
3. The stable region M_2 with alternating dynamics, given by the inequalities $1 - a < b < 5/3 - a$.
4. The unstable region $\mathbf{H}$ with alternating dynamics which lies above the line $b = 5/3 - a$.

We partition the parameter space into regions which exhibit either a single fixed point or three fixed points, as well as a single point where every value $p \in [0,1]$ is a fixed point.

1. The region in which every value $p \in [0,1]$ is a fixed point is $\mathbf{F}_\infty := \{(1/3, 0)\}$.
2. The region $\mathbf{F}_1$ with only a single fixed point which is $1/2$ is given by the inequalities $1/3 - a \le b$ and $b > 0$.
3. The region with three different fixed points is the complement $\mathbf{F}_3 := [0,1]^2 \setminus (\mathbf{F}_\infty \cup \mathbf{F}_1)$, i.e. the corner region around the origin and the a-axis excluding $(1/3, 0)$.

The regimes of the model can now be determined by forming intersections of the regions given above.

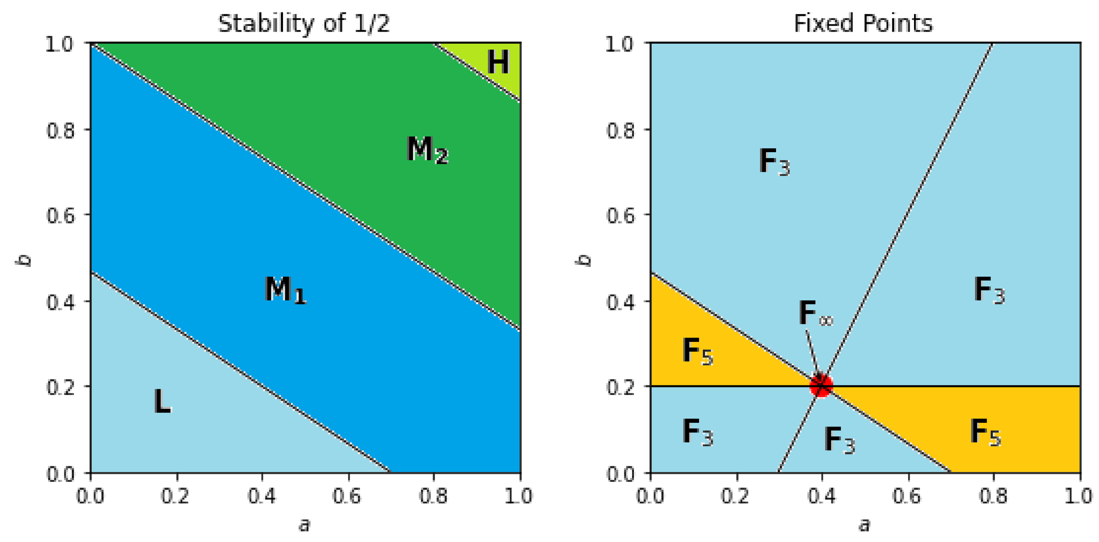
1. The region $\mathbf{F}_\infty$ is a regime of its own. Here the dynamics of the model are stationary.
2. The region $\mathbf{L} \subset \mathbf{F}_3$ is a regime characterised by having three fixed points. $1/2$ is unstable and there are two additional fixed points located at
$$p_\pm := \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1-3a-3b}{1-3a+b}}.$$
Note that in the fraction both the numerator and the denominator are positive. The points $p_\pm$ are attractors and their values are not $0, 1$ if and only if $b > 0$. The location not being at $0, 1$ is a new feature when compared to $b = 0$ and similar to the contrarian model where $a = b > 0$.
3. The region $\mathrm{M}_1 \cap \mathbf{F}_1$ exhibits a single fixed point $1/2$ that is a global attractor. The dynamics of the model starting at any p_0 is monotonic convergence towards $1/2$.
4. The region $\mathrm{M}_1 \cap \mathbf{F}_3$, which is the section of the a-axis with $a > 1/3$, has three fixed points, $0, 1/2, 1$, with $1/2$ being an attractor and $0, 1$ repellers.
5. The region $\mathrm{M}_2 \subset \mathbf{F}_1$ also has a single fixed point $1/2$ that is a global attractor. However, the dynamics of the model are dampened oscillations: Starting at any p_0, the orbits alternate around the limit $1/2$.
6. The region $\mathbf{H} \subset \mathbf{F}_1$ is characterised by having a single fixed point $1/2$ which acts as a repeller. The dynamics are oscillatory, so there are wild swings from one round to the next from large majorities for A to large majorities of B and back. The subsequences p_{2n} and p_{2n-1} converge and their limits can be calculated explicitly:
$$\omega_\pm := \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{-10+6a+6b}{-2+6a-2b}}.$$
These accumulation points are created at the centre, $p = 1/2$, when we cross over from M_2 to $\mathbf{H}$ and they wander towards the corners $0, 1$ as the parameter values (a, b) go toward the corner $(1, 1)$. There is a period-doubling bifurcation when passing from M_2 to $\mathbf{H}$.



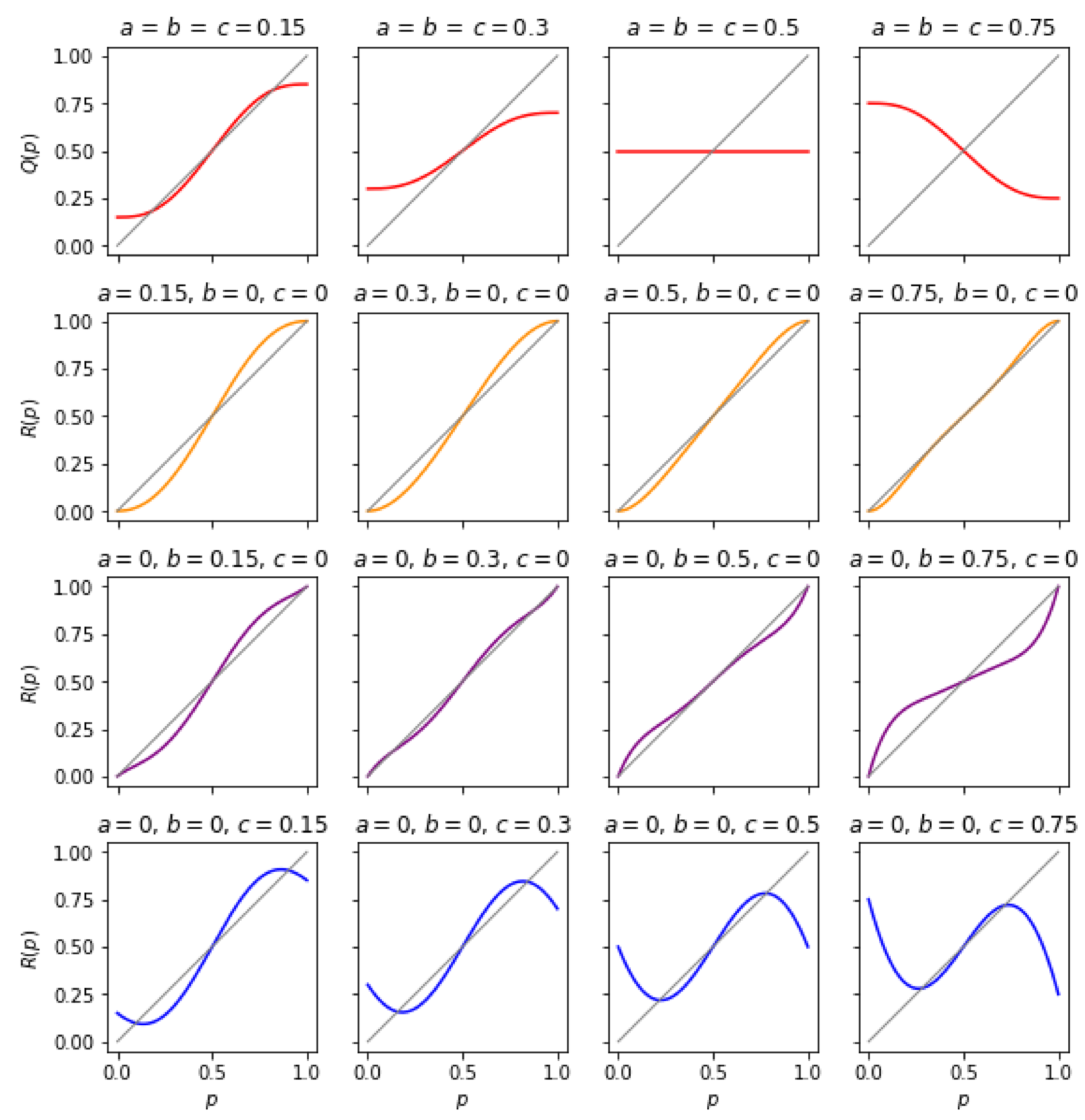
Group Size $r = 5$

Let $a = a_3, b = a_4, c = a_5$. This is a 3-parameter model. We can, however, study some special cases with some of the parameters equal to 0. If we set $c = 3$, then flips are only possible if the majority is not unanimous. We can analyse this model as the one for $r = 3$. The regimes are:

1. The region $\mathbf{F}_\infty$ is a regime of its own. Here the dynamics of the model are stationary.
2. The region $\mathbf{L} \cap \mathbf{F}_3$ is a regime characterised by having the three fixed points $0, 1/2, 1$. $1/2$ is unstable and $0, 1$ are stable. This is the regime most similar in behaviour to the basic model since here the flip probabilities are low.
3. The region $\mathbf{L} \cap \mathbf{F}_5$ has five fixed points: There are two additional fixed points located at
$$p_\pm := \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{7-10a-15b}{3-10a+5b}}. \qquad (1)$$
Note that in the fraction both the numerator and the denominator are positive. As $1/2$ is unstable, the points $p_\pm$ are attractors and $0, 1$ are repellers. This is a feature the one-parameter model with $b = c = 0$ does not exhibit: If $b = 0$, then the points $p_\pm$ are unstable.
4. The region $\mathrm{M}_1 \cap \mathbf{F}_3$ has the fixed points $0, 1/2, 1$. Here, $1/2$ is stable and the dynamics are monotonic.
5. The region $\mathrm{M}_1 \cap \mathbf{F}_5$ has five fixed points, $0, 1/2, 1$, and $p_\pm$ given by the formula above. $0, 1/2, 1$ act as attractors and $p_\pm$ as repellers.
6. The region $\mathrm{M}_2 \subset \mathbf{F}_3$ also has the fixed points $0, 1/2, 1$ with the same stability properties as $\mathrm{M}_1 \cap \mathbf{F}_3$. However, the dynamics of the model starting at any p_0 are now alternating with limit $1/2$.
7. The region $\mathbf{H} \subset \mathbf{F}_3$ is characterised by having the fixed points $0, 1/2, 1$ which acts as a repeller. The dynamics are alternating, so there are wild swings from one round to the next from large majorities for A to large majorities of B and back. $0, 1$ are fixed points contrary to the corresponding regime for $r = 3$, so if p_0 is even slightly different than 0 or 1, the dynamics tend to large swings. There are accumulation points of the dynamics in this regime provided $p_0 \in (0, 1)$. Similarly to $r = 3$, these are symmetric $\omega_- \in (0, 1/2)$ and $\omega_+ = 1 - \omega_-$. However, as $0, 1$ are fixed points, not even in the extreme case $(a, b) = (1, 1)$ do we see the complete unanimity alternating between A and B. Instead, if the initial $p_0 > 0$ is very close to the origin, then there is some lead time before the oscillation between majorities for A and B starts. The discrepancy between the group sizes 3 and 5 is because for the latter we are not allowing flips for unanimous configurations.



We can also compare 1-parameter models for $r = 5$ to the contrarian model first introduced in [2]. We plot the update function Q for the contrarian model and compare it to 1-parameter versions of the local flip model.



Scenarios Explained by the Model

The local flip model may provide a possible explanation of a series of social dynamics triggered by one or a few individuals acting against larger local majorities, such as:

- At a party, initially people are standing around, having conversations. Then at some point in time, the first person starts dancing and others follow. After a short time, some fixed proportion of all people in attendance are dancing.
- The first person to applaud after a concert or a play will trigger others to join.
- A patient arrives at the waiting room of a hospital. If all or nearly all people already there face the same direction, chances are high the new arrival will do so as well. However, if a larger proportion (but not necessarily close to half) faces a different direction, then new arrivals will select their orientation at random – some conforming to the majority, while others go against it. As more patients arrive, in the long run, the people will be evenly split, simply because there was no strong enough initial pressure to conform to the group's behaviour.
- During a demonstration, one person starts throwing objects at the police and others close by follow suit. Rioters move among demonstrators and drag more and more people into violence.
- Sharing some confidential or sensational piece of information, be it political, societal, or financial in nature, can cause a few people to shift their opinion. They will keep their new stance after the meeting and subsequently even use the new information to convince others. This includes the spreading of fake news.

References

[1] Toth, G. and Galam, S.: Deviations from the Majority: A Local Flip Model, arXiv:2107.09344 (2021)
[2] Galam, S.: Minority Opinion Spreading in Random Geometry, European Physical Journal B 25 Rapid Note, 403-406 (2002)