Efimov Scenario In The Presence Of Spin-Orbit Coupling

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Lecture 3

Discussion of one few-body technique:
Stochastic variational approach with explicitly correlated Gaussians.

Efimov effect:
Three identical bosons with short-range interactions (beautifully introduced in Pascal Naidon’s lectures)

- unequal-mass three-boson and three-fermion systems
- larger equal-mass bosonic systems with short-range interactions
- nuclear systems with finite range (beautiful lectures by Alejandro Kievsky)
- three equal-mass bosons with 1D spin-orbit coupling (this lecture)
Overview Of Lecture 3

Efimov scenario of three equal-mass bosons with 1D spin-orbit coupling

Review of Efimov effect for equal-mass bosons with short-range interactions.
What is 1D spin-orbit coupling (system lives in three-dimensional space) and how does it differ from “conventional” situation?
• One-body problem.
• Intermezzo: Numerical approach.
• Two-body problem.
• Three-body problem.

Unlike lectures 1 and 2:
• Consider stationary system (no time dependence).
• No experimental data yet on generalized radial scaling law, even though spin-orbit coupling has been realized in cold atom systems.
Review: 
Three-Boson Hamiltonian

\[ H = \frac{\vec{p}_{12}^2}{2\mu_{12}} + \frac{\vec{p}_{12,3}^2}{2\mu_{12,3}} + \sum_{j<k} g_2 \delta(\vec{r}_{jk}) + g_3 \delta(\vec{r}_1 - \vec{r}_2) \delta(\vec{r}_2 - \vec{r}_3). \]

Question: What units do \( g_2 \) and \( g_3 \) have?
Review: Three-Boson Hamiltonian

\[ H = \frac{\vec{p}_{12}^2}{2\mu_{12}} + \frac{\vec{p}_{12,3}^2}{2\mu_{12,3}} + \sum_{j<k} g_2 \delta(\vec{r}_{jk}) + g_3 \delta(\vec{r}_1 - \vec{r}_2) \delta(\vec{r}_2 - \vec{r}_3). \]

Question: What units do \( g_2 \) and \( g_3 \) have?

\( g_2 \): energy times length\(^3\).
\( g_3 \): energy times length\(^6\).

Follow-up question: Given this answer, what are the most general expressions for \( g_2 \) and \( g_3 \)?
Review: Three-Boson Hamiltonian

\[ H = \frac{\vec{p}_{12}^2}{2\mu_{12}} + \frac{\vec{p}_{12,3}^2}{2\mu_{12,3}} + \sum_{j<k} g_2 \delta(\vec{r}_{jk}) + g_3 \delta(\vec{r}_1 - \vec{r}_2) \delta(\vec{r}_2 - \vec{r}_3). \]

Question: What units do \( g_2 \) and \( g_3 \) have?

\( g_2 \): energy times length³.  
\( g_3 \): energy times length⁶.

Follow-up question: Given this answer, what are the most general expressions for \( g_2 \) and \( g_3 \)?

\[ g_2 = \# \frac{\hbar^2 \text{length}}{m}, \quad g_3 = \# \frac{\hbar^2 \text{length}^4}{m} \quad (\text{since energy} = \frac{\hbar^2}{m \text{length}^2}). \]
Peculiar Three-Boson Efimov States

\[ H = \frac{\vec{p}_{12}^2}{2\mu_{12}} + \frac{\vec{p}_{12,3}^2}{2\mu_{12,3}} + \sum_{j<k} g_2 \delta(\vec{r}_{jk}) + g_3 \delta(\vec{r}_1 - \vec{r}_2) \delta(\vec{r}_2 - \vec{r}_3). \]

\[ g_2 = \frac{4\pi\hbar^2 a_s}{m} \] and \[ g_3 = \frac{\hbar^2 \kappa_*^4}{m}, \] where \[ E_{\text{unit}} = \frac{\hbar^2 \kappa_*^2}{m}. \]

Time-dependent SE for \( H \) possesses continuous scaling symmetry:
\[ t \to \lambda^2 t; \quad \vec{r} \to \lambda \vec{r}; \quad a_s \to \lambda a_s; \quad E \to \lambda^{-2} E; \quad \kappa_* \to \lambda^{-1} \kappa_. \]

Time-dependent SE for \( H \) also possesses discrete scaling symmetry (\( \lambda_0 \approx 22.7 \)):
\[ t \to \lambda_0^2 t; \quad \vec{r} \to \lambda_0 \vec{r}; \quad a_s \to \lambda_0 a_s; \quad E \to \lambda_0^{-2} E; \quad \kappa_* \to \kappa_. \]
Let s-Wave Scattering Length Be Infinitely Large

Hypperradial and hyperangular motion separate exactly:
\[ \Psi = F(R_{\text{hyper}}) \Phi(\Omega) ; \quad R_{\text{hyper}}^2 \propto r_{12}^2 + r_{13}^2 + r_{23}^2 . \]

\[ L^H = 0^+ \text{ hyperangular equation yields eigenvalue } \ell s_0 , \]
where \( s_0 = 1.006 \ldots \)

Hyperangular eigenvalue enters into Schroedinger-like hyperradial equation:
\[ H_{\text{radial}} F(R_{\text{hyper}}) = E_3 F(R_{\text{hyper}}) , \]
where \( H_{\text{radial}}(R_{\text{hyper}}) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial R_{\text{hyper}}^2} + \frac{\hbar^2((\ell s_0)^2 - \frac{1}{4})}{2mR_{\text{hyper}}^2} \).

If \( F(R_{\text{hyper}}) \) is a solution with energy \( E_3^{(n)} \), then \( F(\lambda_0 R_{\text{hyper}}) \)
with \( \lambda_0 = \exp \left( \frac{\pi}{s_0} \right) = 22.7 \ldots \) is a solution with energy \( \lambda_0^{-2} E_3^{(n)} \).
Finite s-Wave Scattering Length: Universally Linked States

Numerical test for two-body plus three-body Gaussian potential: Perfect “collapse” of neighboring energy levels (see lectures by Naidon and Kievsky).

Spectrum is determined by $a_s$ and three-body parameter $\kappa_*$ (radial scaling law).
Borromean rings: The blue ring lies under the green ring (the “blue-green dimer” is unbound). If the red ring is cut open, the trimer flies apart.
Enhanced losses when trimer is degenerate with three free atoms. Ratio of $\lambda_0 = 21.6$ (compared to 22.7)! Confirmation of discrete scaling symmetry.
Efimov Scenario In The Presence Of Spin-Orbit Coupling?

If the single-particle dispersion is modified and the s-wave scattering length is large, what happens to the discrete scaling symmetry/Efimov physics?

Fermions with 3D SOC:

Two-parameter radial scaling law does not hold.

Generalized multiple-parameter radial scaling law holds. Discrete scaling symmetry survives.

BBB with 1D SOC:
Guan, Blume: PRX 8, 021057 (2018).

Conjecture: Should hold for any type of SOC.
Looking Ahead…

\[ H = \left( \frac{\vec{p}_{12}^2}{2\mu_{12}} + \frac{\vec{p}_{12,3}^2}{2\mu_{12,3}} + \sum_{j<k} g_2 \delta(\vec{r}_{jk}) + g_3 \delta(\vec{r}_1 - \vec{r}_2)\delta(\vec{r}_2 - \vec{r}_3) \right) I_8 \]

\[ + \frac{\hbar k_{so}}{m} (...) + \Omega(...) + \tilde{\delta}(...) \text{ changes and extra terms due to SOC} \]

Continuous scaling symmetry (easy to check)!
\[ t \rightarrow \lambda^2 t; \vec{r} \rightarrow \lambda \vec{r}; a_s \rightarrow \lambda a_s; k_{so} \rightarrow \lambda^{-1} k_{so}; \Omega \rightarrow \lambda^{-2} \Omega; \]
\[ \tilde{\delta} \rightarrow \lambda^{-2} \tilde{\delta}; E \rightarrow \lambda^{-2} E; \kappa_* \rightarrow \lambda^{-1} \kappa_* \]

Discrete scaling symmetry?
\[ t \rightarrow \lambda_0^2 t; \vec{r} \rightarrow \lambda_0 \vec{r}; a_s \rightarrow \lambda_0 a_s; k_{so} \rightarrow \lambda_0^{-1} k_{so}; \Omega \rightarrow \lambda_0^{-2} \Omega; \]
\[ \tilde{\delta} \rightarrow \lambda_0^{-2} \tilde{\delta}; E \rightarrow \lambda_0^{-2} E; \kappa_* \rightarrow \kappa_*; \lambda_0 \approx 22.7 \]
Start With Single-Particle Dispersion

Conventionally, kinetic energy:

\[ H = \frac{1}{2m} \overline{p^2} = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2). \]

Say, the particle has two spin states:

Now let the spin states have different momenta:

For this to be interesting, must couple states.
Single-Particle Dispersion

For 1D spin-orbit coupling (equal mixture of Rashba and Dresselhaus coupling): $H = \left(\frac{p_x^2 + p_y^2}{2m} + \frac{p_z^2}{2m}\right)I_2 + \frac{\hbar k_{so}}{m} p_z \sigma_z + \Omega \sigma_x$

$$E_{\pm} = \frac{p_x^2 + p_y^2}{2m} + \frac{p_z^2}{2m} \pm \sqrt{\left(\frac{\hbar k_{so} p_z}{m}\right)^2 + \frac{\Omega^2}{4}}$$

Add detuning $\delta$:

$$H = \left(\frac{p_x^2 + p_y^2}{2m} + \frac{p_z^2}{2m}\right)I_2 + \frac{\hbar k_{so}}{m} p_z \sigma_z + \delta \sigma_z.$$ 

$$E_{\pm} = \frac{p_x^2 + p_y^2}{2m} + \frac{p_z^2}{2m} \pm \sqrt{\left(\frac{\hbar k_{so} p_z}{m} + \frac{\delta}{2}\right)^2}$$
Modified Single-Particle Dispersion → New Physics

Dirac-like term in Hamiltonian.

Typical parameters:

\[ (\vec{k}_{\text{so}})^{-1} \approx 1,810 \text{Å}; \]
\[ \frac{E_{\text{so}}}{\hbar} \approx 2\pi \times 1.775 \text{ kHz}; \]
\[ \Omega, \delta \in [0,10E_{\text{so}}]. \]
Modified Single-Particle Dispersion $\rightarrow$ New Physics

For 1D spin-orbit coupling (equal mixture of Rashba and Dresselhaus coupling): 

$$H = \left(\frac{p_x^2 + p_y^2}{2m} + \frac{p_z^2}{2m}\right)I_2 + \frac{\hbar k_{so}}{m} p_z \sigma_z + \Omega \sigma_x + \delta \sigma_z$$

$$E_{\pm} = \frac{p_x^2 + p_y^2}{2m} + \frac{p_z^2}{2m} \pm \sqrt{\left(\frac{\hbar k_{so} p_z}{m} + \frac{\delta}{2}\right)^2 + \frac{\Omega^2}{4}}$$

Why is this of interest? Expand $E_-$ for large $\Omega$ and $\delta \neq 0$ around $p^-_{z,\text{min}}$:

$$E_- = \text{const} + \frac{(p_z - p^-_{z,\text{min}})^2}{2m} + \ldots$$

Like a charged particle in a uniform vector potential $\vec{A}^*$: $e\vec{A}^* = p^-_{z,\text{min}} \vec{e}_z$!

Possibility to simulate physics of charged particles (e.g., fractional quantum Hall effect) with neutral atoms!
Two Bosons With One-Dimensional Spin-Orbit Coupling

3D system with 1D SOC (spin-orbit coupling + Raman coupling + detuning) → Two-body bound state or not?

Rewrite Hamiltonian in relative coordinates ($\vec{r}$ and $\vec{p}$ with reduced mass $\mu$) and center-of-mass coordinates ($\vec{R}$ and $\vec{P}$ with total mass $M$):

$$H = H_{\text{rel}} + H_{\text{cm}}$$

$$H_{\text{rel}}(P_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2\mu} I_2^{(1)} \otimes I_2^{(2)} + \frac{\hbar k_{so} p_z}{\mu} \left( \sigma_z^{(1)} \otimes I_2^{(2)} - I_2^{(1)} \otimes \sigma_z^{(2)} \right)$$

$$+ \Omega \left( \sigma_x^{(1)} \otimes I_2^{(2)} + I_2^{(1)} \otimes \sigma_x^{(2)} \right) + \left( \delta + \frac{\hbar k_{so} P_z}{M} \right) \left( \sigma_z^{(1)} \otimes I_2^{(2)} + I_2^{(1)} \otimes \sigma_z^{(2)} \right)$$

$$+ V_{2b}(r) I_2^{(1)} \otimes I_2^{(2)}$$

$$[H_{\text{rel}}, P_z] = 0$$

parametric dependence on CoM momentum (call $\delta$)
Demands On Numerical Method

- Must be...
  - ...able to treat multiple channels.
  - ...able to describe small and large length scales.
  - ...able to describe eigen energies and eigen states that vary by orders of magnitude.
  - ...applicable to two and three particles.

Our method of choice: Stochastic variational approach with explicitly correlated Gaussian basis function (see also contributed talk yesterday by Roy Yaron).
Basis Set Expansion: Variational Approach

Let $\Phi_j$ with $j = 0, 1, \ldots$ be an orthonormal complete set.

Any eigen state $\varphi_l$ with energy $E_l$ of $H$ can be expanded as

$$\varphi_l = \sum_{j=0}^{\infty} c_j^{(l)} \Phi_j.$$ 

In reality: $\phi_l = \sum_{j=0}^{N_b} c_j^{(l)} \Phi_j$ ($N_b < \infty$; $\phi_l$ is an approximation to $\varphi_l$).

Form matrix $\hat{C}$ with matrix elements $C_{j\ell} = c_j^{(l)}$.

Eigenvalues $\varepsilon_\ell$ of matrix equation $\hat{H} \hat{C} = \varepsilon \hat{C}$ have the following property:

$$E_0 \leq \varepsilon_0, E_1 \leq \varepsilon_1, \ldots \text{ (variational upper bounds).}$$
Basis Set Expansion: Variational Approach

Question: What changes if $\Phi_j$ with $j = 0, 1, \ldots$ are not orthogonal?
Basis Set Expansion: Variational Approach

Now: Allow $\Phi_j$ with $j = 0, 1, \cdots$ to be linearly dependent (but not too much).

Expand $\phi_l = \sum_{j=0}^{N_b} c_j^{(l)} \Phi_j$ ($N_b < \infty$; $\phi_l$ is an approximation to exact eigen state $\varphi_l$).

Form matrix $\mathcal{C}$ with matrix elements $C_{jl} = c_j^{(l)}$.

The eigenvalues $\varepsilon_l$ of generalized eigen value equation $\mathcal{H} \mathcal{C} = \mathcal{E} \mathcal{O} \mathcal{C}$, where $O_{jl} = \langle \Phi_j | \Phi_l \rangle$, have the following property:

$E_0 \leq \varepsilon_0, E_1 \leq \varepsilon_1, \cdots$ (variational upper bounds).
Basis Set Expansion: Variational Approach

Take advantage of the fact that the basis functions $\Phi_j$ can be “anything”.

Pick $\Phi_j$ such that integrals have compact analytical expressions.

Pick $\Phi_j$ such that the different length scales of the system are covered.

Take advantage of the fact that low-energy Hamiltonian can be constructed using different functional forms for interaction potential:

$$ H = \sum_j T_j + V_{soc,j} + \sum_{j<k} V_{2b,jk} + \sum_{j<k<l} V_{3b,jkl} $$

Require $r_0 \ll$ other scales:

Need to resolve multiple scales. Use $\Phi_j$ with “different widths.”

$V_{2b,jk} = v_0 \exp \left( -\frac{r_{jk}^2}{2r_0^2} \right)$

purely repulsive Gaussian (see also talks by Kievsky)
Basis Set Expansion: Stochastic Variational Approach


Idea:

Use basis functions that involve Gaussians with different widths in interparticle distances (correlations).

Large number of non-linear parameters that are being optimized semi-stochastically.

Simplest case: Basis functions with \( L = 0 \) and \( \Pi = +1 \).

\[
\Phi_j = \exp \left( - \sum_{s<t}^N \frac{r_{st}^2}{2d_{j,st}^2} \right) = \exp \left( -\frac{1}{2} \tilde{x}^T \tilde{A} \tilde{x} \right).
\]

\( \tilde{x} \): Denotes Jacobi vectors \( \tilde{\rho}_1, \tilde{\rho}_2, \ldots \).

\( \tilde{A} \): \((N - 1) \times (N - 1)\) matrix with \( N(N - 1)/2 \) independent parameters.
Stochastic Variational Approach: Outline of Algorithm

- Pick basis function $\Phi_1$ and calculate $\varepsilon_1$.

- Goal: Add $\Phi_2$. Procedure:
  - Pick $\Phi_{2,1}, \ldots, \Phi_{2,p}$ ($p \approx 1 - 10000$).
  - Calculate $\varepsilon_{2,1}, \ldots, \varepsilon_{2,p}$. $\varepsilon_{2,j}$ is eigen value of target state if basis function $\Phi_{2,j}$ is added to basis ($j = 1, \ldots, p$).
  - Determine $\Phi_2 = \Phi_{2,j}$ such that $\varepsilon_2 = \varepsilon_{2,j} = \min(\varepsilon_{2,1}, \ldots, \varepsilon_{2,p})$.
  - Diagonalize Hamiltonian matrix to obtain eigenvalues and eigenvectors.

- To add $\Phi_3$, proceed as above.

- Once basis set is “complete”, calculate structural properties.

- Can optimize ground or excited state.

- Can optimize multiple states simultaneously.
Harmonically Trapped Five-Boson System: Convergence

\[ r_0 = 0.01a_{ho} \]
\[ a_s = 0.0096a_{ho} \]

For each \( N_b \), try a few 1000 and keep the best.

Used energy to benchmark effective field theory Hamiltonian: Johnson, Blume, Yin, Flynn, Tiesinga, NJP (2012).
Trapped (3,3) System: Energy And Pair Distribution Function

Lowest energy at unitarity ($1/a_s=0$):

Two-peak structure of up-down pair distribution function:
- Small $r_{ud}$ peak: pair formation.
- Large $r_{ud}$ peak: unpaired.

extrapolate to obtain zero-range energy
A Few More Comments

Basis functions need to be symmetrized: Five identical bosons implies $5!=120$ permutations.

Use physical insight to choose $d_{j,st}$ efficiently: E.g., “2+1” or “1+1+1” configuration.

If parameter windows for non-linear variational parameters are not set properly, a non-converged energy may appear converged…

Basis sets tend to be small (a few 1000); but we work hard to select the basis functions we want.

Beyond $L^I = 0^+$ states? Many possibilities… Global vector approach is quite convenient.
Spin-Orbit Coupling: Need To Account For Spin

\[ \Phi_j = \exp \left( - \sum_{s<t}^{N} \frac{r_{st}^2}{2d_{j,st}^2} \right) + \sum_{t=1}^{N-1} i\vec{S}_{j,t} \cdot \vec{\rho}_t \]

Correlation between spin and spatial degrees of freedom.
Can be rewritten as
\[ \sum_{t=1}^{N} i\vec{S}_{j,t} \cdot \vec{r}_t \]

Spatial two-body correlations

\[ \Psi_{rel} = \sum_{j=1}^{N_b} c_j \psi_j \text{ and } \psi_j = S(\Phi_j(\vec{\rho}_1, \ldots, \vec{\rho}_{N-1}) \chi_j) \]

Matrix elements have compact analytical expressions.

Bound state:
Energy of dimer with CM momentum \( P_z \) is more negative than that of two free atoms with the same \( P_z \).
Energy of trimer with CM momentum \( P_z \) is more negative than that of three free atoms with the same \( P_z \) and that of a dimer and an atom with the same \( P_z \).
Two Bosons With One-Dimensional Spin-Orbit Coupling

3D system with 1D SOC (spin-orbit coupling + Raman coupling + detuning) \( \rightarrow \) Two-body bound state or not?

Rewrite Hamiltonian in relative coordinates (\( \vec{r} \) and \( \vec{p} \) with reduced mass \( \mu \)) and center-of-mass coordinates (\( \vec{R} \) and \( \vec{P} \) with total mass \( M \)):

\[
H = H_{rel} + H_{cm}
\]

\[
H_{rel}(P_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2\mu} I_2^{(1)} \otimes I_2^{(2)} + \frac{\hbar k_{so} P_z}{\mu} \left( \sigma_z^{(1)} \otimes I_2^{(2)} - I_2^{(1)} \otimes \sigma_z^{(2)} \right)
\]

\[
+ \Omega \left( \sigma_x^{(1)} \otimes I_2^{(2)} + I_2^{(1)} \otimes \sigma_x^{(2)} \right) + \left( \delta + \frac{\hbar k_{so} P_z}{M} \right) \left( \sigma_z^{(1)} \otimes I_2^{(2)} + I_2^{(1)} \otimes \sigma_z^{(2)} \right)
\]

\[
+ V_{2b}(r) I_2^{(1)} \otimes I_2^{(2)}
\]

\[
[H_{rel}, P_z] = 0
\]

parametric dependence on CoM momentum (call \( \delta \))
Non-Interacting Relative Dispersion Curves Along $z$

$\delta = \delta + \frac{\hbar k_{so} p_z}{M}$.

Transition from double- to single-minimum regime.
Two Identical Bosons:
\[ \Omega = 2E_{so}; \tilde{\delta} \geq 0 (a_{\uparrow\uparrow} = a_{\uparrow\downarrow} = a_{\downarrow\uparrow} = a_{\downarrow\downarrow}) \]

\[ \tilde{\delta} = \delta + \frac{\hbar k_{so} p_z}{M}. \]

 NI two-particle dispersion

Negative of binding energy; deviations from \( \frac{-\hbar^2}{ma_s^2} \)
Three BB, one FF bound states \((\Omega = 2E_{so})\)

Scattering threshold:

Where does the “shape” come from?
"Shape"?
Simple Qualitative Picture

$$\Omega = 2E_{so} \text{ (numerical)}$$

$$\Omega = 0 \text{ (analytical)}$$
Binding Energy For $\Omega = 2E_{so}$: Lowest BB State

Maximum binding roughly where the dispersion has three global minima

**momentum space**

**real space**

Weakly-bound state for certain negative free-space s-wave scattering lengths.

For FF, see: Shenoy, PRA 88, 033609 (2013). Dong et al., PRA 87, 043616 (2013).
With SOC: Fate Of Three-Boson Efimov States?

\[ H = \left( \frac{\vec{p}^2_{12}}{2\mu_{12}} + \frac{\vec{p}^2_{12,3}}{2\mu_{12,3}} \right) + \sum_{j<k} g_2 \delta(\vec{r}_{jk}) + g_3 \delta(\vec{r}_1 - \vec{r}_2)\delta(\vec{r}_2 - \vec{r}_3) \]

\[ + \frac{\hbar k_{so}}{m} (...) + \Omega (...) + \tilde{\delta}(...). \]

Continuous scaling symmetry (easy to check!)
\[ t \rightarrow \lambda^2 t; \vec{r} \rightarrow \lambda \vec{r}; a_s \rightarrow \lambda a_s; k_{so} \rightarrow \lambda^{-1} k_{so}; \Omega \rightarrow \lambda^{-2} \Omega; \]
\[ \tilde{\delta} \rightarrow \lambda^{-2} \tilde{\delta}; E \rightarrow \lambda^{-2} E; k_* \rightarrow \lambda^{-1} k_* \]

Discrete scaling symmetry?
\[ t \rightarrow \lambda^2_0 t; \vec{r} \rightarrow \lambda_0 \vec{r}; a_s \rightarrow \lambda_0 a_s; k_{so} \rightarrow \lambda_0^{-1} k_{so}; \Omega \rightarrow \lambda_0^{-2} \Omega; \]
\[ \tilde{\delta} \rightarrow \lambda_0^{-2} \tilde{\delta}; E \rightarrow \lambda_0^{-2} E; k_* \rightarrow k_*; \lambda_0 \approx 22.7 \]
Generalized Radial Scaling Law?

\( \tilde{\delta} = 0 \) And \( (\kappa_*)^{-1} = 66r_0 \)

\( (k_{so})^{-1} = 25r_0 \cdot \Omega = 0.0016 \frac{\hbar^2}{mr_0^2} \)

\( (k_{so})^{-1} = \lambda_0^2 25r_0 \cdot \Omega = (0.0016/\lambda_0^2) \frac{\hbar^2}{mr_0^2} \)
Generalized Radial Scaling Law (Five Instead Of Two Axes)

Discrete scaling symmetry \((\lambda_0 \approx 22.7)! \)

\(a_s \rightarrow \lambda_0 a_s;\)
\(k_{so} \rightarrow \lambda_0^{-1} k_{so};\)
\(\Omega \rightarrow \lambda_0^{-2} \Omega;\)
\(\tilde{\delta} \rightarrow \lambda_0^{-2} \tilde{\delta};\)
\(E \rightarrow \lambda_0^{-2} E.\)

\(\kappa_* \rightarrow \kappa_* \).

Collapse of neighboring energy levels!

Solid line (gr. st.): \((\kappa_*)^{-1} = 66r_0.\)
\((k_{so})^{-1} = 25r_0.\)
\(\Omega = 2E_{so}; \tilde{\delta} = 0.\)
Dots (exc. st. of \(H\) with scaled parameters).

Collapse of neighboring energy manifolds!

Solid lines (gr. st. manifold): \((\kappa_*)^{-1} = 66r_0.\)
\((k_{so})^{-1} = 100r_0.\)
\(\Omega = 2E_{so}. \tilde{\delta} = 0.\)
Dots (exc. st. manifold of \(H\) with scaled parameters).
Proposal: Experimental Observability

Using three-body parameter for $^{133}\text{Cs}$. Lowest state in excited state manifold.

$(k_{so})^{-1} \approx 10,160a_0$.

$\frac{k_{so}}{\kappa_*} \approx 1.32$ (exc. state).

$\Omega = 2E_{so}$.

Ground state resonance mostly unchanged.

Excited state resonance: Enhanced losses between $a_s \approx -7,790a_0$ and $a_s \approx -20,190a_0$.

Scattering length window!
Summary

Discussion of one few-body technique: Stochastic variational approach with explicitly correlated Gaussians.

Application of this approach to bosons in the presence of 1D spin-orbit coupling.

Generalized radial scaling law for three identical bosons.
Many Thanks To Collaborators

Debraj Rakshit, Xiangyu (Desmond) Yin, Qingze Guan: ECG approach.

Qingze Guan: ECG approach and generalized radial scaling law.

Three-Boson Spectrum in the Presence of 1D Spin-Orbit Coupling: Efimov’s Generalized Radial Scaling Law

Q. Guan and D. Blume

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Energetics and structural properties of two- and three-boson systems in the presence of one-dimensional spin-orbit coupling

Q. Guan© and D. Blume
Thank You!