The two-body system at low energies:

- scattering length
- effective range
- bound (and virtual) shallow states
- introducing universal behavior

The two-body system

The structure and dynamics of a two-body system is governed by the underlying force and by the character of the particles.

Some general aspects:

- If the force depends only on the $|\vec{r}_1 - \vec{r}_2|$ distance, the angular momentum $\vec{\ell}$ is conserve and ℓ is a good quantum number.
- If the parity is conserved the wave function has well defined parity: $\pi = (-1)^\ell$
- In the case of equal particles, bosons or fermions, the wave function verified an specific permutation symmetry:
- For spin-0 bosons the wave function is symmetric $\rightarrow \ell =$ even
- For spin-1/2 fermions the wave function is antisymmetric $\rightarrow \ell + S$ even
- For spin-isospin-1/2 fermions the wave function is antisymmetric $\rightarrow \ell + S + T$ odd

The two-body system

For equal bosons

$$\Psi_\ell = \frac{u_\ell(r)}{r} Y_{\ell m}(\hat{r}) = \frac{u_\ell(r)}{r} |\ell m>, \;\; \ell \text{ even}$$

For $1/2$-spin fermions

$$\Psi_{J^\pi} = \frac{u^J_{\ell S}(r)}{r} [Y_\ell(\hat{r}) \otimes \chi_S]_{JJ_z} = \frac{u^J_{\ell S}(r)}{r} |\ell S J J_z >, \;\; \ell + S \text{ even}$$

For $1/2$-spin-isospin fermions

$$\Psi^T_{J^\pi}(1,2) = \frac{u^J_{\ell S}(r)}{r} [Y_\ell(\hat{r}) \otimes \chi_S]_{JJ_z} \xi_{TT_z} = \frac{u^J_{\ell S}(r)}{r} |\ell S J J_z >, \;\; \ell + S + T \text{ odd}$$

For example the states with $\ell = 0, 1$:

$$\Psi^1_{0^+}(1,2) = \frac{u^0_{00}(r)}{r} |000 > \qquad \Psi^0_{1^-}(1,2) = \frac{u^1_{10}(r)}{r} |101 >$$

$$\Psi^0_{1^+}(1,2) = \frac{u^1_{01}(r)}{r} |011 > \qquad \Psi^1_{J^-}(1,2) = \frac{u^J_{11}(r)}{r} |11J >$$

1/2-spin-isospin channels

The total spin $\vec{S} = \vec{s}_1 + \vec{s}_2$ can take the values $S = 1, 0$. The functions are symmetric ($S = 1$) or antisymmetric ($S = 0$)

$S = 1$ case:

$$\chi_{11} = \chi_{\frac{1}{2}\frac{1}{2}}(1)\chi_{\frac{1}{2}\frac{1}{2}}(2)$$

$$\chi_{10} = \frac{1}{\sqrt{2}}[\chi_{\frac{1}{2}\frac{1}{2}}(1)\chi_{\frac{1}{2}-\frac{1}{2}}(2) + \chi_{\frac{1}{2}-\frac{1}{2}}(1)\chi_{\frac{1}{2}\frac{1}{2}}(2)]$$

$$\chi_{1-1} = \chi_{\frac{1}{2}-\frac{1}{2}}(1)\chi_{\frac{1}{2}-\frac{1}{2}}(2)$$

$S = 0$ case:

$$\chi_{00} = \frac{1}{\sqrt{2}}[\chi_{\frac{1}{2}\frac{1}{2}}(1)\chi_{\frac{1}{2}-\frac{1}{2}}(2) - \chi_{\frac{1}{2}-\frac{1}{2}}(1)\chi_{\frac{1}{2}\frac{1}{2}}(2)]$$

With similar properties for the isospin wavefunctions ξ_{1T_z} and ξ_{00}

The two-body system with short-range interactions

Let us look to the Schrödinger equation:

$$H\Psi^T_{J^\pi} = \left[-\frac{\hbar^2}{m}\nabla^2 + V(1,2)\right]\Psi^T_{J^\pi} = E\Psi^T_{J^\pi}$$

- $E = -\frac{k_d^2\hbar^2}{m} < 0$ bound or virtual states

  $\Psi^T_{J^\pi}(r \to \infty) \to C_a\frac{e^{-k_d r}}{r}$ (bound) or $C_a\frac{e^{+k_d r}}{r}$ (virtual)

  C_a is called the asymptotic constant

- $E = \frac{k^2\hbar^2}{m} > 0$ scattering states

  $\Psi^T_{J^\pi}(r \to \infty) \to j_\ell(kr) + \tan\delta_\ell y_\ell(kr)$

  δ_ℓ is called the phase-shift

- $E = 0$ is a particular case. For $\ell = 0$, $\Psi^T_{J^\pi}(r \to \infty) \to 1 - \frac{a}{r}$

  a is called the scattering length

Defining low energies

For short-range interactions:

$V(1,2) \to 0$ when $r > r_N$ with r_N the range of the force.

It is possible to contruct the energy

$$E_N = \frac{\hbar^2}{m r_N^2}$$

which is the natural energy of the system.

We consider low energies $E < E_N$ and high energies $E > E_N$. In particular scattering energies at $E < E_N$ are dominated by the $\ell = 0$ wave.

Scattering at low energies

Considering uncoupled channels, the wave function is
$\Psi^T_{J^\pi} = (u_{\ell S}(r)/r)|\ell SJ >$
and the Schroedinger equation results

$$\left[-\frac{\hbar^2}{m}\nabla^2 + V(1,2)\right]\frac{u_{\ell S}(r)}{r}|\ell SJ >= E\frac{u_{\ell S}(r)}{r}|\ell SJ >$$

For $r_N > r$, with r_N the range of the nuclear interaction (not considering the long-range Coulomb interaction), the equation is

$$-\frac{\hbar^2}{m}\left[\frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}\right]u_{\ell S}(r) = E\, u_{\ell S}(r)$$

Different partial waves contribute to the scattering process when
$E > \frac{\hbar^2}{m}\frac{\ell(\ell+1)}{r_N^2}$
s-wave dominate the process when $E << \frac{\hbar^2}{m}\frac{2}{r_N^2}$

Scattering at low energies

Considering only *s*-waves,
$\Psi^T_{J^\pi} = (u_S(r)/r)|0SJ>$,
in the case of 1/2-spin-isospin fermions, there are two channels:
$J^\pi = 0^+ \ (S = 0, T = 1)$ or $J^\pi = 1^+ \ (S = 1, T = 0)$

The Schroedinger equation is:

$$\left[-\frac{\hbar^2}{m}\nabla^2 + V(1,2)\right]\frac{u_S(r)}{r}|0SJ> \rightarrow \left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + V_S(r)\right]u_S(r) = E\, u_S(r)$$

with $V_S(r) =< 0SJ|V(1,2)|0SJ >$. The potential could be different in the two spin channels: $S = 0, 1$

- The behavior of the system in $S = 0, 1$ states clarify this fact:

For example a bound state could appear for $S = 1$ and not for $S = 0 \rightarrow$ in nuclear physics the potential is different in the two spin channels.

Zero-energy scattering

More information about the spin-dependence of the force is obtained looking at the zero-energy equation:

$$\left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + V_S(r)\right] u_S(r) = 0$$

The potential is short-range, $V(r > r_N) = 0$, and for $r > r_N$, the equation is

$$\frac{\partial^2}{\partial r^2} u_S(r) = 0$$

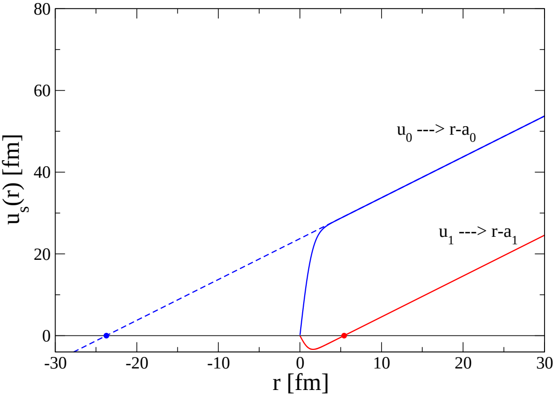
Asymptotically

$$u_S(r) \to r - a_S$$

with a_S the singlet ($S = 0$) or triplet ($S = 1$) scattering length

Zero-energy scattering: *NN* as example

The equation $u''_S(r) - V_S(r)/(\hbar^2/m) = 0$ is integrated from the origin with the boundary condition, $u_S(0) = 0$, up to matching the above linear behavior.



Nuclear experimental data for these quantities are:
$a_0 = -23.74(2)$ fm and $a_1 = 5.42(1)$ fm

Positive energies

At positive energies the *s*-wave Schroedinger equation is:

$$\left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + V_S(r)\right] u_S(r) = E\, u_S(r)$$

or

$$u_S''(r) + \left[k^2 - \frac{mV_S(r)}{\hbar^2}\right] u_S(r) = 0$$

with $E = \hbar^2 k^2/m$ and asymptotically the reduced wave function is

$$u_S(r > r_N) = \sin(kr) + R_S \cos(kr)$$

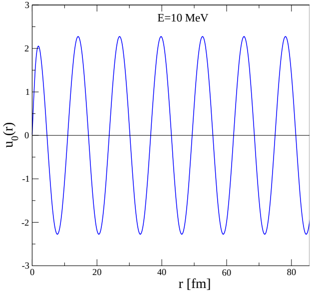
In general

$$u_S(r > r_N) = \begin{cases} \sin(kr) + R_S \cos(kr) & K - matrix \\ e^{-ikr} - S_S\, e^{ikr} & S - matrix \\ \sin kr + T_S\, e^{ikr} & T - matrix \\ R_S^{-1} \sin(kr) + \cos(kr) & K^{-1} - matrix \end{cases}$$

The Phase-shift δ_S

For single channels: $R_S = \tan\delta_S$ with δ_S the phase-shift in spin channel S and $S_S = e^{2i\delta_S}$; $T_S = \sin\delta_S e^{i\delta_S}$

To determine the phase-shift we impose the boundary conditions:
1)$u_S(0) = 0$
2)$u_S(r > r_N) = \sin(kr) + R_S\cos(kr)$



NN scattering at 10 MeV in $S = 0$ channel

Effective Range expansion

Behavior of the phase-shift at low energies.

The asymptotic (reduced) wave function for positive energies is:

$$\phi_S(kr) = R_S^{-1}\sin(kr) + \cos(kr) = \frac{1}{\tan\delta_S}\sin(kr) + \cos(kr)$$

at low energies $k \to 0$, then

$$\frac{1}{\tan\delta_S}\sin(kr) + \cos(kr) \to \frac{kr}{\tan\delta_S} + 1$$

Remembering the asymptotic behavior of the zero-energy (reduced) wave function $u_S = r - a_S = -a_S(-r/a_S + 1)$, we identify

$$\lim_{k\to 0} k\cot\delta_S = -\frac{1}{a_S}$$

Effective Range expansion

The scattering length and the phase-shift are related through the effective range expansion, valid at low energies:

$$k \cot \delta_S = -\frac{1}{a_S} + \ldots = -\frac{1}{a_S} + \frac{1}{2} r_{eff}^{(S)} k^2 + \ldots$$

where $\hbar^2 k^2/m = E$ is the energy of the process and $r_{eff}^{(S)}$ is the effective range in spin channel S:

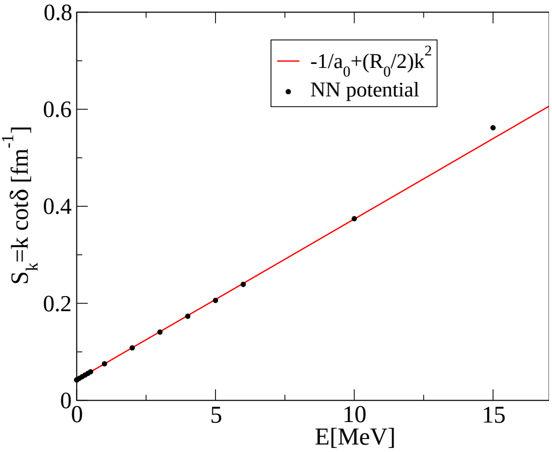
$$r_{eff}^{(S)} = 2 \int_0^\infty \left[\phi_S^2 - u_S^2(k=0) \right] \, dr$$

with $\phi_S = 1 - r/a_S$ and $u_S(k=0)$ the zero-energy solution normalized in such a way that

$$u_S(r > r_N) \to 1 - r/a_S$$

In the above integral, the integrand goes to zero very quickly as ϕ_S and $u_S(k=0)$ becomes equal.

Effective Range expansion



The effective range

For s-wave scattering the equation at two different energies

$$\frac{\partial^2 u_1(r)}{\partial r^2} - \left(\frac{mV(r)}{\hbar^2} - k_1^2\right) u_1(r) = 0$$

$$\frac{\partial^2 u_2(r)}{\partial r^2} - \left(\frac{mV(r)}{\hbar^2} - k_2^2\right) u_2(r) = 0$$

Multiplying the equations by u_2 and u_1, substracting and integrating

$$\int_r^{r'} (u_2 u_1'' - u_1 u_2'') dr = (k_2^2 - k_1^2) \int_r^{r'} u_1 u_2 dr$$

$$(u_2 u_1' - u_1 u_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} u_1 u_2 dr$$

The effective range

We can start from the non-interacting equation

$$\frac{\partial^2 \phi_1(r)}{\partial r^2} - k_1^2 \phi_1(r) = 0$$
$$\frac{\partial^2 \phi_2(r)}{\partial r^2} - k_2^2 \phi_2(r) = 0$$

Multiplying the equations by ϕ_2 and ϕ_1, substracting and integrating

$$\int_r^{r'} (\phi_2 \phi_1'' - \phi_1 \phi_2'') dr = (k_2^2 - k_1^2) \int_r^{r'} \phi_1 \phi_2 dr$$

$$(\phi_2 \phi_1' - \phi_1 \phi_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} \phi_1 \phi_2 dr$$

Substructing the equations for the interacting and non-interacting systems

$$(\phi_2 \phi_1' - \phi_1 \phi_2')|_r^{r'} - (u_2 u_1' - u_1 u_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} (\phi_1 \phi_2 - u_1 u_2) dr$$

The effective range

$$(\phi_2\phi_1' - \phi_1\phi_2')|_r^{r'} - (u_2u_1' - u_1u_2')|_r^{r'} = (k_2^2 - k_1^2)\int_r^{r'}(\phi_1\phi_2 - u_1u_2)dr$$

- making $r' > r_N \rightarrow u_i(r') = \phi_i(r')$
- making $r \rightarrow 0$ the wave functions $u_1, u_2 \rightarrow 0$
- we choose $\phi_i = \cot\delta_i \sin(k_i r) + \cos(k_i r)$
- $\phi_i(r \rightarrow 0) = \phi_i(0) = 1$
- $\phi_i'(0) = k_i \cot\delta_i$

Therefore, the only remaining term is

$$\phi_2\phi_1' - \phi_1\phi_2' \xrightarrow{r\rightarrow 0} k_2\cot\delta_2 - k_1\cot\delta_1 = (k_2^2 - k_1^2)\int_0^{\infty}(\phi_1\phi_2 - u_1u_2)$$

The effective range

$$\phi_2\phi_1' - \phi_1\phi_2' \xrightarrow{r\to 0} k_2 \cot\delta_2 - k_1 \cot\delta_1 = (k_2^2 - k_1^2)\int_0^\infty (\phi_1\phi_2 - u_1 u_2)$$

making $k_1 \to 0$, remembering

$$\lim_{k\to 0} k\cot\delta = -\frac{1}{a}$$

defining $k_2 \equiv k$ and introducing the dependence on the spin channel

$$k\cot\delta_S = -\frac{1}{a_S} + \frac{1}{2} r_{\rm eff}^{(S)} k^2 + \ldots$$

with

$$r_{\rm eff}^{(S)} = 2\int_0^\infty \left[\phi_S^2 - u_S^2(k=0)\right] \, dr$$

The pole of the S-matrix

The S-matrix is defined as

$$S = e^{2i\delta} = \frac{e^{i\delta}}{e^{-i\delta}} = \frac{cos\delta + i\sin\delta}{cos\delta - i\sin\delta} = \frac{cot\delta + i}{cot\delta - i}$$

and $kcot\delta = ik$ is a pole of the S-matrix. The extension to the complex plane (at the immaginary axis) corresponds to bound states: $k = i\kappa_d$. Resulting in $i\kappa_d cot\delta = -\kappa_d$. Using the effective range expansion, valid only for bound states close to the threshold (shallow states), we have:

$$\kappa_d = \frac{1}{a_S} + \frac{1}{2} r_{eff}^{(S)} \kappa_d^2 + \dots$$

from where, in the case of shallow states, it would be possible to extract the bound state energy using scattering properties.

Shallow bound states

Experimental data:

		deuteron(1^+)
np	a_1	5.419(7) fm
	$r_{\rm eff}^{(1)}$	1.753(8) fm
	E_d	2.22456 MeV

	Helium dimer(0^+)
a	189.45 a_0
$r_{\rm eff}$	13.85 a_0
$E_{\rm He}$	1.303 mK

Using a and $r_{\rm eff}$ the binding energies can be estimated as:

$$\kappa_d = \frac{1}{r_{\rm eff}}\left(1-\sqrt{1-2r_{\rm eff}/a}\right)$$

$E_d = \hbar^2\kappa_d^2/m = 2.223$ MeV
$\hbar^2/m = 41.47\ \mathrm{MeV\ fm}^2$
$\hbar^2/mr_N^2 \approx 10\ \mathrm{MeV}$
$r_{\rm eff}^{(1)}/a_1 \approx 0.3$

$E_{\rm He} = \hbar^2\kappa_d^2/m = 1.303$ mK
$\hbar^2/m = 43.281\ \mathrm{K}a_0^2$
$\hbar^2/mr_N^2 \approx 250\ \mathrm{mK}$
$r_{\rm eff}/a \approx 0.07$

The nn and np virtual states

Experimental data:

np	a_0	-23.740(20) fm	
	$r_{eff}^{(0)}$	2.77(5) fm	
nn	a_0	-18.90(40) fm	
	$r_{eff}^{(0)}$	2.75(11) fm	
pp	a_0	-7.8063(26) fm	≈ -17.3 fm without EM
	$r_{eff}^{(0)}$	2.794(14) fm	

Using a_0 and $r_{eff}^{(0)}$ the energy of the nn and np virtual states are

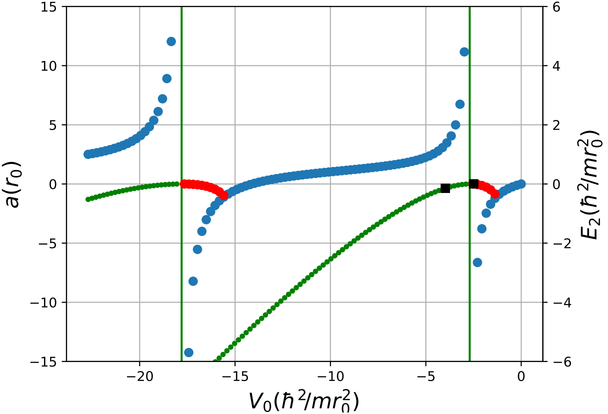
$$\kappa_v = \frac{1}{r_{eff}^{(0)}} \left(1 - \sqrt{1 - 2 r_{eff}^{(0)} / a_0} \right)$$

nn: $E_v = \hbar^2 \kappa_v^2 / m = 0.102$ MeV.
np: $E_v = \hbar^2 \kappa_v^2 / m = 0.066$ MeV.

The sign of the scattering length

$V(1,2) = V_0 e^{-r^2/r_0^2}$

- • scattering length, • bound state, • virtual state



The universal window

When a shallow bound state verifies: $k_d = 1/a + r_{\text{eff}} k_d^2/2$
In this region $r_{\text{eff}}/a >> 1$, this relation defines the universal window.

Moreover, defining $a_B = 1/k_d$ and $r_B = a - a_B$, the above relation results

$$r_{\text{eff}} a = 2 r_B a_B$$

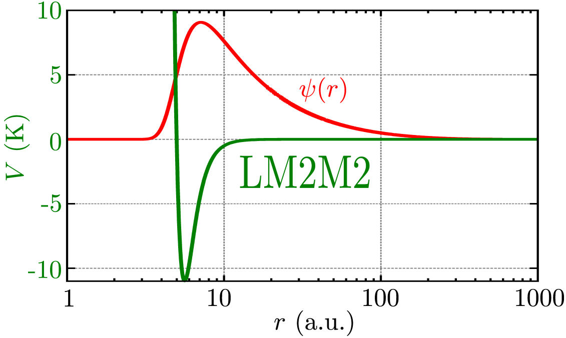
At the unitary limit $a \to \infty$, $r_{\text{eff}} = 2r_B$ and the effective range expansion results

$$\cot\delta = r_B k$$

Inside the unitary window, $r_{\text{eff}}/a << 1$, the dynamics is constrained by the strict relation between the low energy parameters $\rightarrow$ universal behavior.

The universal window

During the analysis of the two-body system at low energies, I mentioned two very different systems: the two-nucleon system and the dimer of two helium atoms. Let us take this system as example:



$N = 2$: $\psi(r) \to 0$ if $r < r_N$ $N > 2$: $\psi(\dots r_{ij} \dots) \to 0$ if $r_{ij} < r_N$

The many body system is strongly correlated since $\psi \to 0$ when two particles are close independently of the position of the other particles

From correlations to universality

Weakly bound systems

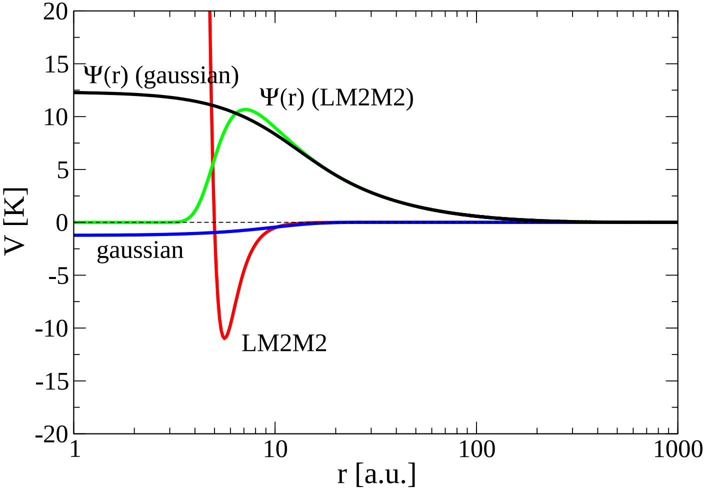
- When a system is weakly bound the particles are most of the time outside the intercation range
- A new type of correlation appears
- To see this we define the following potential:

$$V(r) = V_0 \; e^{-r^2/r_0^2}$$

and fix the strength V_0 to describe the binding energy B of the weakly system:

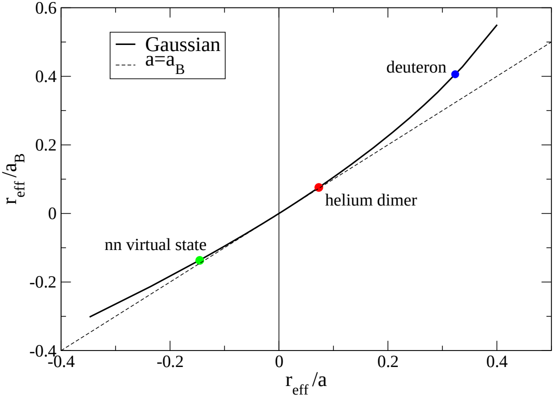
potential	T(mK)	V(mK)	B(mK)	$a(a_0)$	$r_e(a_0)$	$P(r < r_{eff})$
LM2M2	99.4	-100.7	1.3	189.4	13.8	0.07
gaussian	42.2	-43.5	1.3	189.4	13.8	0.07

From correlations to universality



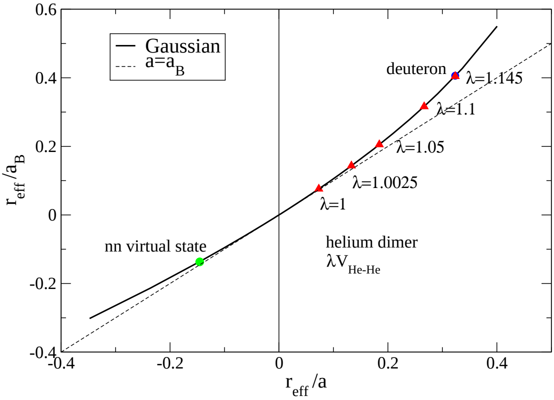
Universal behavior in few-body systems

- When a shallow state exists, a Gaussian potential gives a reasonable description of the low energy regime, bound and scattering states.



Walking around the universal window

- Varying the strength of the potential we can move allong the gaussian trajectory



Continuous Scale Invariance

- For $\lambda \approx 1.145$ the helium dimer and the deuteron overlap:

λV_{He}	B(mK)	$a(a_0)$	$r_{eff}(a_0)$	r_{eff}/a	$r_0(a_0)$
1.000	1.303	189.42	13.845	0.07	10.03
1.025	5.027	99.935	13.290	0.13	9.99
1.050	11.137	69.448	12.792	0.18	9.94
1.100	30.358	44.792	11.937	0.27	9.88
1.145	55.408	34.919	11.299	0.32	9.86

Studying the CSI: C_a and $< r^2 >$

$$\langle r^2 \rangle = \frac{r_0^2}{4} \int_0^\infty dz\, z^2 \phi_B(z)^2 = \frac{a^2}{8}(1 + \left(\frac{r_B}{a}\right)^2 + \mathcal{O}(\left(\frac{r_B}{a}\right)^3)) \simeq \frac{a_B^2}{8} e^{2r_B/a_B}$$

The ANC is defined: $\phi_B(r > r_N) \to C_a e^{-r/a_B}$ and results

$$C_a^2 \simeq \frac{2}{a_B} \frac{1}{1 - r_e/a_B} = \frac{2}{a_B} e^{2r_B/a_B}$$

Continuous Scale Invariance