# Low energy nuclear systems: The two-nucleon system

# Introduction

- It is more than 100 years that we have research in nuclear physics
- At present times it is a very active field of research
- It is a strong interacting many-body system:
  - Very complicate to describe
  - Not only for discrete states but also continuum states
- The nuclear interaction is still under active research
  - It is a residual interaction
  - Its long range part is known: the OPEP
  - The two-nucleon system is inside the universal window

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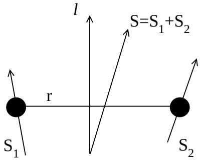
# The NN potential

The potential can depends on

- the distance $\longrightarrow$ $V(r)$
- the spins angle $\longrightarrow$ $V_\sigma(r)\vec{\sigma}_1 \cdot \vec{\sigma}_2$
- the spin $\vec{S}$ and $\vec{\ell}$ angle $\longrightarrow$ $V_{ls}(r)\,\vec{\ell} \cdot \vec{S}$
- the spins and $\vec{r} = \vec{r}_1 - \vec{r}_2$ angles $\longrightarrow$ $V_T(r)\,S_{12}$
  with the tensor operator $S_{12} = 3(\vec{\sigma}_1 \cdot \hat{r})\,(\vec{\sigma}_2 \cdot \hat{r}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2$
- the different isospin channels $\longrightarrow$ $\vec{\tau}_1 \cdot \vec{\tau}_2$
- $V(1,2) = V(r) + V_\sigma(r)\,\vec{\sigma}_1 \cdot \vec{\sigma}_2 + V_t(r)\,\vec{\tau}_1 \cdot \vec{\tau}_2 + V(r)_{\sigma\tau}\,(\vec{\sigma}_1 \cdot \vec{\sigma}_2)\,(\vec{\tau}_1 \cdot \vec{\tau}_2) +$
  $+ V_{\ell s}(r)\,\vec{\ell} \cdot \vec{S} + V_{\ell s\tau}(r)\,(\vec{\tau}_1 \cdot \vec{\tau}_2)\vec{\ell} \cdot \vec{S} + V_T(r)\,S_{12} + V_{T\tau}(r)\,(\vec{\tau}_1 \cdot \vec{\tau}_2) S_{12} + \ldots$



# The Two-Nucleon System, spin-isospin channels

A nucleon is identified by its position $\boldsymbol{r}_1$, its spin $\chi_{\frac{1}{2}s_z}(1)$ and its isospin $\xi_{\frac{1}{2}t_z}(1)$, with $t_z = \frac{1}{2}$ for protons and $t_z = -\frac{1}{2}$ for neutrons.

The following properties are verified:

$$s^2\chi_{\frac{1}{2}s_z}(1) = s(s+1)\chi_{\frac{1}{2}s_z}(1) = \frac{3}{4}\chi_{\frac{1}{2}s_z}(1)$$
$$s_z\chi_{\frac{1}{2}s_z}(1) = s_z\chi_{\frac{1}{2}s_z}(1)$$

and

$$t^2\xi_{\frac{1}{2}t_z}(1) = t(t+1)\xi_{\frac{1}{2}t_z}(1) = \frac{3}{4}\xi_{\frac{1}{2}t_z}(1)$$
$$t_z\xi_{\frac{1}{2}t_z}(1) = t_z\xi_{\frac{1}{2}t_z}(1)$$

To be noticed the relations:
$\vec{\sigma} = 2\vec{s}$ and $\vec{\tau} = 2\vec{t}$

# The Two-Nucleon System, spin-isospin channels

The two-nucleon system is identified by the relative position $\mathbf{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$ and the two-nucleon spin ($\vec{S} = \vec{s}_1 + \vec{s}_2$) and isospin ($\vec{T} = \vec{t}_1 + \vec{t}_2$) functions.

$$\chi_{SS_z} = \sum_{s_{z1},s_{z2}} (\frac{1}{2}s_{z1}\frac{1}{2}s_{z2}|SS_z)\chi_{\frac{1}{2}s_{z1}}(1)\chi_{\frac{1}{2}s_{z2}}(2)$$

$$\xi_{TT_z} = \sum_{t_{z1},t_{z2}} (\frac{1}{2}t_{z1}\frac{1}{2}t_{z2}|TT_z)\xi_{\frac{1}{2}t_{z1}}(1)\xi_{\frac{1}{2}t_{z2}}(2)$$

with the properties:

$S^2\chi_{SS_z} = S(S+1)\chi_{SS_z}$

$S_z\chi_{SS_z} = S_z\chi_{SS_z}$

$T^2\xi_{TT_z} = T(T+1)\xi_{TT_z}$

$T_z\xi_{TT_z} = T_z\xi_{TT_z}$

# The Two-Nucleon System, spin-isospin channels

The total spin $\vec{S} = \vec{s}_1 + \vec{s}_2$ can take the values $S = 1, 0$. The functions are symmetric ($S = 1$) or antisymmetric ($S = 0$)

$S = 1$ case:

$$\chi_{11} = \chi_{\frac{1}{2}\frac{1}{2}}(1)\chi_{\frac{1}{2}\frac{1}{2}}(2)$$

$$\chi_{10} = \frac{1}{\sqrt{2}}[\chi_{\frac{1}{2}\frac{1}{2}}(1)\chi_{\frac{1}{2}-\frac{1}{2}}(2) + \chi_{\frac{1}{2}-\frac{1}{2}}(1)\chi_{\frac{1}{2}\frac{1}{2}}(2)]$$

$$\chi_{1-1} = \chi_{\frac{1}{2}-\frac{1}{2}}(1)\chi_{\frac{1}{2}-\frac{1}{2}}(2)$$

$S = 0$ case:

$$\chi_{00} = \frac{1}{\sqrt{2}}[\chi_{\frac{1}{2}\frac{1}{2}}(1)\chi_{\frac{1}{2}-\frac{1}{2}}(2) - \chi_{\frac{1}{2}-\frac{1}{2}}(1)\chi_{\frac{1}{2}\frac{1}{2}}(2)]$$

With similar properties for the isospin wavefunctions $\xi_{1T_z}$ and $\xi_{00}$

# The Two-Nucleon System, spin-isospin channels

$T = 1$ case:

$$\xi_{11} = \xi_{\frac{1}{2}\frac{1}{2}}(1)\xi_{\frac{1}{2}\frac{1}{2}}(2)$$

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Specific isospin configuration

$$\xi_{\frac{1}{2}t_{z1}}(1)\xi_{\frac{1}{2}t_{z2}}(2) = \sum_{T,T_z}(\frac{1}{2}t_{z1}\frac{1}{2}t_{z2}|TT_z)\xi_{TT_z}$$

# The Two-Nucleon System: Quantum Numbres

- The total angular momentum $\vec{J} = \vec{\ell} + \vec{S}$
- The nuclear interaction conserves ...
- ... parity, accordingly nuclei have well defined parity $\pi$
- What about isospin? Does the nuclear force conserves isospin?
- The electromagnetic force $V_{EM}(i,j) = e^2 \frac{(t_z(i)+1/2)(t_z(j)+1/2)}{r_{ij}} + \dots$

does not conserves isospin. Only in the $A = 2$ system. However in light nuclear systems where the EM interactions is weak compared to the nuclear force, isospin is almost a good quantum number.

- In general the wave function of a nucleus is indicated by:

$$\Psi_{J^\pi}$$

In light nuclear systems the total isospin $T$ is almost conserved

$$\Psi^T_{J^\pi}$$

with $T = T_z$

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# The Two-Nucleon wave function

The two-nucleon wave function has total $J$, $T$ and parity quantum numbers. In partial wave decomposition it is:

$$\Psi^T_{J^\pi}(1,2) = \sum_{\ell S} \frac{u_{\ell S}(r)}{r} \left[Y_\ell(\hat{r}) \otimes \chi_S\right]_{JJ_z} \xi_{TT_z}$$

with

$$\chi_{SS_z} = \sum_{s_1,s_2} = (\frac{1}{2}s_1\frac{1}{2}s_2|SS_z)\chi_{\frac{1}{2}s_1}(1)\chi_{\frac{1}{2}s_2}(2)$$

$$\xi_{TT_z} = \sum_{t_1,t_2} = (\frac{1}{2}t_1\frac{1}{2}t_2|TT_z)\xi_{\frac{1}{2}t_1}(1)\xi_{\frac{1}{2}t_2}(2)$$

and

$$\left[Y_\ell(\hat{r}) \otimes \chi_S\right]_{JJ_z} = \sum_{\ell_z S_z} (\ell\ell_z SS_z|JJ_z) Y_{\ell\ell_z}(\hat{r})\chi_{SS_z}$$

# The Two-Nucleon wave function

Antisymmetrization:

$$\Psi^{T}_{J^\pi}(1,2) = -\Psi^{T}_{J^\pi}(2,1) \Rightarrow \ell + S + T \equiv \text{odd}$$

$\rightarrow$ *excercise*

Parity:

$$\Psi^{T}_{J^\pi}(\boldsymbol{r}) = \pi\Psi^{T}_{J^\pi}(-\boldsymbol{r}) \Rightarrow \pi = (-1)^{\ell}$$

$\rightarrow$ *excercise*

The lower quantum numbers are:

| $\ell$ | $S$ | $T$ | $J^\pi$ | |
|---|---|---|---|---|
| 0 | 0 | 1 | $0^+$ | |
| 0,2 | 1 | 0 | $1^+$ | $\rightarrow$ deuteron channel |
| 1 | 0 | 0 | $1^-$ | |
| 1 | 1 | 1 | $0^-$ | |
| 1 | 1 | 1 | $1^-$ | |
| 1,3 | 1 | 1 | $2^-$ | $\rightarrow$ coupled channel |
| 2 | 0 | 1 | $2^+$ | |

# The two-nucleon wave function

The structure of the two-nucleon system is governed by the NN force. Let us look to the Schroedinger equation:

$$H\Psi^T_{J^\pi} = \left[-\frac{\hbar^2}{m}\nabla^2 + V(1,2)\right]\Psi^T_{J^\pi} = E\Psi^T_{J^\pi}$$

Defining

$$\Psi^T_{J^\pi}(1,2) = \sum_\ell \frac{u_{\ell S}(r)}{r}\left[Y_\ell(\hat{r}) \otimes \chi_S\right]_{JJ_z} \xi_{TT_z} = \sum_\ell \frac{u_{\ell S}(r)}{r}|\ell SJ>$$

In addition to $J$ and $\pi$ also $S$ and $T$ are good quantum numbers. For example the states in which $\ell = 0$ is present are $J^\pi = 0^+$ and $J^\pi = 1^+$

$\Psi^1_{0^+}(1,2) = \frac{u_{00}(r)}{r}|000>$

$\Psi^0_{1^+}(1,2) = \frac{u_{01}(r)}{r}|011> + \frac{u_{21}(r)}{r}|211>$

# Spin-Isospin projectors

Let us define the following operators:

$$P_0^\sigma = \frac{1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2}{4}, \qquad P_1^\sigma = \frac{3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2}{4}$$

and

$$P_0^\tau = \frac{1 - \vec{\tau}_1 \cdot \vec{\tau}_2}{4}, \qquad P_1^\tau = \frac{3 + \vec{\tau}_1 \cdot \vec{\tau}_2}{4}$$

with the properties

$P_0^\sigma + P_1^\sigma = 1$ and $P_0^\tau + P_1^\tau = 1$

$P_S^\sigma \, \chi_{S' S_z'} = \delta_{SS'} \, \chi_{SS_z'}$ and $P_T^\tau \, \chi_{T' T_z'} = \delta_{TT'} \, \chi_{TT_z'}$

Demonstration:

considering that $\vec{\sigma} = \vec{\sigma}_1 + \vec{\sigma}_2$ then $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{\sigma^2 - \sigma_1^2 - \sigma_2^2}{2}$

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 \, \chi_{SS_z} = \frac{\sigma^2 - 6}{2} \chi_{SS_z} = \begin{cases} -3\chi_{SS_z} & S = 0 \\ 1\chi_{SS_z} & S = 1 \end{cases}$$

# Spin-Isospin projectors

considering that $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$ then $\vec{\tau}_1 \cdot \vec{\tau}_2 = \frac{\tau^2 - \tau_1^2 - \tau_2^2}{2}$

$$\vec{\tau}_1 \cdot \vec{\tau}_2 \, \xi_{TT_z} = \frac{\tau^2 - 6}{2} \xi_{TT_z} = \begin{cases} -3\chi_{TT_z} & T = 0 \\ 1\chi_{TT_z} & T = 1 \end{cases}$$

We have demonstrated that:

$P_S^\sigma \, \chi_{S'S_z'} = \delta_{SS'} \, \chi_{SS_z'}$ and $P_T^\tau \, \chi_{T'T_z'} = \delta_{TT'} \, \chi_{TT_z'}$

Important consequence:

The four operators $[1 \otimes 1, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \otimes 1, 1 \otimes \vec{\tau}_1 \cdot \vec{\tau}_2, (\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{\tau}_1 \cdot \vec{\tau}_2)]$ are linear combinations of the four operators $[P_S^\sigma \otimes P_T^\tau]$.

Exercise: calculate the four relations

# The deuteron

The deuteron is the only *pn* bound system, it has $J^\pi = 1^+, S = 1, T = 0$

Some static properties:

- $E = 2.22457$ MeV
- $r_d = 1.975$ fm
- $Q_d = 0.2859$ fm$^2$
- $\mu_d = 0.8574\mu_0$
- $C_a = 0.8781$ fm$^{-1/2}$

Its wave function is:

$$\Psi^0_{1^+}(1,2) = \frac{u_0(r)}{r}|011> + \frac{u_2(r)}{r}|211>$$

# deuteron equations

The Schroedinger equations is:

$$(H-E)\Psi^0_{1^+} = \left[-\frac{\hbar^2}{m}\nabla^2 + V(1,2) - E\right]\Psi^0_{1^+} = 0$$

or

$$(H-E)\Psi^0_{1^+} = \left[-\frac{\hbar^2}{m}\nabla^2 + V(1,2) - E\right]\frac{u_0(r)}{r}|011> + \frac{u_2(r)}{r}|211> = 0$$

Projecting the equations in the two angular states

$$\begin{cases} \left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + <011|V(1,2)|011> - E\right]\frac{u_0}{r} = -<011|V(1,2)|211>\frac{u_2}{r} \\ \left[-\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial r^2} - \frac{6}{r^2}\right) + <211|V(1,2)|211> - E\right]\frac{u_2}{r} = -<211|V(1,2)|011>\frac{u_0}{r} \end{cases}$$

The next step is to calculate the matrix elements of the potential

# Evidence for the *D*-state of the deuteron

Evidences for the deuteron *D*-state are coming from the quadrupole moment $Q_d$ (different from zero) and from the magnetic moment $\mu_d$ (different from summing the proton and deuteron magnetic moments).

A) The quadrupole moment:

$$Q_d = \left(\frac{16\pi}{5}\right)^{1/2} \int [\Psi^0_{1+}]^* Q_{20} \Psi^0_{1+} \, d\vec{r}$$

where

$$Q_{20} = e \sum_{j=1}^{A} r_j^2 Y_{20}(\theta_j)(\frac{1}{2} + t_{zj}) = e \left(\frac{r}{2}\right)^2 Y_{20}(\theta)$$

and the deuteron wave function written as

$$\Psi^0_{1+} = \cos\epsilon \, u_0(r)|011 > + \sin\epsilon \, u_2(r)|211 >$$

with $J_z = J$

# Evidence for the *D*-state of the deuteron

The $\ell = 0$ component does not contribute

$$Q_d = \left(\frac{16\pi}{5}\right)^{1/2} \frac{e}{\sqrt{4\pi}} \cos\epsilon \sin\epsilon \,(2011|11) \int_0^\infty u_0(r)u_2(r)r^2 dr - \sin^2\epsilon \ldots$$

$$Q_d = \frac{1}{10}\left[\sqrt{2}\cos\epsilon\sin\epsilon \int_0^\infty u_0(r)u_2(r)r^2 dr - \frac{1}{2}\sin^2\epsilon \int_0^\infty u_2^2(r)r^2 dr\right]$$

the experimental value is $Q_d = 0.2859(15)\,\text{fm}^2$. This results can be reproduced with $\sin^2\epsilon \approx 0.04$ (a *D*-state probability of about 4%). A pure *D* state will produce a negative value of Q

Exercise:

- derive the above expresion

# Evidence for the $D$-state of the deuteron

B) The magnetic moment $\mu_d$:
Is the mean value of the $z$-component of the magnetic moment operator

$$\vec{M} = \mu_0 \left( \sum_{k=1}^{A} (\frac{1}{2} + t_3^k)(\vec{\ell}_k + g_k^p \vec{s}_k) + \sum_{k=1}^{A} (\frac{1}{2} - t_3^k) g_k^n \vec{s}_k \right)$$

where the giroscope factors are:
$g^p/2 = 2.792782(17)$
$g^n/2 = -1.913148(66)$.

The magnetic moment is define as:

$$\mu_d = < \Psi_{J^+}^0 | M_z | \Psi_{J^+}^0 >$$

where $\Psi_{J^+}^0$ is the deuteron wave function with $J_z = J$.

# Evidence for the *D*-state of the deuteron

Defining the scalar $\mu_s = (g^p + g^n)/2$ and vector $\mu_v = (g^p - g^n)/2$ moments, the operator is

$$\vec{M} = \mu_0 \left( \sum_k^A (\frac{1}{2} + t_3^k)\vec{\ell}_k + \mu_s \vec{S} + \mu_v \sum_k^A t_3^k \vec{s}_k \right)$$

which for the deuteron reduces to $(\mu_0/2)\vec{\ell} + \mu_s \vec{S}$. Therefore:

$$M_z/\mu_0 = \frac{1}{2}\ell_z + \mu_s S_z$$

A pure *s*-wave state predicts $\mu_d/\mu_0 = \mu_s = 0.879$. Whereas the experimental results is $\mu_d = 0.857\mu_0$.

In order to consider the *D*-component we use the followng relation (exercise)

$$M_z/\mu_0 = \frac{1}{2}\left[ (\frac{1}{2} + \mu_s) + \frac{1}{2}(\mu_s - \frac{1}{2})(S^2 - \ell^2) \right]$$

# Evidence for the *D*-state of the deuteron

$$M_z/\mu_0 = \frac{1}{2}\left[(\frac{1}{2} + \mu_s) + \frac{1}{2}(\mu_s - \frac{1}{2})(S^2 - \ell^2)\right]$$

Considering the deuteron wave function

$$\Psi^0_{1^+} = \cos\epsilon\, u_0(r)|011 > + \sin\epsilon\, u_2(r)|211 >$$

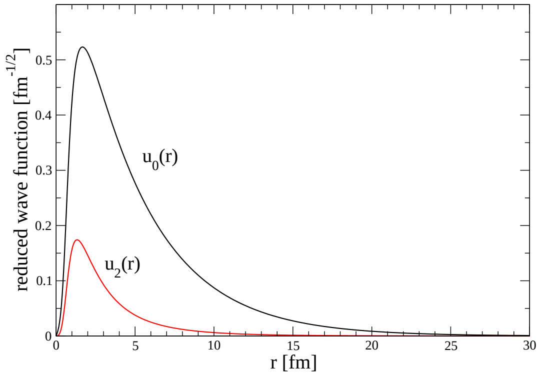
the magnetic moment results

$$\mu_d/\mu_0 = \cos^2\epsilon < 011|M_z|011 > + \sin^2\epsilon < 211|M_z|211 >$$

$$\mu_d/\mu_0 = \mu_s + \sin^2\epsilon\,(\frac{3}{4} - \frac{3\mu_s}{2}) = 0.879 - 0.569\sin^2\epsilon$$

the measured value is $\mu_d/\mu_0 = 0.857406(1)$, implying a *D*-state probability of around 4%.

# Deuteron wave function




with the normalization $\int_0^\infty (u_0^2 + u_2^2) dr = 1$

and a *D*-state probability $P_d = 100 \times \int_0^\infty u_2^2 \, dr \approx 5\%$

# Matrix elements of the nuclear potential

Let us return to the projected equations in the two angular states

$$\begin{cases} \left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + <011|V(1,2)|011> - E\right]\frac{u_0}{r} = -<011|V(1,2)|211>\frac{u_2}{r} \\ \left[-\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial r^2} - \frac{6}{r^2}\right) + <211|V(1,2)|211> - E\right]\frac{u_2}{r} = -<211|V(1,2)|011>\frac{u_0}{r} \end{cases}$$

The two radial components $u_0$ and $u_2$ are determined by the potential in $S, T = 1, 0$ channel.

$$V_{10}(1,2) = V_c(r) + V_{\ell s}(r)\vec{\ell}\cdot\vec{s} + V_T(r)S_{12} + V_{\ell^2}(r)\ell^2 + \ldots$$

# Matrix elements of the potential $V(1,2)$

Let us consider a general state $|LSJ>$. The central potential is very simple

$$< LSJ|V(1,2)|L'S'J >= \delta_{SS'}\delta_{LL'}V_c(r)$$

The spin-orbit interaction is

$$< LSJ|\vec{L}\cdot\vec{S}|L'S'J >= \delta_{SS'}\delta_{LL'}\frac{1}{2}[J(J+1)-L(L+1)-S(S+1)]$$

and the $L^2$ interactions is

$$< LSJ|L^2|L'S'J >= \delta_{SS'}\delta_{LL'}L(L+1)$$

# Matrix elements of the potential $V(1,2)$

The tensor operator $S_{12}$ acts in $S = 1$ and can coupled different $L$ states:

$$< J\, 1 J | S_{12} | J\, 1 J >= 2$$

$$< J-1\; 1\; J-1 | S_{12} | J-1\; 1\; J-1 >= -2\frac{(J-1)}{2J+1}$$

$$< J+1\; 1\; J | S_{12} | J-1\; 1\; J >= 6\frac{\sqrt{J(J+1)}}{2J+1}$$

$$< J+1\; 1 J | S_{12} | J+1\; 1 J >= -2\frac{(J+2)}{2J+1}$$

exercise!

Remember: $S_{12} = 3(\sigma_1 \cdot \hat{r})(\sigma_2 \cdot \hat{r}) - \sigma_1 \cdot \sigma_2$

# Solving the radial equations

Ingredients of the Schroedinger equation:

For the *np* case:

- $\frac{\hbar^2}{m} = \frac{c^2\hbar^2}{mc^2} \approx \frac{197.327053}{m_{nd}} \approx 41.471$ MeV fm$^2$
- The knowledge of the potential $V(1,2)$ in the proper $S$, $T$ channel

$$\begin{cases} \left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + <011|V(1{,}2)|011> - E\right]\frac{u_0}{r} = -<011|V(1{,}2)|211>\frac{u_2}{r} \\ \left[-\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial r^2} - \frac{6}{r^2}\right) + <211|V(1{,}2)|211> - E\right]\frac{u_2}{r} = -<211|V(1{,}2)|011>\frac{u_0}{r} \end{cases}$$

# Solving the radial equations

For example, in $S, T = 1, 0$, the deuteron channles, the force is:

$$V(1,2) = V_c(r) + V_T(r)S_{12} + V_{\ell S}(r)\vec{\ell} \cdot S + V_{\ell^2}(r)\ell^2 + \dots$$

The equations are

$$\begin{cases} \left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + <011|V(1{,}2)|011> - E\right]\frac{u_0}{r} = -<011|V(1{,}2)|211>\frac{u_2}{r} \\ \left[-\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial r^2} - \frac{6}{r^2}\right) + <211|V(1{,}2)|211> - E\right]\frac{u_2}{r} = -<211|V(1{,}2)|011>\frac{u_0}{r} \end{cases}$$

$$\begin{cases} \left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial r^2} + V_c(r) - E\right]\frac{u_0}{r} = -\sqrt{8}V_T(r)\frac{u_2}{r} \\ \left[-\frac{\hbar^2}{m}\left(\frac{\partial^2}{\partial r^2} - \frac{6}{r^2}\right) + V_c(r) - 2V_T(r) - 3V_{\ell S}(r) + 6V_{\ell^2}(r) - E\right]\frac{u_2}{r} = -\sqrt{8}V_T(r)\frac{u_0}{r} \end{cases}$$

# *pp* scattering

Two protons have isospin $T = 1$, the wave function is

$$\Psi^1_{J^\pi}(1,2) = \sum_{\ell S} \frac{u_{\ell S}(r)}{r} \left[ Y_\ell(\hat{r}) \otimes \chi_S \right]_{JJ_z} \xi_{11}$$

Outside the nuclear range the equations are not coupled. For energy $E = \hbar^2 k^2/m$

$$\left( \frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} - \frac{me^2}{\hbar^2 r} + k^2 \right) u_{\ell S}(r) = 0$$

defining the dimensionless variable $z = kr$

$$u''_{\ell S}(z) + \left( 1 - \frac{2\eta}{z} - \frac{\ell(\ell+1)}{z^2} \right) u_{\ell S}(z) = 0$$

with $2\eta = me^2/\hbar^2 k$ or $2k\eta = me^2/\hbar^2 = 1/R_p$ with $R_p$ the protonic Bohr radius

# *pp* scattering

The solutions are combinations of the regular and irregular Coulomb wave functions $F_\ell(\eta, z)$, $G_\ell(\eta, z)$ with the following asymptotic behavior

$$F_\ell(\eta, z) \stackrel{z\to\infty}{\longrightarrow} \sin(z - \eta \ln 2z - \ell\pi/2 + \sigma_\ell)$$

$$G_\ell(\eta, z) \stackrel{z\to\infty}{\longrightarrow} \cos(z - \eta \ln 2z - \ell\pi/2 + \sigma_\ell)$$

with the Coulomb phase-shift $\sigma_\ell = arg\Gamma(\ell + 1 + i\eta)$

The asymptotic form of the wave function is the combination

$$u_{\ell S}(r > r_N) \propto F_\ell(\eta, z) + \tan\delta_{\ell,N}\, G_\ell(\eta, z)$$

with $\delta_{\ell,N}$ the nuclear phase-shift with respect to the Coulomb field

# *pp* effective range

The effective range formula can be obtained similar to the *np* case. For s-wave scattering the equation at two different energies

$$\frac{\partial^2 u_1(r)}{\partial r^2} - \left(\frac{me^2}{\hbar^2 r} + \frac{mV(r)}{\hbar^2} - k_1^2\right) u_1(r) = 0$$

$$\frac{\partial^2 u_2(r)}{\partial r^2} - \left(\frac{me^2}{\hbar^2 r} + \frac{mV(r)}{\hbar^2} - k_2^2\right) u_2(r) = 0$$

substracting and integrating the equations the equations we have

$$(u_2 u_1' - u_1 u_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} u_1 u_2 dr$$

or using the asymptotic form $\phi_i = C_{0i}[G_0(k_i r) + \cot\delta F_0(k_i r)]$

$$(\phi_2 \phi_1' - \phi_1 \phi_2')|_r^{r'} = (k_2^2 - k_1^2) \int_r^{r'} \phi_1 \phi_2 dr$$

# *pp* effective range

making $r' > r_N$ and substracting the two equations, the asymptotic part is canceled. For $r \to 0$ the wave functions $u_1, u_2 \to 0$. The behavior of the functions $\phi_i$ as $r \to 0$ can be obtained from

$$F_0(z) \stackrel{z\to 0}{\longrightarrow} C_0 z(1+\eta z+\ldots)$$

$$G_0(z) \stackrel{z\to 0}{\longrightarrow} \frac{1}{C_0}\{1+2\eta z\,[\ln(2\eta z)+2\gamma-1+h(\eta)]+\ldots\}$$

with $\gamma = 0.577215\ldots$ the Euler constant, $C_0^2 = 2\pi\eta/(e^{2\pi\eta}-1)$ and

$$h(\eta) = \eta^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2+\eta^2)} - \ln\eta - \gamma$$

The only remaining term is

$$\phi_2\phi_1' - \phi_1\phi_2' \stackrel{r\to 0}{\longrightarrow} \frac{1}{R_p}[h(\eta_2)-h(\eta_1)] + C_0^2 k_2 \cot\delta_2 - C_0^2 k_1 \cot\delta_1$$

# *pp* effective range

In fact, using the asymptotic form $\phi_i = C_{0i}[G_0(k_i r) + \cot\delta F_0(k_i r)]$

$$\phi_i(z \to 0) \to 1$$

the derivative term is

$$k_i \frac{d\phi_i}{dz} = \frac{1}{R_p}\left(\ln\frac{r}{R_p} + 2\gamma\right) + \frac{1}{R_p} h(\eta_i) + C_{0i}^2 k_i \cot\delta_i$$

Therefore, as $r \to 0$

$$\phi_2\phi_1' - \phi_1\phi_2' \xrightarrow{r\to 0} \frac{1}{R_p}[h(\eta_2) - h(\eta_1)] + C_0^2 k_2 \cot\delta_2 - C_0^2 k_1 \cot\delta_1$$

# *pp* effective range

For $k_1 \to 0$ we define

$$\lim_{k\to 0} C_0^2 k \cot\delta(k) + \frac{1}{R_p} h(\eta) = -\frac{1}{a_{pp}}$$

and the effective range expansion results

$$C_0^2 k \cot\delta(k) + \frac{1}{R_p} h(\eta) = -\frac{1}{a_{pp}} + k^2 \int_0^\infty (\phi\phi_0 - uu_0) dr$$

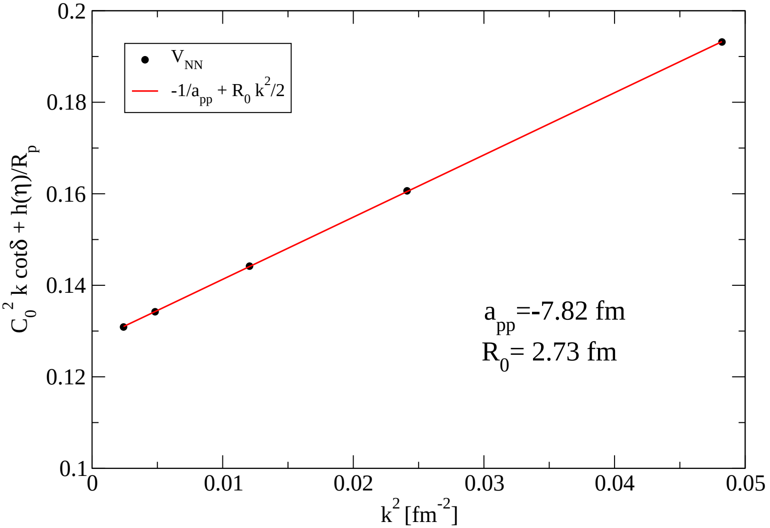
defining again

$$R_0 = 2\int_0^\infty (\phi_0^2 - u_0^2) dr$$

the effective range expansion is

$$C_0^2 k \cot\delta(k) + \frac{1}{R_p} h(\eta) \approx -\frac{1}{a_{pp}} + \frac{1}{2} k^2 R_0$$

# *pp* effective range



# Relation between *pp* and *nn* effective ranges

Taking the proper limits, the *nn* scattering length in the same nuclear potential is related to the *pp* scattering length by the following approximate relation

$$\frac{1}{a_{pp}} = \frac{1}{a_{nn}} + \frac{1}{R_p}\left(\ln\frac{R_0}{R_p} + 0.330\right)$$

where the number 0.330 is obtained from a combination of the Euler constant, $\ln\pi$ and the cosine integral $C_i(\pi)$.

Remembering the experimental values: $a_{pp} = -7.8063(26)$ fm and $R_0 = 2.794(14)$ fm. We can calculated the corrected value: $\tilde{a}_{pp} = -17.137$ fm. To be compare to the $a_{nn} = -18.90(40)$ fm.

There is a small difference between the *pp* and the *nn* forces.