Applications to few- and many-body systems

Zero- and finite-range description of two-particles

- In the zero-range theory

$$k \cot\delta = -1/a \rightarrow S(k) = -\frac{k + i/a}{k - i/a}$$

- In the finite-range theory

$$k \cot\delta = -1/a + r_{\text{eff}} k^2/2 \rightarrow S(k) = \frac{k + i/a_B}{k - i/a_B}\frac{k + i/r_B}{k - i/r_B}$$

with $r_B = a - a_B$

- In the finite-range theory a two-parameter potential is used to reproduce the two parameters of the theory $\rightarrow$ correct pole structure of the S-matrix

Zero- and finite-range description of two-particles

- In the zero-range theory the extension to more particles goes through the solution of the zero-range FY equations (for three particles the STM equation)
  - Efimov spectrum
  - Scale invariance
  - Two-level structure for $N > 3$
- In the finite-range theory the two-parameter potential used to reproduce the low energy S-matrix behavior is taken to solve the Schroedinger equation for more than two particles
- In the two-body system we saw how to make things independent of the choice of the two-parameter potential $\rightarrow$ charaterization of the universal window
- What happens in the three- and more particle systems?
- Could we introduce a finite-range parameter? how?
- And how this description (universal regime) deviates as N increases (non-universal regime)

Gaussian characterization of the universal window for two particles

We characterize the universal window with a Gaussian potential:

$$V(r) = V_0 e^{-r^2/r_0^2}$$

where r is the interparticle distance, while the strength V_0 and range r_0 are parameters useful to explore the low-energy dynamics associated with the existence of one (bound or virtual) state close to threshold.
For bound states, the wave function is obtained by solving the s-wave Schrödinger equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{m r_0^2 V_0}{\hbar^2} e^{-z^2} - \frac{r_0^2}{a_B^2} \right) \phi_B(z) = 0$$

where $z = r/r_0$ and $\phi_B(z)$ is the reduced wave function and

$$E = -\frac{\hbar^2}{m a_B^2}$$

Gaussian characterization of the universal window

For zero-energy the wave function, ϕ_0, is obtained by solving

$$\left(\frac{\partial^2}{\partial z^2} - \frac{m r_0^2 V_0}{\hbar^2} e^{-z^2} \right) \phi_0(z) = 0$$

with $\phi_0(z \to \infty) \to 1 - z r_0/a$, from which the scattering length a is extracted and the effective range is

$$r_{\rm eff} = 2 r_0 \int_0^\infty \left[(1 - z r_0/a)^2 - \phi_0^2 \right] dz$$

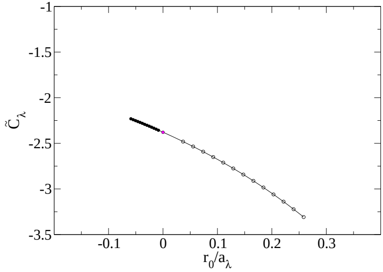
n	$r_{\rm eff}/r_0$	$V_0/(\hbar^2/m r_0^2)$
0	1.43522	2.6840
1	2.41303	17.7957
2	2.89034	45.5735
3	3.20006	85.9632

RG trajectory

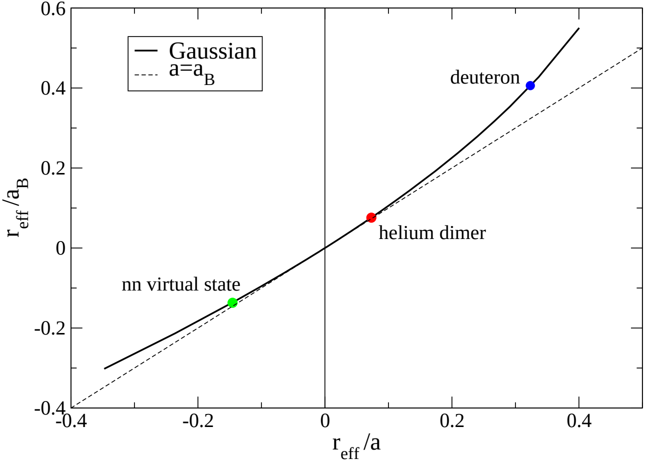
Inside the universal limit, the strength parameter $\widetilde{C}_\lambda = \frac{\sqrt{\pi}}{2}\frac{mr_0^2}{\hbar^2}V_0$ can be expanded in powers of the small parameters r_0/a as

$$\widetilde{C}_\lambda = \frac{\sqrt{\pi}}{2}\frac{mr_0^2}{\hbar^2}V_0 = C_\infty\left(1+\alpha_1\frac{r_0}{a}+\alpha_2(\frac{r_0}{a})^2+\dots\right) \tag{1}$$

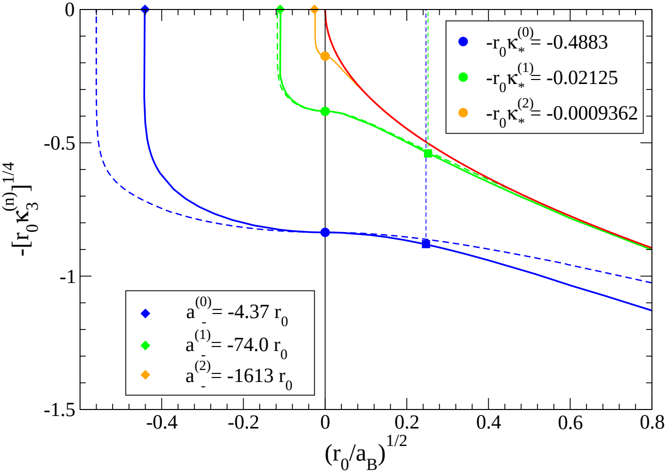
The pure number $C_\infty = 2.379$ is the same for all gaussians. The above equation maps the renormalization group (RG) trajectories as the interaction approaches the scaling limit.



Gaussian characterization



Gaussian characterization of the universal window for three bosons



Notable points

The values at the unitary limit:

$-r_0\kappa_*^{(0)} = -0.4883$	$\kappa_*^{(n)}/\kappa_*^{(n+1)}$	e^{π/s_0}
$-r_0\kappa_*^{(1)} = -0.02125,$	22.979	
$-r_0\kappa_*^{(2)} = -0.0009362,$	22.696	$\to 22.694$

Values at $\kappa_3^{(n)} = 0$

$-a_-^{(0)} = -4.37r_0$	$a_-^{(n+1)}/a_-^{(n)}$	e^{π/s_0}
$-a_-^{(1)} = -74.0r_0$	19.93	
$-a_-^{(2)} = -1613r_0$	21.80	$\to 22.7$

and the almost model independent quantities:

$\kappa_*^{(0)}a_-^{(0)} = -2.14$ for van der Waals systems ≈ -2.2

$\kappa_*^{(1)}a_-^{(1)} = -1.57$

$\kappa_*^{(2)}a_-^{(2)} = -1.51$ the zero-range theory ≈ -1.507

van der Waals universality

Using the LM2M2 helium trimers: $E_3^{(0)} = 126.4\,\mathrm{mK}$ and $E_3^{(1)} = 2.27\,\mathrm{mK}$ and the dimer: $E_2 = 1.303\,\mathrm{mK}$, the position of these data on the plot can be located through the angle θ defined as $E_3^{(n)}/E_2 = \tan^2\theta$. The axis value is $r_0/a_B = 0.061$ corresponding to a Gaussian range $r_0^{(0)} = 11.15\,a_0$ with which a Gaussian potential reproduces the dimer and ground state trimer energies. From that value, the three-body parameters of the helium trimer, ground and excited states, can be estimated

$$E_*^{(0)} = \frac{\hbar^2}{m}\left[\frac{\gamma_0}{r_0^{(0)}}\right]^2 = 83.1\,\mathrm{mK} \rightarrow 84\text{ mK for the LM2M2 potential}$$

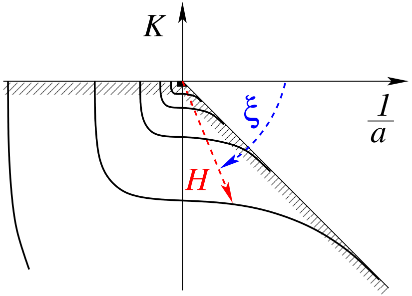
$$E_*^{(1)} = \frac{\hbar^2}{m}\left[\frac{\gamma_1}{r_0^{(0)}}\right]^2 = 0.157\,\mathrm{mK} \rightarrow 0.157\text{ mK for the LM2M2 potential}$$

At the three-atom continuum the characteristic range predicts the value $a_-^{(0)} = -48.7\,a_0$. The Gaussian trajectory predicts $a_-^{(0)}/\tilde{r}_{vdW} \approx -9.6$, in close agreement with the universal value observed in van der Waals species.

Efimov radial law

$$\frac{E_3^{(n)}}{E_2} = \tan^2 \xi$$

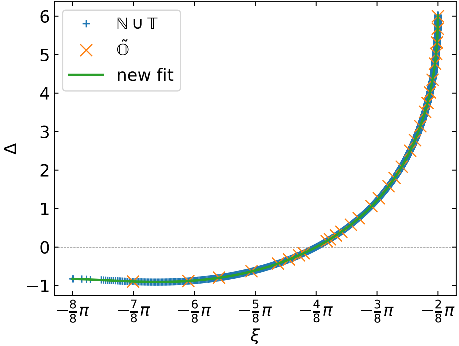
$$-\frac{\hbar^2}{m} H^2 = E_3^{(n)} + E_2 = e^{-2(n-n_*)\pi/s_0}\, e^{\Delta(\xi)/s_0} E_*$$



Efimov radial law

where $\Delta(\xi)$ is the zero-range universal function.

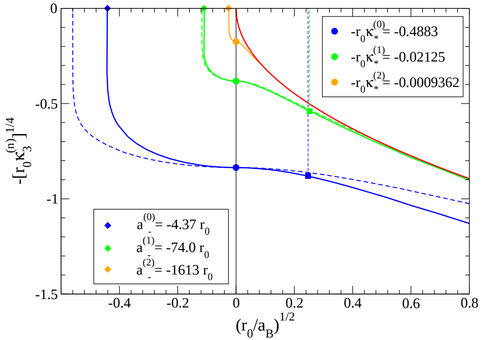
• The same for all levels



Finite-range Efimov radial law

$$\frac{E_3^{(n)}}{E_2} = \tan^2 \xi$$

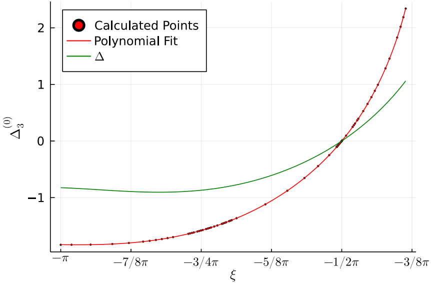
$$-\frac{\hbar^2}{m} H^2 = E_3^{(n)} + E_2 = e^{\Delta_3^{(n)}(\xi)/s_0} E_*^{(n)}$$

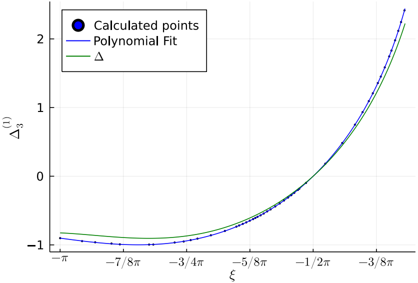


Finite-range Efimov radial law

where $\Delta_3^n(\xi)$ is the finite-range level function.

- $\Delta_3^n(\xi) = s_0 \log \frac{E_3^{(n)} + E_2}{E_*^{(n)}} \to \Delta(\xi)$ for $n \geq 2$





Finite-range Efimov radial law

The results for a Gaussian potential of range r_0 with variable strength can be summarized in the following equations

$$a_B\kappa_3^{(n)} = \tan\xi$$

$$r_0\kappa_3^{(n)} = \gamma_3^{(n)} e^{\Delta_3^{(n)}(\xi)/2s_0} \sin\xi$$

with $\gamma_3^{(n)} = r_0\kappa_*^{(n)}$ and $E_3^{(n)} = \hbar^2[\kappa_3^{(n)}]^2/m$.

• The pure numbers $r_0\kappa_*^{(n)} = \gamma_3^{(n)}$, and $\Delta_3^{(n)}$ are the same for all Gaussian potentials.

The finite-range equation can be related to the zero-range equation as

$$r_0\kappa_3^{(n)} = \gamma_3^{(n)} e^{\Delta(\xi)/2s_0} \sin\xi \left(1 + \frac{r_0\Gamma_3^{(n)}}{\gamma_3^{(n)} a_b}\right)$$

with $\Gamma_3^{(n)}$ the finite-range parameter of level n, $\Gamma_3^{(n)} \to 0$ as $n > 2$

Gaussian characterization of the window for $N, A > 3$

- The Gaussian characterization of the universal window can be extended to describe systems composed by more than three particles.
- The DSI, which emerges in the three-body sector and gives rise to the Efimov spectrum, strongly constrains the $N > 3$ (bosons) or $A > 3$ (nucleons) energy spectrum.
- For equal bosons, where the spatial wave function is symmetric, DSI can be observed well beyond three particles.
- In the case of A nucleons, the spatial-symmetric wave function is dominant only up to four particles
- Deviations from the bosonic-Efimov scenario appear for the $A > 4$ levels
- It is interesting to explore how the energy levels emerge from the unitary limit.

Gaussian characterization for $N > 3$

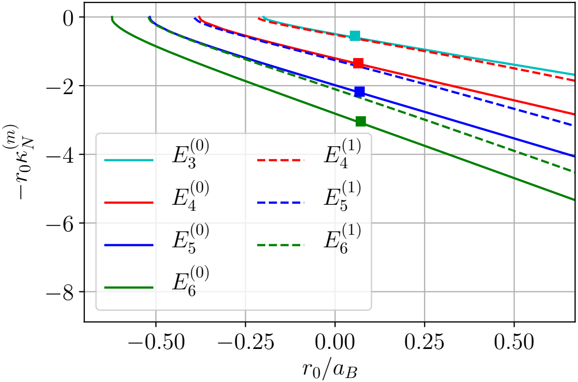
- Defining the angle: $a_B \kappa_N^{(m)} = \tan\xi$
  the spectrum is determined by the equation

$$r_0 \kappa_N^{(m)} = \gamma_N^{(m)} e^{\Delta_N^{(m)}(\xi)/2s_0} \sin\xi$$

  with $m = 0$ being the N-body ground state and $m = 1$ the excited state
- The pure numbers $\gamma_N^{(m)} = r_0 \kappa_{*,N}^{(m)}$, determining the energies at the unitary limit, $E_{*,N}^{(m)}$, are characteristic of every Gaussian potential.
- The energy of the level m is $E_N^{(m)} = \hbar^2 [\kappa_N^{(m)}]^2 / m$ and $\Delta_m^N(\theta)$ is the Gaussian level function for N bosons in the states $m = 0, 1$:

$$\Delta_N^{(m)}(\theta) = s_0 \log \frac{E_N^{(m)} + E_2}{E_{*,N}^{(m)}}$$

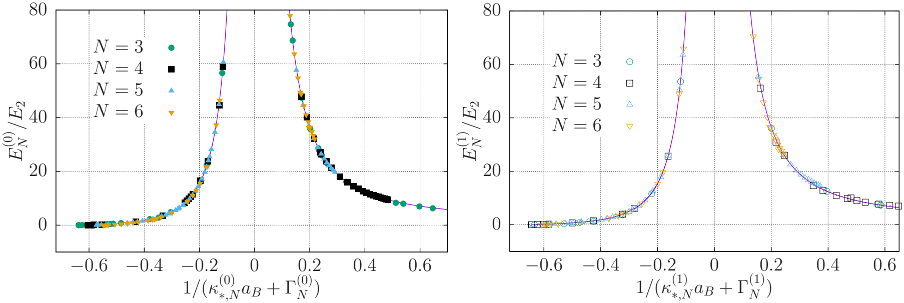
Gaussian characterization for $N \leq 6$



Gaussian characterization for $N \leq 6$

The N bosons spectrum can be put in the following way:

$$\kappa_N^{(m)} a_B = \tan\xi\,, \quad \kappa_{*,N}^{(m)} a_B + \Gamma_N^{(m)} = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

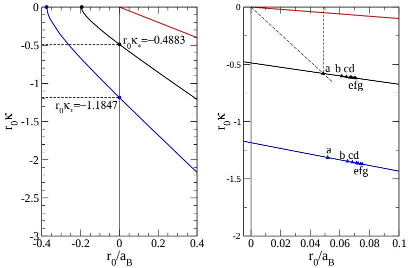


Gaussian characterization of the universal window

- The gaussian charaterization for three bosons does not show the Thomas collapse
- It introduces a finite-range parameter $\Gamma_3^{(n)}$ and for $N > 3$ $\Gamma_N^{(m)}$, with $m = 0, 1$
- For $N = 3$ the gaussian charaterization captures important properties observed in experiments as the van der Waals universality
- It is able to describe (theoreticaly) movements of the helium trimer along the window
- Can this description be extended for general $N > 3$
- We can expect a break down of the universal plus finite-raneg description as $N >> 3$
- What emerges is a correlation between few- and many-body dynamics

Universal window for N bosons

Potential	E_2(mK)	E_3(mK)	E_4(mK)	$r_0^{(3)}(a_0)$	$r_0^{(4)}(a_0)$
HFD-HE2	0.8301	117.2	535.6	11.146	11.840
LM2M2	1.3094	126.5	559.2	11.150	11.853
HFD-B3-FCH	1.4475	129.0	566.1	11.148	11.853
CCSAPT	1.5643	131.0	571.7	11.149	11.851
PCKLJS	1.6154	131.8	573.9	11.148	11.852
HFD-B	1.6921	133.1	577.3	11.149	11.854
SAPT96	1.7443	134.0	580.0	11.147	11.850



Universal window for N bosons

• The different trimer predictions are on top of the curve predicted by the following potential:

• for $N = 3$

$$V^{(3)}(r_{ij}) = V_0^{(3)} e^{-r_{ij}^2/(r_0^{(3)})^2}$$

this potential, with variable strength describes the motion along the universal particles for the helium dimer and trimer

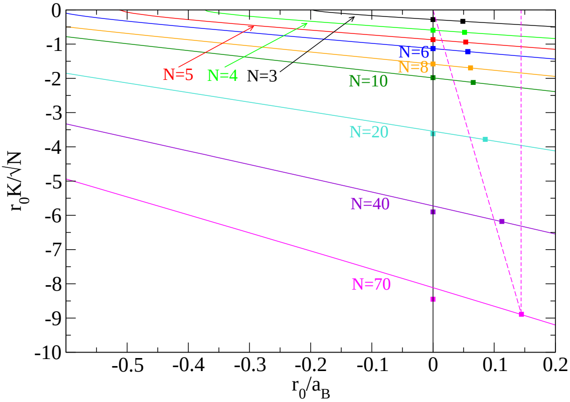
• for $N = 4$

$$V^{(4)}(r_{ij}) = V_0^{(4)} e^{-r_{ij}^2/(r_0^{(4)})^2}$$

this potential, with variable strength describes the motion along the universal particles for the helium dimer and tetramer

• Up to which number of particles this behavior will hold?

Universal window for N bosons



This behavior degraded between 10 to 20 bosons →
a non universal behavior appears!

From Universal to non-universal behavior

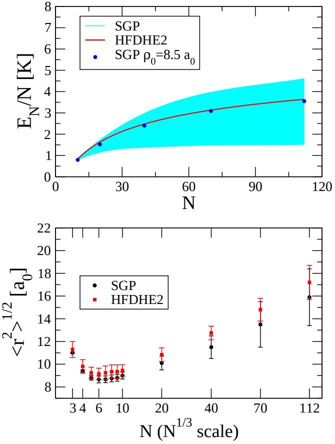
- The gaussian characterization has been done using an attractive gaussian interaction
- As $N \to \infty$ the ground state energy per particle $E_N/N \to \infty$
- To stabilize the system a repulsion is needed
- Based on the results of EFT at LO the following potential can be studied

$$V = V_0 \sum_{i<j} e^{-r_{ij}^2/r_0^2} + W_0 \sum_{i<j<k} e^{-2\rho_{ijk}^2/\rho_0^2}$$

with $\rho_{ijk}^2 = (2/3)(r_{ij}^2 + r_{jk}^2 + r_{ki}^2)$

- r_0 and V_0 are fixed to reproduce the two-body physics: dimer energy and scattering length
- W_0 could be fixed to reproduce the trimer energy
- The range of the three-body fore ρ_0 has to be analyzed

From Universal to non-universal behavior



From Universal to non-universal behavior

	physical point		unitary point	
	SGP	HFD-HE2	SGP	HFD-HE2
$r_0[a_0]$	10.0485		10.0485	
$V_0[K]$	1.208018		1.150485	
$\rho_0[a_0]$	8.4853		8.4853	
$W_0[K]$	3.011702		3.014051	
$E_4[K]$	0.536	0.536	0.440	0.440
$E_5[K]$	1.251	1.266	1.076	1.076
$E_6[K]$	2.216	2.232	1.946	1.963
$E_{10}/10[K]$	0.792(2)	0.831(2)	0.714(2)	0.746(2)
$E_{20}/20[K]$	1.525(2)	1.627(2)	1.389(2)	1.491(2)
$E_{40}/40[K]$	2.374(2)	2.482(2)	2.170(2)	2.308(2)
$E_{70}/70[K]$	3.07(1)	3.14(1)	2.80(1)	2.92(1)
$E_{112}/112[K]$	3.58(2)	3.63(2)	3.30(2)	3.40(2)
$E_N/N(\infty)[K]$	7.2(3)*	7.14(2)	6.8(3)*	6.72(2)
HFD-B [K]		7.33(2)		6.73(2)

• the ρ_0 parameter works as a non-universal parameter to fix the correct amount of repulsion needed to describe the curve E_N/N
• It takes into account non-universal physics

The three-nucleon system

- The two-nucleon states $J^\pi = 0^+$ and 1^+ belongs to the universal window.
- The 0^+ state is an *s*-wave state
- The 1^+ has a dominant *s*-wave component, $\approx 95\%$
- The lightest nuclei, ^{2}H, ^{3}H, ^{3}He and ^{4}He have large probabilities to be in $L = 0$, we expect to observe universal properties.
- Important questions to be clarified are the lack of excited states in the three- and four-nucleon systems.
- The doublet neutron-deuteron scattering length, $^2a_{nd} \approx 0.65\,$fm is very small value compared to the triplet neutron-proton scattering length $a_{np} \approx 5.2$ fm.
- Data for low energy neutron-deuteron scattering reveal the presence of a triton virtual state.
- All these properties can be traced back to the position of the nuclear system inside the universal window.

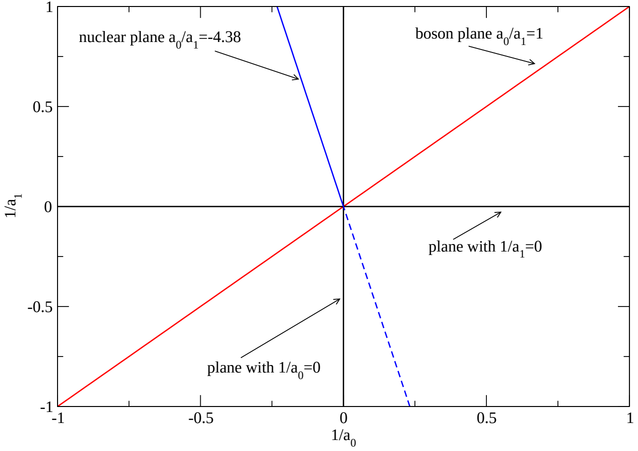
The three-nucleon system

• The study of the universal window in the case of three nucleons has to consider the two different values of the singlet and triplet scattering lengths, a_s and a_t.
• the nuclear plane is defined when the ratio a_s/a_t is close to the experimental value, $a_s/a_t = -4.38$
• To characterize the universal window we construct a spin-dependent Gaussian potential with different strengths and ranges in the spin-isospin channels $S, T = 0, 1$ and $1, 0$

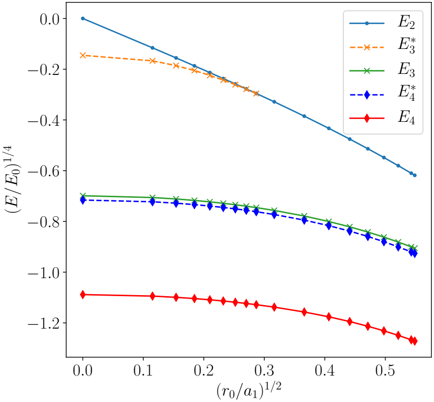
$$V(r) = V_0 e^{-r^2/r_0^2} \mathcal{P}_0 + V_1 e^{-r^2/r_1^2} \mathcal{P}_1$$

• $\mathcal{P}_0$ and $\mathcal{P}_1$ are projectors on the $S, T = 0, 1$ and $S, T = 1, 0$ channels
• In the following we study the spectrum of the three-nucleon $J^\pi = 1/2^+$ state considering $r_0 = r_1$, for which choice, at the unitary limit, the spectrum coincides with the boson case.
• V_0 and V_1 are varied maintaining $a_s/a_t = -4.38$

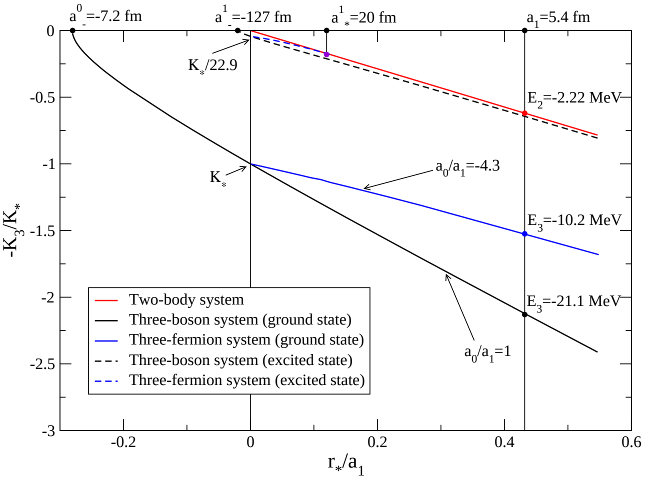
The nuclear plane



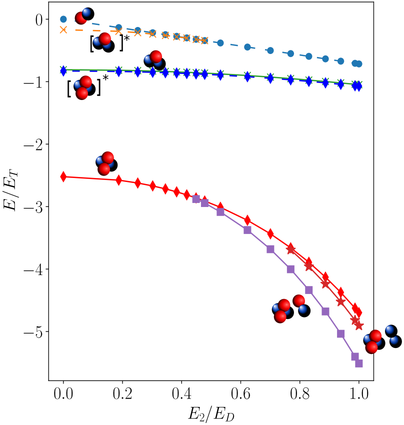
The universal window for $A = 2, 3, 4$



The universal window for $A = 2, 3$



Gaussian characterization for $A \leq 6$



Gaussian characterization for $A \leq 6$

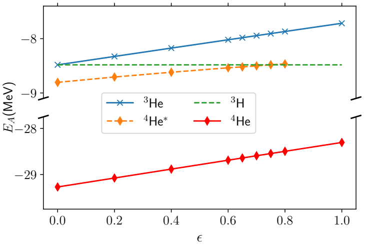
a_1(fm)	E_2(MeV)	E_3(MeV)	E_3^*(MeV)	E_4(MeV)	E_4^*(MeV)	^{6}He(MeV)	^{6}Li(MeV)
5.4802	-2.2255	-10.2455	-	-39.843	-11.19	-41.60	-46.74
5.5980	-2.1098	-10.0056	-	-39.221	-10.93	-40.87	-45.82
6.0683	-1.7270	-9.1903	-	-37.093	-10.01	-38.36	-42.71
6.6607	-1.3762	-8.4054	-	-35.017	-9.14	-35.95	-39.67
7.4310	-1.0593	-7.6526	-	-32.997	-8.31	-33.58	-36.77
8.4756	-0.77842	-6.9333	-	-31.035	-7.52	-31.31	-33.95
9.9750	-0.53599	-6.2493	-	-29.135	-6.78	-	-31.23
12.3136	-0.33466	-5.6023	-	-27.300	-6.08	-	-28.62
16.4715	-0.17736	-4.9945	-	-25.536	-5.43	-	-26.17
20.0638	-0.11633	-4.7058	-0.1168	-24.682	-5.13	-	-24.96
22.6041	-0.09038	-4.5654	-0.0920	-24.262	-4.98	-	-24.41
25.9589	-0.06756	-4.4278	-0.0705	-23.847	-4.83	-	-
30.5953	-0.04794	-4.2927	-0.0530	-23.437	-4.69	-	-
37.4216	-0.03158	-4.1605	-0.0385	-23.032	-4.55	-	-
48.4699	-0.01855	-4.0311	-0.0270	-22.633	-4.42	-	-
69.4131	-0.00891	-3.9044	-0.0182	-22.238	-4.28	-	-
124.3314	-0.00273	-3.7807	-0.0119	-21.850	-4.15	-	-
∞	0	-3.6322	-0.0068	-21.378	-4.00	-	-

Gaussian characterization for $A \leq 6$

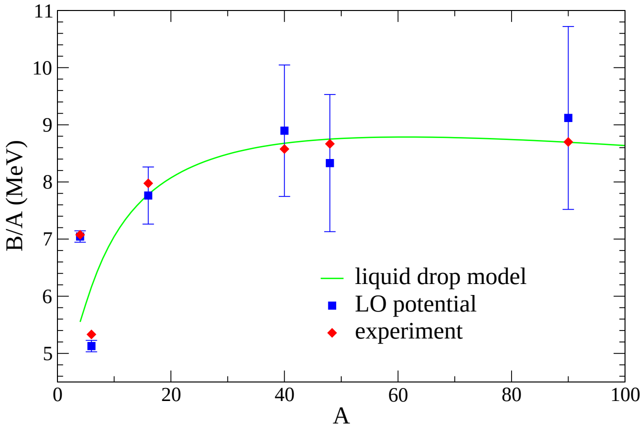
- The energies are described only quantitatively
- Based on EFT at LO (and the results on bosons), we introduce a three-body force

$$W(\rho) = W_0\, e^{-(r_{12}^2 + r_{13}^2 + r_{23}^2)/\rho_3^2}$$

- with the strength and range fixed to describe E_3 and E_4



Gaussian characterization for $A >> 3$



From few to many bodies

- Inside the universal window we observe universal behavior encoded in two quantities: the dimer energy and the scattering length
- To describe these two quantities we contruct a two-parameter potential used to characterize the universal window
- For more than two bodies this potential is able to capture many dynamical properties
- Increasing the number of particles non-universal behavior could appear
- It can be taken into account by fixing with detail the range of the three-body repulsion
- This parameter can be fixed in the few-body sector to describe for example E_3 and E_4
- The complete potential is parametrized by four low energy observables
- This implies a strong correlation between the few-body and many-body dynamics

The $2+1$ scattering length

- In the two-body system the scattering length and the dimer energy are stricted correlated inside the universal window: $k_d = 1/a + r_{eff} k_d^2/2$
- The atom-dimer scattering length is stricted correlated to the discrete spectrum
- The functional form in zero-range theory was derived by Efimov

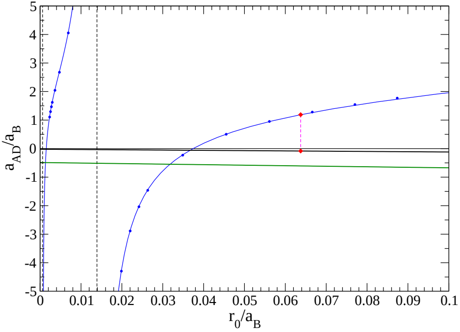
$$a_{AD}/a_B = d_1 + d_2 \tan[s_0 \ln(\kappa_* a_B) + d_3]$$

- d_1, d_2 and d_3 are universal numbers
- κ_* is the three-body parameter belonging to one of the three-body energy branches
- In the case of finite-range interactions

$$a_{AD}/a_B = d_1 + d_2 \tan[s_0 \ln(\kappa_*^{(n)} r_0 (a_B/r_0) + \Gamma_3^{(n)}) + d_3]$$

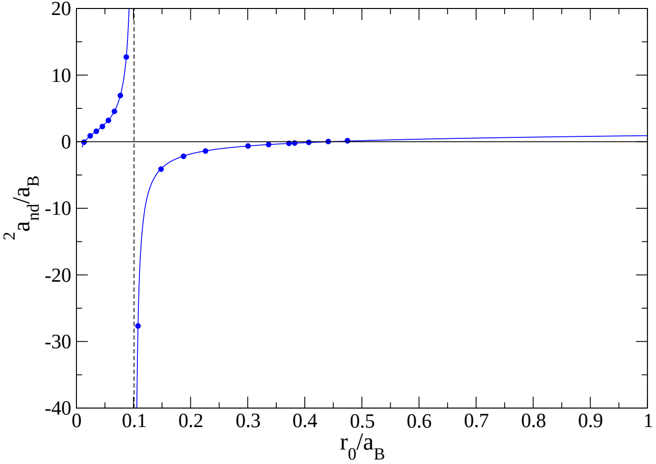
- $\kappa_*^{(n)} r_0 = \gamma_3^{(n)}$, is used as the driving term
- $\Gamma_3^{(n)}$ is a finite-range three-body parameter

The $2 + 1$ scattering length

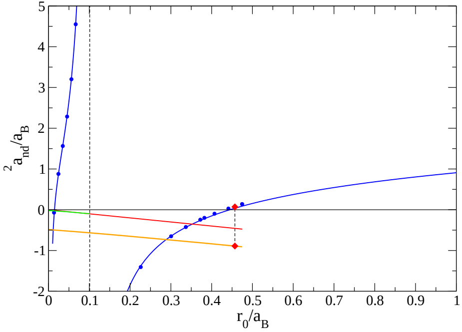


Putting numbers: At $r_0/a_B = 0.0637 \rightarrow a_{AD}/a_B = 1.19$
Using the LM2M2 value, $a_B = 182.22\, a_0 \rightarrow a_{AD} = 217\, a_0$
The LM2M2 value for this quantity of $218.4\, a_0$!

The doublet *nd* scattering length



The doublet *nd* scattering length



Putting numbers: At $r_0/a_B = 0.457 \rightarrow a_{nd}/a_B = 0.08$
Using the deuteron value, $a_B = 4.3\,\text{fm} \rightarrow a_{nd} = 0.4\,\text{fm}$
The experimental value for this quantity of 0.65 fm!