Applications to few- and many-body systems
Zero- and finite-range description of two-particles

- In the zero-range theory
  \[ k \cot \delta = -1/a \rightarrow S(k) = -\frac{k + i/a}{k - i/a} \]

- In the finite-range theory
  \[ k \cot \delta = -1/a + r_{\text{eff}} k^2/2 \rightarrow S(k) = \frac{k + i/a_B k + i/r_B}{k - i/a_B k - i/r_B} \]

  with \( r_B = a - a_B \)

- In the finite-range theory a two-parameter potential is used to reproduce the two parameters of the theory → correct pole structure of the S-matrix
Zero- and finite-range description of two-particles

- In the zero-range theory the extension to more particles goes through the solution of the zero-range FY equations (for three particles the STM equation)
  - Efimov spectrum
  - Scale invariance
  - Two-level structure for $N > 3$

- In the finite-range theory the two-parameter potential used to reproduce the low energy $S$-matrix behavior is taken to solve the Schroedinger equation for more than two particles

- In the two-body system we saw how to make things independent of the choice of the two-parameter potential → characterization of the universal window

- What happens in the three- and more particle systems?
- Could we introduce a finite-range parameter? how?
- And how this description (universal regime) deviates as $N$ increases (non-universal regime)
Gaussian characterization of the universal window for two particles

We characterize the universal window with a Gaussian potential:

\[ V(r) = V_0 e^{-r^2/r_0^2} \]

where \( r \) is the interparticle distance, while the strength \( V_0 \) and range \( r_0 \) are parameters useful to explore the low-energy dynamics associated with the existence of one (bound or virtual) state close to threshold.

For bound states, the wave function is obtained by solving the \( s \)-wave Schrödinger equation

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{mr_0^2 V_0}{\hbar^2} e^{-z^2} - \frac{r_0^2}{a_B^2} \right) \phi_B(z) = 0
\]

where \( z = r/r_0 \) and \( \phi_B(z) \) is the reduced wave function and

\[ E = -\frac{\hbar^2}{ma_B^2} \]
Gaussian characterization of the universal window

For zero-energy the wave function, \( \phi_0 \), is obtained by solving

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{mr_0^2 V_0}{\hbar^2} e^{-z^2} \right) \phi_0(z) = 0
\]

with \( \phi_0(z \to \infty) \to 1 - zr_0/a \), from which the scattering length \( a \) is extracted and the effective range is

\[
r_{\text{eff}} = 2r_0 \int_0^{\infty} \left[ (1 - zr_0/a)^2 - \phi_0^2 \right] dz
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( r_{\text{eff}}/r_0 )</th>
<th>( V_0/(\hbar^2/mr_0^2) )</th>
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RG trajectory

Inside the universal limit, the strength parameter $\tilde{C}_\lambda = \frac{\sqrt{\pi}}{2} \frac{m r_0^2}{\hbar^2} V_0$ can be expanded in powers of the small parameters $r_0/a$ as

$$\tilde{C}_\lambda = \frac{\sqrt{\pi}}{2} \frac{m r_0^2}{\hbar^2} V_0 = C_\infty \left( 1 + \alpha_1 \frac{r_0}{a} + \alpha_2 \left( \frac{r_0}{a} \right)^2 + \ldots \right)$$  \hspace{1cm} (1)

The pure number $C_\infty = 2.379$ is the same for all gaussians. The above equation maps the renormalization group (RG) trajectories as the interaction approaches the scaling limit.
Gaussian characterization

-0.4 -0.2 0 0.2 0.4
\[ r_{\text{eff}} / a \]

-0.4
-0.2
0
0.2
0.4
0.6
\[ r_{\text{eff}} / a_B \]

Gaussian

\[ \text{deuteron} \]

\[ \text{helium dimer} \]

\[ \text{nn virtual state} \]

Applications

4-8 October 2021

(4rd lesson)
Gaussian characterization of the universal window for three bosons

- $r_0/\alpha_B^{1/2}$
- $-r_0 \kappa_3^{(n)}$  
  - $-r_0 \kappa_3^{(0)} = -0.4883$
  - $-r_0 \kappa_3^{(1)} = -0.02125$
  - $-r_0 \kappa_3^{(2)} = -0.0009362$
- $a_0^{(0)} = -4.37 r_0$
- $a_0^{(1)} = -74.0 r_0$
- $a_0^{(2)} = -1613 r_0$
Notable points

The values at the unitary limit:

\[-r_0 \kappa_*^{(0)} = -0.4883\]
\[-r_0 \kappa_*^{(1)} = -0.02125,\]
\[-r_0 \kappa_*^{(2)} = -0.0009362,\]

Values at \(\kappa_3^{(n)} = 0\)

\[-a_0^{(0)} = -4.37r_0\]
\[-a_1^{(1)} = -74.0r_0\]
\[-a_2^{(2)} = -1613r_0\]

and the almost model independent quantities:

\[\kappa_*^{(0)} a_0^{(0)} = -2.14\] for van der Waals systems \(\approx -2.2\)
\[\kappa_*^{(1)} a_1^{(1)} = -1.57\]
\[\kappa_*^{(2)} a_2^{(2)} = -1.51\] the zero-range theory \(\approx -1.507\)
van der Waals universality

Using the LM2M2 helium trimers: $E_3^{(0)} = 126.4 \text{ mK}$ and $E_3^{(1)} = 2.27 \text{ mK}$ and the dimer: $E_2 = 1.303 \text{ mK}$, the position of these data on the plot can be located through the angle $\theta$ defined as $E_3^{(n)}/E_2 = \tan^2 \theta$. The axis value is $r_0/a_B = 0.061$ corresponding to a Gaussian range $r_0^{(0)} = 11.15 a_0$ with which a Gaussian potential reproduces the dimer and ground state trimer energies. From that value, the three-body parameters of the helium trimer, ground and excited states, can be estimated

$$E_*^{(0)} = \frac{\hbar^2}{m} \left[ \frac{\gamma_0}{r_0^{(0)}} \right]^2 = 83.1 \text{ mK} \rightarrow 84 \text{ mK} \text{ for the LM2M2 potential}$$

$$E_*^{(1)} = \frac{\hbar^2}{m} \left[ \frac{\gamma_1}{r_0^{(0)}} \right]^2 = 0.157 \text{ mK} \rightarrow 0.157 \text{ mK} \text{ for the LM2M2 potential}$$

At the three-atom continuum the characteristic range predicts the value $a^{(0)}_\rightarrow = -48.7 a_0$. The Gaussian trajectory predicts

$$a^{(0)}_\rightarrow /\tilde{r}_{vdW} \approx -9.6$$

in close agreement with the universal value observed in van der Waals species.
Efimov radial law

\[ \frac{E_3^{(n)}}{E_2} = \tan^2 \xi \]

\[ -\frac{\hbar^2}{m} H^2 = E_3^{(n)} + E_2 = e^{-2(n-n_*)\pi/s_0} e^{\Delta(\xi)/s_0} E_* \]
Efimov radial law

where $\Delta(\xi)$ is the zero-range universal function.

- The same for all levels

![Graph showing the Efimov radial law](image_url)
Finite-range Efimov radial law

\[ \frac{E_3^{(n)}}{E_2} = \tan^2 \xi \]

\[ -\frac{\hbar^2}{m} H^2 = E_3^{(n)} + E_2 = e^{\Delta_3^{(n)}(\xi)/s_0} E_*^{(n)} \]

\[ (r_0/a_B)^{1/2} \]

\[ -[r_0 \kappa_3^{(n)}]^{1/4} \]

\[ \begin{align*}
  -r_0 \kappa_3^{(0)} &= -0.4883 \\
  -r_0 \kappa_3^{(1)} &= -0.02125 \\
  -r_0 \kappa_3^{(2)} &= -0.0009362
\end{align*} \]

\[ \begin{align*}
  a_0 &= -4.37 r_0 \\
  a_1 &= -74.0 r_0 \\
  a_2 &= -1613 r_0
\end{align*} \]
Finite-range Efimov radial law

where $\Delta_n^3(\xi)$ is the finite-range level function.

- $\Delta_n^3(\xi) = s_0 \log \frac{E_3^{(n)} + E_2}{E^{(n)}_*} \to \Delta(\xi)$ for $n \geq 2$
Finite-range Efimov radial law

The results for a Gaussian potential of range $r_0$ with variable strength can be summarized in the following equations

\[ a_B \kappa_3^{(n)} = \tan \xi \]

\[ r_0 \kappa_3^{(n)} = \gamma_3^{(n)} e^{\Delta_3^{(n)}(\xi)/2s_0} \sin \xi \]

with $\gamma_3^{(n)} = r_0 \kappa_*^{(n)}$ and $E_3^{(n)} = \hbar^2 [\kappa_3^{(n)}]^2 / m$.

- The pure numbers $r_0 \kappa_*^{(n)} = \gamma_3^{(n)}$, and $\Delta_3^{(n)}$ are the same for all Gaussian potentials.

The finite-range equation can be related to the zero-range equation as

\[ r_0 \kappa_3^{(n)} = \gamma_3^{(n)} e^{\Delta(\xi)/2s_0} \sin \xi \left( 1 + \frac{r_0 \Gamma_3^{(n)}}{\gamma_3^{(n)} a_B} \right) \]

with $\Gamma_3^{(n)}$ the finite-range parameter of level $n$, $\Gamma_3^{(n)} \rightarrow 0$ as $n > 2$.
Gaussian characterization of the window for $N, A > 3$

- The Gaussian characterization of the universal window can be extended to describe systems composed by more than three particles.
- The DSI, which emerges in the three-body sector and gives rise to the Efimov spectrum, strongly constrains the $N > 3$ (bosons) or $A > 3$ (nucleons) energy spectrum.
- For equal bosons, where the spatial wave function is symmetric, DSI can be observed well beyond three particles.
- In the case of $A$ nucleons, the spatial-symmetric wave function is dominant only up to four particles.
- Deviations from the bosonic-Efimov scenario appear for the $A > 4$ levels.
- It is interesting to explore how the energy levels emerge from the unitary limit.
Gaussian characterization for $N > 3$

- Defining the angle: $a_B \kappa_N^{(m)} = \tan \xi$
  
  the spectrum is determined by the equation

$$r_0 \kappa_N^{(m)} = \gamma_N^{(m)} e^{\Delta_N^{(m)}(\xi)/2s_0 \sin \xi}$$

with $m = 0$ being the $N$-body ground state and $m = 1$ the excited state.

- The pure numbers $\gamma_N^{(m)} = r_0 \kappa_N^{(m)}$, determining the energies at the unitary limit, $E_{*,N}^{(m)}$, are characteristic of every Gaussian potential.

- The energy of the level $m$ is $E_N^{(m)} = \hbar^2 [\kappa_N^{(m)}]^2 / m$ and $\Delta_N^{(m)}(\theta)$ is the Gaussian level function for $N$ bosons in the states $m = 0, 1$: 

$$\Delta_N^{(m)}(\theta) = s_0 \log \frac{E_N^{(m)} + E_2}{E_{*,N}^{(m)}}$$
Gaussian characterization for $N \leq 6$

Applications

4-8 October 2021
Gaussian characterization for $N \leq 6$

The $N$ bosons spectrum can be put in the following way:

$$\kappa_N^{(m)} a_B = \tan \xi, \quad \kappa_{*,N}^{(m)} a_B + \Gamma_N^{(m)} = \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}$$
Gaussian characterization of the universal window

- The gaussian characterization for three bosons does not show the Thomas collapse.
- It introduces a finite-range parameter $\Gamma_3^{(n)}$ and for $N > 3$ $\Gamma_N^{(m)}$, with $m = 0, 1$.
- For $N = 3$ the gaussian characterization captures important properties observed in experiments as the van der Waals universality.
- It is able to describe (theoreticaly) movements of the helium trimer along the window.
- Can this description be extended for general $N > 3$?
- We can expect a break down of the universal plus finite-range description as $N >> 3$.
- What emerges is a correlation between few- and many-body dynamics.
# Universal window for $N$ bosons

<table>
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<tr>
<th>Potential</th>
<th>$E_2$(mK)</th>
<th>$E_3$(mK)</th>
<th>$E_4$(mK)</th>
<th>$r_0^{(3)}(a_0)$</th>
<th>$r_0^{(4)}(a_0)$</th>
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<td>HFD-HE2</td>
<td>0.8301</td>
<td>117.2</td>
<td>535.6</td>
<td>11.146</td>
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<td>1.3094</td>
<td>126.5</td>
<td>559.2</td>
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<td>11.853</td>
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<tr>
<td>HFD-B3-FCH</td>
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<td>129.0</td>
<td>566.1</td>
<td>11.148</td>
<td>11.853</td>
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<tr>
<td>CCSAPT</td>
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<td>11.851</td>
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<tr>
<td>PCKLJS</td>
<td>1.6154</td>
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<td>573.9</td>
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<td>11.852</td>
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<tr>
<td>HFD-B</td>
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<td>577.3</td>
<td>11.149</td>
<td>11.854</td>
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<tr>
<td>SAPT96</td>
<td>1.7443</td>
<td>134.0</td>
<td>580.0</td>
<td>11.147</td>
<td>11.850</td>
</tr>
</tbody>
</table>

- $r_0^{(3)}(a_0) = -0.4883$
- $r_0^{(4)}(a_0) = -1.1847$
Universal window for $N$ bosons

• The different trimer predictions are on top of the curve predicted by the following potential:
  
  • for $N = 3$
    
    $$V^{(3)}(r_{ij}) = V_0^{(3)} e^{-r_{ij}^2/(r_0^{(3)})^2}$$

  this potential, with variable strength describes the motion along the universal particles for the helium dimer and trimer

  • for $N = 4$
    
    $$V^{(4)}(r_{ij}) = V_0^{(4)} e^{-r_{ij}^2/(r_0^{(4)})^2}$$

  this potential, with variable strength describes the motion along the universal particles for the helium dimer and tetramer

• Up to which number of particles this behavior will hold?
Universal window for $N$ bosons

This behavior degraded between 10 to 20 bosons → a non universal behavior appears!
From Universal to non-universal behavior

- The gaussian characterization has been done using an attractive gaussian interaction
- As $N \to \infty$ the ground state energy per particle $E_N/N \to \infty$
- To stabilize the system a repulsion is needed
- Based on the results of EFT at LO the following potential can be studied

$$V = V_0 \sum_{i<j} e^{-r_{ij}^2/r_0^2} + W_0 \sum_{i<j<k} e^{-2\rho_{ijk}^2/\rho_0^2}$$

with $\rho_{ijk}^2 = (2/3)(r_{ij}^2 + r_{jk}^2 + r_{ki}^2)$
- $r_0$ and $V_0$ are fixed to reproduce the two-body physics: dimer energy and scattering length
- $W_0$ could be fixed to reproduce the trimer energy
- The range of the three-body force $\rho_0$ has to be analyzed
From Universal to non-universal behavior

![Graph showing energy per particle vs. particle number for different systems, with annotations for SGP and HFDHE2 models.]
From Universal to non-universal behavior

<table>
<thead>
<tr>
<th></th>
<th>physical point</th>
<th>unitary point</th>
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</thead>
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<tr>
<td></td>
<td>SGP</td>
<td>HFD-HE2</td>
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<tr>
<td>$r_0 [a_0]$</td>
<td>10.0485</td>
<td>10.0485</td>
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<tr>
<td>$V_0 [K]$</td>
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<td>$\rho_0 [a_0]$</td>
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<tr>
<td>$W_0 [K]$</td>
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<td>$E_4 [K]$</td>
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<td>0.536</td>
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<tr>
<td>$E_5 [K]$</td>
<td>1.251</td>
<td>1.266</td>
</tr>
<tr>
<td>$E_6 [K]$</td>
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<td>2.232</td>
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<td>$E_{10}/10 [K]$</td>
<td>0.792(2)</td>
<td>0.831(2)</td>
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<tr>
<td>$E_{20}/20 [K]$</td>
<td>1.525(2)</td>
<td>1.627(2)</td>
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<tr>
<td>$E_{40}/40 [K]$</td>
<td>2.374(2)</td>
<td>2.482(2)</td>
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<td>$E_{70}/70 [K]$</td>
<td>3.07(1)</td>
<td>3.14(1)</td>
</tr>
<tr>
<td>$E_{112}/112 [K]$</td>
<td>3.58(2)</td>
<td>3.63(2)</td>
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<tr>
<td>$E_N/N(\infty) [K]$</td>
<td>7.2(3)*</td>
<td>7.14(2)</td>
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<tr>
<td>HFD-B [K]</td>
<td>7.33(2)</td>
<td></td>
</tr>
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</table>

- the $\rho_0$ parameter works as a non-universal parameter to fix the correct amount of repulsion needed to describe the curve $E_N/N$
- It takes into account non-universal physics
The three-nucleon system

- The two-nucleon states $J^{\pi} = 0^+$ and $1^+$ belongs to the universal window.
- The $0^+$ state is an $s$-wave state
- The $1^+$ has a dominant $s$-wave component, $\approx 95\%$
- The lightest nuclei, $^2$H, $^3$H, $^3$He and $^4$He have large probabilities to be in $L = 0$, we expect to observe universal properties.
- Important questions to be clarified are the lack of excited states in the three- and four-nucleon systems.
- The doublet neutron-deuteron scattering length, $^2a_{nd} \approx 0.65$ fm is very small value compared to the triplet neutron-proton scattering length $a_{np} \approx 5.2$ fm.
- Data for low energy neutron-deuteron scattering reveal the presence of a triton virtual state.
- All these properties can be traced back to the position of the nuclear system inside the universal window.
The three-nucleon system

- The study of the universal window in the case of three nucleons has to consider the two different values of the singlet and triplet scattering lengths, \( a_s \) and \( a_t \).
- The nuclear plane is defined when the ratio \( a_s/a_t \) is close to the experimental value, \( a_s/a_t = -4.38 \).
- To characterize the universal window we construct a spin-dependent Gaussian potential with different strengths and ranges in the spin-isospin channels \( S, T = 0, 1 \) and \( 1, 0 \):

\[
V(r) = V_0 e^{-r^2/r_0^2} P_0 + V_1 e^{-r^2/r_1^2} P_1
\]

- \( P_0 \) and \( P_1 \) are projectors on the \( S, T = 0, 1 \) and \( S, T = 1, 0 \) channels.
- In the following we study the spectrum of the three-nucleon \( J^\pi = 1/2^+ \) state considering \( r_0 = r_1 \), for which choice, at the unitary limit, the spectrum coincides with the boson case.
- \( V_0 \) and \( V_1 \) are varied maintaining \( a_s/a_t = -4.38 \).
The nuclear plane

- **Nuclear plane** $a_0/a_1 = -4.38$
- **Boson plane** $a_0/a_1 = 1$
- **Plane with** $1/a_1 = 0$
- **Plane with** $1/a_0 = 0$
The universal window for $A = 2, 3, 4$
The universal window for $A = 2, 3$

Two-body system
Three-boson system (ground state)
Three-boson system (excited state)
Three-fermion system (ground state)
Three-fermion system (excited state)

$K_*/22.9$
$a_0 = -7.2$ fm
$a_1 = -127$ fm
$a_1^* = 20$ fm
$a_1 = 5.4$ fm

$E_2 = -2.22$ MeV
$E_3 = -10.2$ MeV
$E_3 = -21.1$ MeV
$a_0/a_1 = -4.3$
$a_0/a_1 = 1$
Gaussian characterization for $A \leq 6$
Gaussian characterization for $A \leq 6$

<table>
<thead>
<tr>
<th>$a_1$(fm)</th>
<th>$E_2$(MeV)</th>
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<td>-4.28</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>124.3314</td>
<td>-0.00273</td>
<td>-3.7807</td>
<td>-0.0119</td>
<td>-21.850</td>
<td>-4.15</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0</td>
<td>-3.6322</td>
<td>-0.0068</td>
<td>-21.378</td>
<td>-4.00</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Gaussian characterization for $A \leq 6$

- The energies are described only quantitatively.
- Based on EFT at LO (and the results on bosons), we introduce a three-body force

$$W(ρ) = W_0 \, e^{-\left(\frac{r_{12}^2 + r_{13}^2 + r_{23}^2}{ρ_3^2}\right)}$$

- with the strength and range fixed to describe $E_3$ and $E_4$. 

![Graph showing the energy levels for different isotopes]
Gaussian characterization for $A \gg 3$
From few to many bodies

- Inside the universal window we observe universal behavior encoded in two quantities: the dimer energy and the scattering length.
- To describe these two quantities we construct a two-parameter potential used to characterize the universal window.
- For more than two bodies this potential is able to capture many dynamical properties.
- Increasing the number of particles non-universal behavior could appear.
- It can be taken into account by fixing with detail the range of the three-body repulsion.
- This parameter can be fixed in the few-body sector to describe for example $E_3$ and $E_4$.
- The complete potential is parametrized by four low energy observables.
- This implies a strong correlation between the few-body and many-body dynamics.
The 2 + 1 scattering length

- In the two-body system the scattering length and the dimer energy are strictly correlated inside the universal window:
  \[ k_d = 1/a + r_{\text{eff}} k_d^2 / 2 \]
- The atom-dimer scattering length is strictly correlated to the discrete spectrum
- The functional form in zero-range theory was derived by Efimov
  \[ a_{AD}/a_B = d_1 + d_2 \tan [s_0 \ln(\kappa_* a_B) + d_3] \]
- \( d_1, d_2 \) and \( d_3 \) are universal numbers
- \( \kappa_* \) is the three-body parameter belonging to one of the three-body energy branches
- In the case of finite-range interactions
  \[ a_{AD}/a_B = d_1 + d_2 \tan [s_0 \ln(\kappa_*^{(n)} r_0(a_B/r_0) + \Gamma_3^{(n)}) + d_3] \]
- \( \kappa_*^{(n)} r_0 = \gamma_3^{(n)} \), is used as the driving term
- \( \Gamma_3^{(n)} \) is a finite-range three-body parameter
The $2 + 1$ scattering length

Putting numbers: At $r_0/a_B = 0.0637 \rightarrow a_{AD}/a_B = 1.19$

Using the LM2M2 value, $a_B = 182.22 \, a_0 \rightarrow a_{AD} = 217 \, a_0$

The LM2M2 value for this quantity of $218.4 \, a_0$!
The doublet $nd$ scattering length

![Graph showing the scattering length for $nd$ as a function of $r_0/a_B$. The graph indicates a rapid increase at small $r_0/a_B$ values, followed by a slower approach to a constant value.](image-url)
The doublet $nd$ scattering length

Putting numbers: At $r_0/a_B = 0.457 \rightarrow a_{nd}/a_B = 0.08$
Using the deuteron value, $a_B = 4.3 \text{ fm} \rightarrow a_{nd} = 0.4 \text{ fm}$
The experimental value for this quantity of $0.65 \text{ fm}$!