A continuous-opinion model inspired by the physics of granular gases

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Motivation

**Analogy** between continuous-opinion models & granular gases in 1D

**Evolution toward consensus** is important
Opinion model

System:
- $N$ agents on a network with adjacency matrix $A_{ij}$
- The state/opinion of agent $i$ is $s_i \in (-\infty, \infty)$
- The state of the system at a given time is $S \equiv \{s_i\}_{i=1}^N$
Opinion model

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- The state/opinion of agent \( i \) is \( s_i \in (\mathbb{R}, \mathbb{R}) \)
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Dynamics:
- Choose two agents \( i \) and \( j \) with a rate
  \[
  \pi(s_i, s_j) = rA_{ij} \rho(s_i - s_j)
  \]
  where \( r > 0, \rho(x) \geq 0 \)
- Interaction/collision:
  \[
  s_i \rightarrow b_{ij} = s_i' \equiv s_i + \mu(s_j - s_i)
  \]
  \[
  s_j \rightarrow b_{ij} = s_j' \equiv s_j - \mu(s_j - s_i)
  \]
  where \( \mu \in [0, 1] \) is the persuasibility
Relation with other models

**Deffuant** model is recovered when:
- $A_{ij} = 1$
- $\mu \in [0, 1/2]$
- Initially $s_i \in [0, 1]$ for all $i$
- And

$$
\rho(s_i - s_j) = \begin{cases} 
1 & \text{if } |s_i - s_j| \leq \epsilon \\
0 & \text{if } |s_i - s_j| > 0 
\end{cases}
$$

$\epsilon \equiv$ bound of confidence
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**“Voter”** model
- \( \mu = 1 \)
- \( \rho = 1 \)
- Trivial limit: both agents copy simultaneously
Properties

(i) Conservation of the “total opinion”:

\[ s_i' + s_j' = s_i + s_j \]
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(ii) One consensus state:

\[ s_i = \bar{s} \equiv \frac{1}{N} \sum_{k=1}^{N} s_k, \quad i = 1, \ldots, N \]
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(iii) Dissipation of “opinion energy”:

\[ (s_i'^2 + s_j'^2) - (s_i^2 + s_j^2) = -2\mu(1 - \mu)(s_i - s_j)^2 \leq 0 \]
Analogy/difference with a 1D granular gas

- One-dimensional granular gas with

  \[ s \rightarrow \text{velocity} \]

  \[ \mu = \frac{1 + \alpha}{2}; \quad \alpha \in [-1, 1] \text{ coefficient of normal restitution} \]
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  (i) Conservation of linear momentum
  (ii) Zero-temperature state
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  (i) Conservation of linear momentum
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- But “stochastic dynamics”
Master equation

- \( p(S, t) \equiv \) the probability density of state \( S \) at time \( t \)

\[
\partial_t p = \sum_{i > j} (|\alpha|^{-1} b_{ij}^{-1} - 1) \pi(s_i, s_j)p
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where \( b_{ij}^{-1} \) is the inverse of \( b_{ij} \)
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- \( p_i(s, t) \equiv \langle \delta(s - s_i) \rangle \) probability density of \( s_i \) at time \( t \):

\[
\partial_t p_i(s_i, t) = \sum_{\{j|j\neq i\}} \int ds_j (|\alpha|^{-1} b_{ij}^{-1} - 1) \pi(s_i, s_j) p_{ij}(s_i, s_j, t)
\]
Opinion temperature

Mean opinion is conserved:

\[ \bar{s} \equiv \frac{1}{N} \sum_i \langle s_i \rangle \implies \frac{d}{dt} \bar{s} = 0 \]

Opinion temperature:

\[ T \equiv \frac{1}{N} \sum_i \langle (s_i - \bar{s})^2 \rangle \implies \frac{d}{dt} T = -\zeta T \]

with \( \zeta \equiv \) cooling rate

\[ \zeta \equiv \frac{\mu(1 - \mu)}{NT} \sum_{\{i,j|i\neq j\}} \int ds_i ds_j (s_i - s_j)^2 \pi(s_i, s_j) p_{ij}(s_i, s_j, t) \geq 0 \]
Absorbing and consensus states

- $T$ is a decreasing function of time
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- $S$ is a consensus state $\iff T = 0$
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- An absorbing state ($\frac{dT}{dt} = 0$) is not necessarily a consensus state ($T = 0$)

* If $\pi(s_i, s_j) > 0$ when $s_i \neq s_j$, then $S$ is a consensus state $\iff S$ is an absorbing state
Mean field

Two approximations:

1. Homogeneity (exact for fully-connected networks):

   \[ A_{ij} = 1, \]
   \[ p_i(s, t) \approx p(s, t) \]

2. Mean-field approximation ("Molecular chaos"): 

   \[ p_{ij}(s_i, s_j, t) \approx p_i(s_i, t)p_j(s_j, t) \]
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Boltzmann kinetic equation for \( f(s, t) \equiv Np(s, t) \):

\[ \partial_t f(s_i, t) \approx \int ds_j (|\alpha|^{-1} b_{ij}^{-1} - 1) \pi(s_i, s_j)f(s_i, t)f(s_j, t) \]
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1D granular gas and the opinion model have the same mesoscopic description
Mean field for $\pi(s_i, s_j) = r |s_i - s_j|^\beta$

- The system reaches consensus for any $\beta > 0$ and $\alpha \in (-1, 1)$

$$f(s, t) \rightarrow N\delta(s - \bar{s})$$
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- **Approach** to consensus:
  - **Scaling** solution:

$$f(s, t) = Ns_0^{-1} \phi(c); \quad s_0 \equiv \sqrt{2T(t)}; \quad c \equiv \frac{s}{s_0}$$
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  - For $\beta = 0$:
    $$\phi(c) = \frac{2\sqrt{2}}{\pi [1 + 2c^2]^2}$$
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  - For $\beta = 0$:
    $$\phi(c) = \frac{2\sqrt{2}}{\pi [1 + 2c^2]^2}$$
  - For $\beta > 0$, $\phi$ can be approximated by the sum of two Gaussian distributions
Theory & simulations

\[ \beta = 0 \]

\[ |\alpha| = 0.7; 0.8; 0.9; \beta = 1 \]

\[ |\alpha| = 0.8; \beta = 0.5; 1; 1.5 \]
Phase diagram

\[ \beta \]

\[ |\alpha| \]

\[ \phi(c) \]

Unimodal

Multimodal
Conclusions

1. The **opinion temperature** is an useful concept (absorbing and consensus states)

2. If the probability of two voters having different opinions is nonzero then the absorbing state coincides with the unique consensus state

3. For $\pi(s_i, s_j) = r|s_i - s_j|^\beta$ the system always reaches consensus
   - For $|\alpha| \leq |\alpha_c| (\beta)$: 1 group of voters reaches consensus
   - For $|\alpha| \geq |\alpha_c| (\beta)$: 2 groups of voters reach consensus

   - **Coexistence** dominates consensus when $|\alpha|$ is big enough (weak interaction)

   - **Steady state** if agents tend to be more radical with time

4. Straightforward generalization to higher dimensions