A continuous-opinion model inspired by the physics of granular gases

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Motivation

Analogy between

continuous-opinion models

&

granular gases in 1D

Evolution toward consensus is important

Opinion model

System:

- N agents on a network with adjacency matrix A_{ij}
- The state/opinion of agent i is $s_i \in (-\infty, \infty)$
- The state of the system at a given time is $S \equiv \{s_i\}_{i=1}^{N}$

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Dynamics:

- Choose two agents i and j with a rate

$$\pi(s_i, s_j) = rA_{ij}\rho(s_i - s_j)$$

  where $r > 0$, $\rho(x) \geq 0$

- Interaction/collision:

$$\begin{aligned} s_i &\to b_{ij}s_i \equiv s_i' \equiv s_i + \mu(s_j - s_i) \\ s_j &\to b_{ij}s_j \equiv s_j' \equiv s_j - \mu(s_j - s_i) \end{aligned}$$

  where $\mu \in [0, 1]$ is the persuasibility

Relation with other models

Deffuant model is recovered when:

- $A_{ij} = 1$
- $\mu \in [0, 1/2]$
- Initially $s_i \in [0, 1]$ for all i
- And

$$\rho(s_i - s_j) = \begin{cases} 1 & \text{if} \quad |s_i - s_j| \leq \epsilon \\ 0 & \text{if} \quad |s_i - s_j| > 0 \end{cases}$$

$\epsilon \equiv$ bound of confidence

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"Voter" model

- $\mu = 1$
- $\rho = 1$
- Trivial limit: both agents copy simultaneously

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(iii) Dissipation of "opinion energy":

$$(s_i'^2 + s_j'^2) - (s_i^2 + s_j^2) = -2\mu(1-\mu)(s_i - s_j)^2 \leq 0$$

Analogy/difference with a 1D granular gas

- One-dimensional granular gas with

$$s \longrightarrow \text{velocity}$$

$$\mu = \frac{1+\alpha}{2}; \qquad \alpha \in [-1, 1] \quad \text{coefficient of normal restitution}$$

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- But "stochastic dynamics"

Master equation

- $p(S,t) \equiv$ the probability density of state S at time t

$$\partial_t p = \sum_{i>j} (|\alpha|^{-1} b_{ij}^{-1} - 1)\pi(s_i, s_j)p$$

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- $p_i(s,t) \equiv \langle \delta(s - s_i) \rangle$ probability density of s_i at time t:

$$\partial_t p_i(s_i, t) = \sum_{\{j|j\neq i\}} \int ds_j \; (|\alpha|^{-1} b_{ij}^{-1} - 1)\pi(s_i, s_j) p_{ij}(s_i, s_j, t)$$

Opinion temperature

Mean opinion is conserved:

$$\bar{s} \equiv \frac{1}{N}\sum_i \langle s_i \rangle \qquad \Rightarrow \qquad \frac{d}{dt}\bar{s} = 0$$

Opinion temperature:

$$T \equiv \frac{1}{N}\sum_i \langle (s_i - \bar{s})^2 \rangle \qquad \Rightarrow \qquad \frac{d}{dt}T = -\zeta T$$

with $\zeta \equiv$ cooling rate

$$\zeta \equiv \frac{\mu(1-\mu)}{NT} \sum_{\{i,j|i\neq j\}} \int ds_i ds_j \; (s_i - s_j)^2 \pi(s_i, s_j) p_{ij}(s_i, s_j, t) \geq 0$$

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- An absorbing state ($\frac{d}{dt}T = 0$) is not necessarily a consensus state ($T = 0$)

    * If $\pi(s_i, s_j) > 0$ when $s_i \neq s_j$, then S is a consensus state $\Leftrightarrow$ S is an absorbing state

Mean field

Two **approximations**:

1. Homogeneity (exact for fully-connected networks):

$$A_{ij} = 1,$$
$$p_i(s,t) \simeq p(s,t)$$

2. Mean-field approximation ("Molecular chaos"):

$$p_{ij}(s_i, s_j, t) \simeq p_i(s_i, t) p_j(s_j, t)$$

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Boltzmann kinetic equation for $f(s,t) \equiv Np(s,t)$:

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1D granular gas and the opinion model have the same **mesoscopic description**

Mean field for $\pi(s_i, s_j) = r|s_i - s_j|^\beta$

- The system reaches consensus for any $\beta > 0$ and $\alpha \in (-1, 1)$

$$f(s,t) \to N\delta(s - \bar{s})$$

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$$f(s,t) = N s_0^{-1} \phi(c); \qquad s_0 \equiv \sqrt{2T(t)}; \qquad c \equiv \frac{s}{s_0}$$

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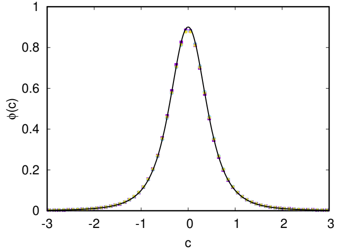
  - For $\beta > 0$, ϕ can be approximated by the sum of **two Gaussian distributions**

Theory & simulations



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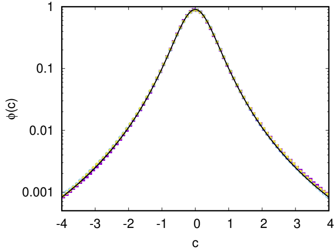

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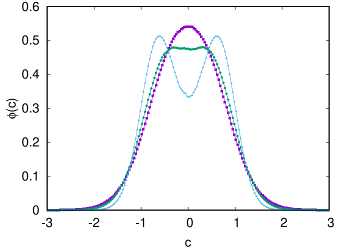
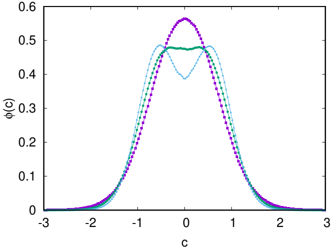
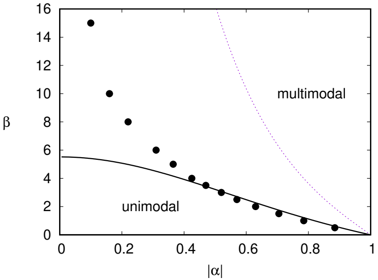


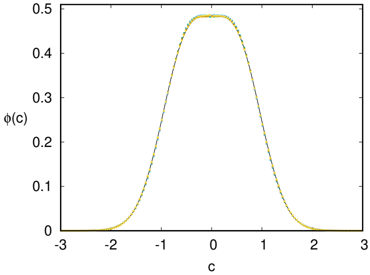
$|\alpha| = 0.7; 0.8; 0.9;\ \beta = 1$



$|\alpha| = 0.8;\ \beta = 0.5; 1; 1.5$

Phase diagram





Conclusions

1. The **opinion temperature** is an useful concept (absorbing and consensus states)
2. If the probability of two voters having different opinions is nonzero then the absorbing state coincides with the unique consensus state
3. For $\pi(s_i, s_j) = r|s_i - s_j|^\beta$ the system always reaches consensus
   - For $|\alpha| \leq |\alpha_c|(\beta)$: 1 group of voters reaches consensus
   - For $|\alpha| \geq |\alpha_c|(\beta)$: 2 groups of voters reach consensus

   - **Coexistence** dominates consensus when $|\alpha|$ is big enough (weak interaction)
   - **Steady state** if agents tend to be more radical with time
4. Straightforward generalization to higher dimensions