Coupled dynamics of agents and link states on the random geometric graph: phase and topological transitions

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Introduction

- Social dynamics is the study of the social macroscopic properties of human communities determined by the interactions between people within this group.
- Usually a group of individuals in a model have different states that change via interactions with his/her neighbors, defined by a network where the individuals are the nodes and interaction between them is the edges (or links).
- In this work we also consider the time evolution of the state of the link between two agents, not only the agent state. The coupling between these two properties defines the dynamics of the ensemble [1].

Theoretical Model

- We consider a set with M agents connected by a RGG (see Fig. 5a) with two different opinions: blue and red. Each pair of agents can be connected by a friendly or non-friendly link. Both agents and links have a time evolution for its states and the control parameter is the probability p.
- There are 6 different pair combinations of agents and links, named as satisfying and unsatisfying as depicted on Fig. 1.
- The density of satisfying and unsatisfying pairs are ρ_s and ρ_u respectively, been the latter the order parameter of the model. They evolve with time and the system comes to a halt when $\rho_u = 0.0$, which is an absorbing state. The active state occurs when the absorbing state is not reached at all, when ρ_u remains greater than zero for $t \to \infty$.
- We propose the modularity (or assortativity coefficient) as another order parameter. It is defined as (Eq. 7.69 on Ref. [2]):

$$\Lambda = \frac{1}{2L}\sum_{ij}\left(A_{ij} - \frac{\kappa_i\kappa_j}{2L}\right)\delta(c_i, c_j),$$

where $\delta(i,j)$ is the Kronecker delta, κ_i is the degree of vertex i and c_i is the class or type of the vertex i. Here there are two classes: red ($c_i = 1$) and blue ($c_i = 2$) agents. This definitions is such that $|\Lambda| < 1.0$ and it is positive if there are more edges between vertices of the same type than expected by chance, and negative if there are less.

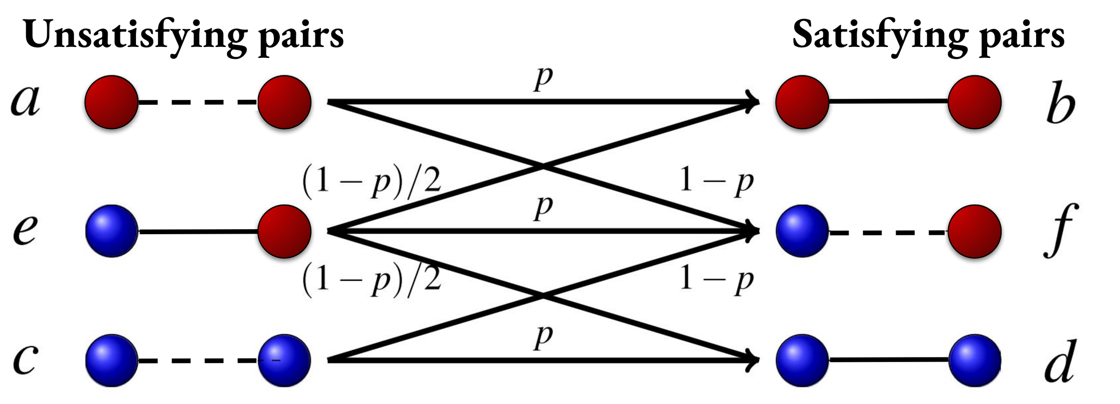

Figure 1: Six possible confgurations of pairs constructed by two types of link states, friendly (solid line) and non friendly (dashed line), and two types of node states, blue and red opinion. The pairs a, c and e are unsatisfying while pairs b, d and f are satisfying. p is the probability of link update and $1-p$ is the node update one. Adapted from Ref. [1]. The density of type e pairs is ρ_e.

Results

Phase diagram

- The density ρ_u is the immediate order parameter, from which we can build the phase diagram on the space (α, p) shown of Fig. 3. The radius α is the interaction distance on the definition of the RGG, which becomes the other control parameter of the system.
- Larger p value (more updates on the states) and low connectivity (low α, less interacting neighbors) enable the system to easyly gets the absorbing state.
- The absorbing state is composed of only satisfying pairs (see Fig. 2). In this phase all friendly edges connect equal nodes and non friendly ones connect different ones. This feature is captured by the modularity Λ.

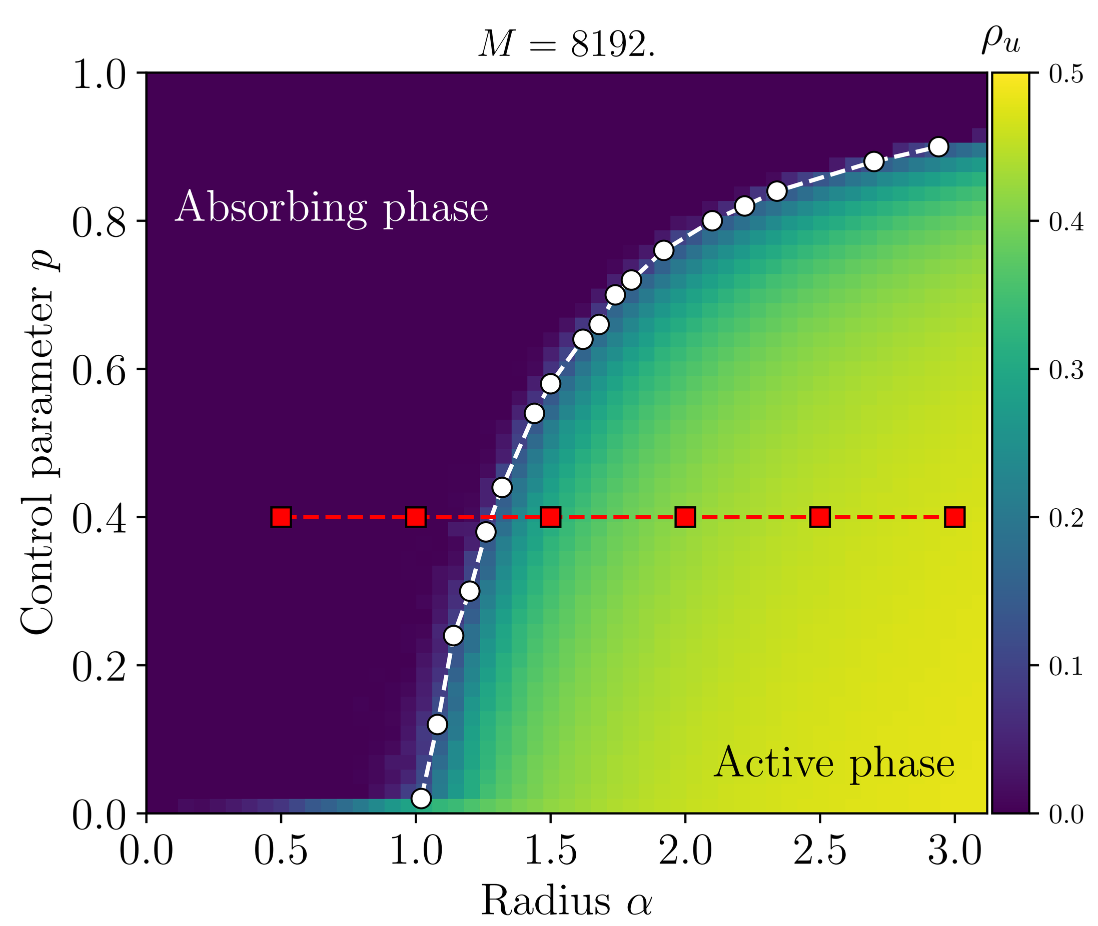

Figure 2: Phase diagram of the model. Density ρ_u in the (α, p) space. Red squares indicate the region for the Λ calculation on Fig. 3a) and white circles indicate the estimated phase transition (where $\rho_e = 0.06$). Absorbing phase: dark blue region with $\rho_u \sim 0.0$. Active phase: yellow-green region with $\rho_u > 0.0$.

Modularity

- Considering the associated network with the friendly edges only, the modularity Λ_f should be maximum on the absorbing state, while the non-friendly network (containing non friendly edges only) should have a minimum modularity Λ_n.
- Fig. 3a shows the two (normalized) modularities calculated on the red horizontal line indicated on Fig. 1. $\Lambda = 1$ (maximum) means a perfect assortative network (all edges connect same nodes types) and $\Lambda = -1$ (minimum) means a perfect disassortative one (all edges connect different nodes types). Indeed in the absorbing state $\Lambda_f \simeq 1.0$ and $\Lambda_n \simeq -1.0$ as expected.
- Fig. 3b shows the full network in an absorbing state with different nodes and edges. Fig. 3c shows the associated friendly network and Fig. 3d shows the non-friendly one.

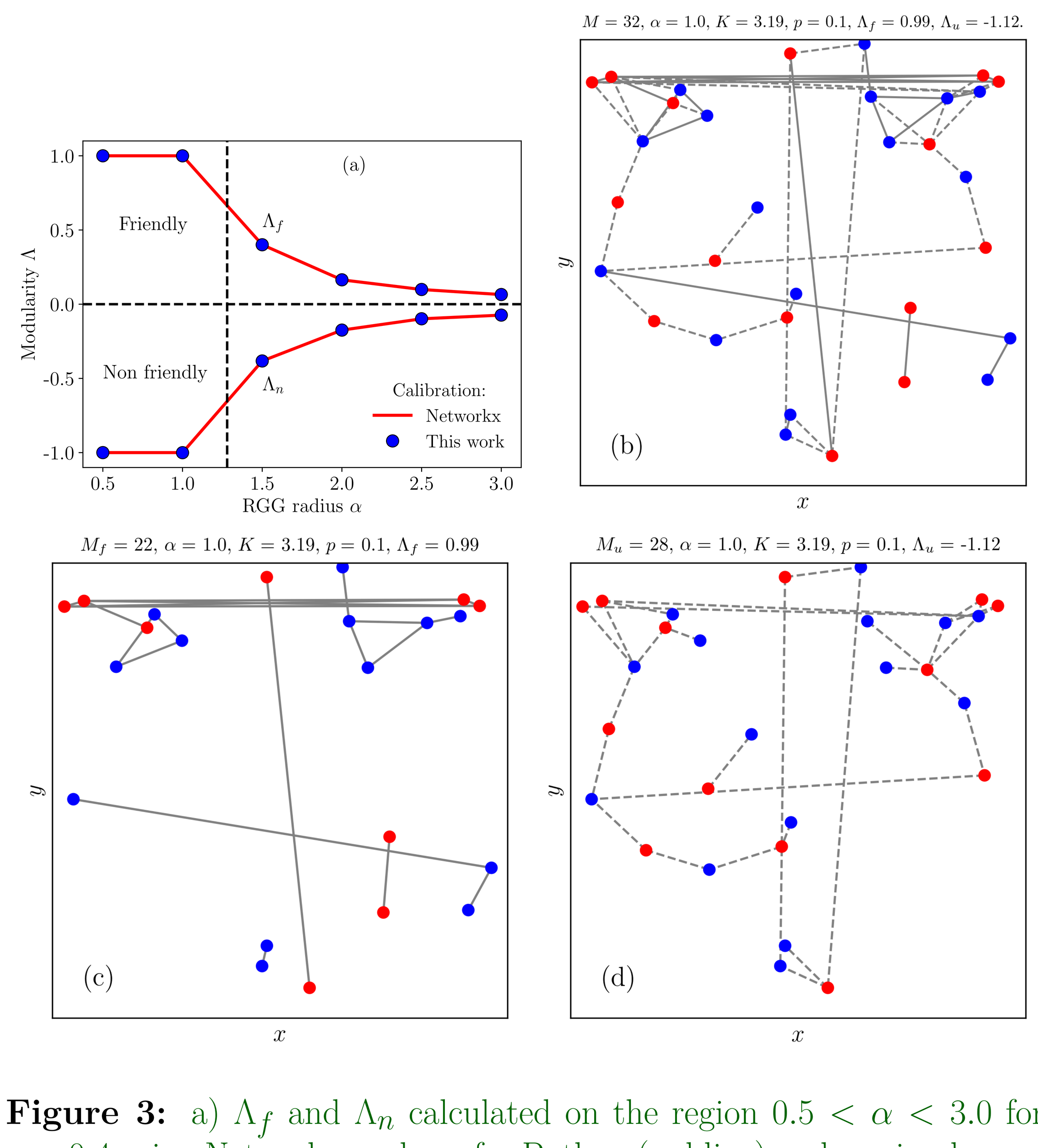

Figure 3: a) Λ_f and Λ_n calculated on the region $0.5 < \alpha < 3.0$ for $p = 0.4$ using Networkx package for Python (red line) and our implementation (blue circles). Vertical black dashed line at $\alpha = 1.28$ roughly indicates the phase transition: $\alpha < 1.28$ is the absorbing state and $\alpha > 1.28$ is the active one. b) Full network on the absorbing phase at the point $(\alpha, p) = (1.0, 0.1)$. The node colors (blue and red) indicate its states. Solid and dashed lines indicate friendly and non friendly edges. c) Friendly network (friendly edges only). d) Non friendly network (non friendly edges only).

- Fig. 4 shows the modularity phase diagram for both associated networks. In the absorbing state the modularities reach an extreme.
- In the active phase both modularities are rougly zero, because there are all 6 kinds of pairs, with no correlation between edges and nodes.

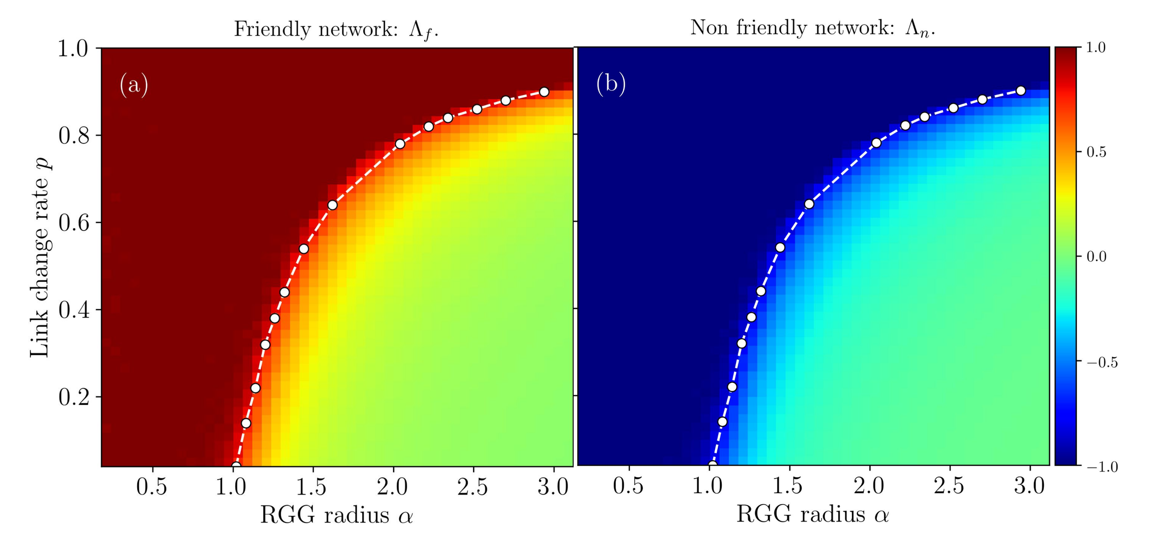

Figure 4: Phase diagrams in the (α, p) space of the modularities a) Λ_f of the friendly network and b) Λ_n of the non friendly network. The colorbar on the right works for both graphics.

- Our strategy to identify the phase transition using short time simulations is to find the points (α, p) where the unstatisfying density follows a power law $\rho_e(t) = at^b$.
- The quality of the fit is measured by the squared of the sample's Pearson correlation coefficient (see Eq. 1.1 of Ref. [3]):

$$c_r = \frac{\left[\sum x_i y_i - \frac{1}{N}\left(\sum x_i\right)\left(\sum y_i\right)\right]^2}{\left[\sum x_i^2 - \frac{1}{N}\left(\sum x_i\right)^2\right]\left[\sum y_i^2 - \frac{1}{N}\left(\sum y_i\right)^2\right]},$$

where the sum $\sum$ goes through $i = t_i, t_i+1, t_i+2, t_i+3\ldots, t_i+N-1 = t_f$, $x = \ln t$ and $y = \ln \rho_e$.
- We found that the phase transition can be roughly estimated by the right border of the region where $c_r \lesssim 1.0$ (see Fig. 5b).

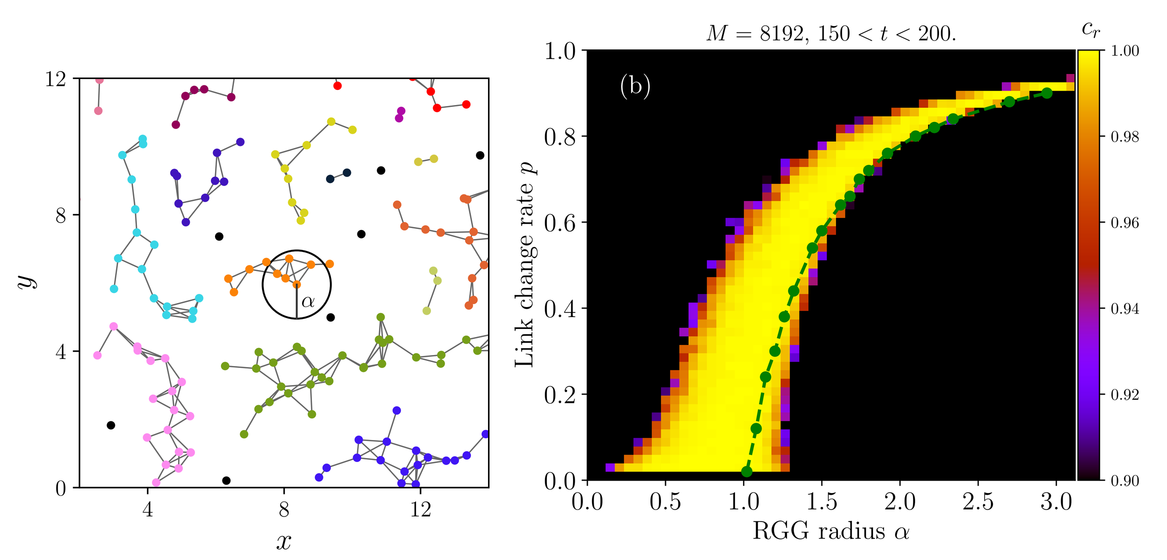

Figure 5: a) Detail of a RGG with $M = 256$ nodes and radius $\alpha = 1.0$. b) Map of the correlation coefficient c_r on the (α, p) space. The power law fit was performed on the interval $150 < t < 200$ where time is measured in Monte Carlo steps. The green circles (with a dashed line) is the same curve represented as white circles on Figs. 2 and 4.

Conclusions

- In summary, we have performed steady-state and out of equilibrium Monte Carlo simulations in order to investigate the phase transitions of the edge node coupled dynamics on the Random Geometric Graph.
- We showed that the modularity for each associated network can be used as order parameter.
- We also showed that the phase transition can be detected using short time simulations.

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