Geometry at the Quantum Scale

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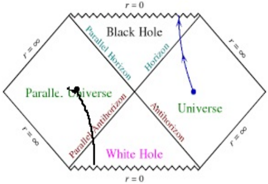
1-Affinities and Connections
2-Renormalizing Geometry
3-Geometry in Quantum gravity
4-Quantum Gravity from Quantum Fields

In a recent seminar, G. 't Hooft, used a Penrose embedding diagram to describe trajectories of particles falling inside a quantum black hole.



"I commented that such simplified description of embedded universes requires the definition of geometry at the quantum scale of events (in which the quantum principles applies). Therefore the question whether or not the basic axioms of geometry hold at that scale. Here we show:

- That the perturbative quantization of gauge fields ('t Hooft-Veltman 1972), **necessarily implies that the affine geometry of the internal spaces of quantum fields is well defined.**
- That the non-renormalizable gravitation in Einstein' GR ('t Hooft-Veltman 1974), together with the recent CMBR data, imply that **Einstein's gravitation cannot be limited to ordinary matter (lumps of atoms).**

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Affine geometry was introduced by Brunelleschi in 1415 and it was applied to gauge field theory by Emmy Noether in 1918, when she defined the "*gauge covariant derivative*" $D_\mu = \partial_\mu + \mathcal{A}_\mu$, with components

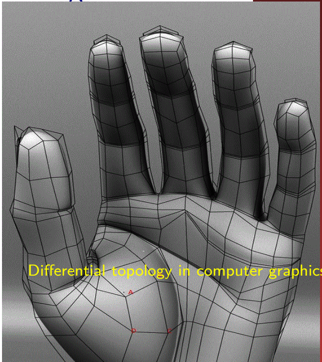
$$(D^a_b)_\mu = (\delta^a_b)\partial_\mu + (\mathcal{A}^a_b)_\mu$$

so as to obtain the gauge conserved quantities*.

Due to such somewhat "ad hoc" definition, we require a proof of its existence. This was already proved by the Frobenius theorem on differential topology stating that:

> *An affine connection in a manifold exists if the associated* ***curvature is integrable****.*

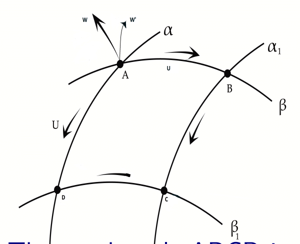
Differential topology shows how to construct differentiable manifolds out of a differentiable wire mesh.



* Leite Lopes: Classical symmetries (Download from the CBPF Library)

Curvature in Affine Manifolds

Consider a differentiable manifold covered by a diferentiable wire made of geodesics with respect to a covariant derivative $D_\mu = \partial_\mu + \mathcal{A}_\mu$. Take two geodesics α and β with respective tangent vectors u and v, crossing at the point A: $D_u u = 0$ and $D_v v = 0$. By parallel transporting v along u and u along v, and using the closing condition $D_u v = D_v u$ at the point D we



The quadrangle ABCD in the hand manifold

obtain a closed quadrangle. Finally, drag an arbitrary vector field w along the quadrangle from A, back to A.

The local curvature of the hand manifold at A is the difference

$$w' - w = \mathcal{C}(u,v)w \overset{def}{=} [D_u, D_v]w - D_{[u,v]}w$$

In a tangent basis $\{e_\mu\}$, the curvature components are

$$\mathcal{C}(e_\mu, e_\nu) = [D_\mu, D_\nu] = (\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu) + [\mathcal{A}_\mu, \mathcal{A}_\nu] = F_{\mu\nu}$$

In the renormalization procedure, redefine $\mathcal{A}_\mu = \mathbf{g} A_\mu$, where A_μ is the gauge potential satisfying the Yang-Mills equations. Then, $\mathbf{g}$ is expanded in powers of the Planck energy $\varepsilon = \hbar\nu$, where the coefficients are adjusted so that $\mathbf{g}A_\mu$ remain finite for any subsequent terms of the expansion. Translating this in therms of geometry, the Frobenius theorem says that a normalizable gauge potential exists if the field force (the curvature) is integrable at the quantum scale

$$F_{\mu\nu} = \mathbf{g}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \mathbf{g}^2 [A_\mu, A_\nu]$$

In other words, the renormalization of gauge fields is equivalent to say in the geometric language that the concepts of parallel lines, parallel transport, geodesics and curvature, remain consistent at the quantum scale of events.

Would the space-time geometry be also consistent with quantum mechanics?

The 't Hooft- Veltman result of 1974 (shows that the gravitational field of RG is non- renormalizable), is due to the presence of the Newtonian coupling constant G in Einstein's Equations: More specifically in the presence of the term $[M]^{-1}$ in the physical dimensions of G

$$[G] = \frac{[L]^3[T]^2}{[M]}$$

$\nearrow$ the problem is here

Recall that the constant G was originally conceived and measured for the gravitational attraction between lumps made of ordinary matter (atoms), while the recent CMBR measurements persistently confirm that ordinary matter is responsible for only $\approx 4.5\%$ of the estimated total energy of the universe.

However, the presence of Einstein's gravitation is given by the the eigenvalues of the space-time curvature measured in the 3-dimensional hypersurface of the observers:

$$R_{\mu\nu\rho\sigma}X^{\rho\sigma} = \lambda_{\mu\nu}X^{\rho\sigma}$$

There are five non-trivial such eigenvalues or "degrees of freedom" (dof=5). Equivalently, the spin-statistics theorem says that

> The fundamental particle of the gravitational field, the gravitation, is a massless particle with
>
> $$spin = (dof - 1)/2 = 2$$

It means that all spin-2 fundamental particles of may also contribute to the gravitational field in the sense of Einstein

The general Lagrangian for a spin-2 field $h_{\mu\nu}$ in Minkowski space-time was described by Fierz and Pauli in 1939 as

$$\mathcal{L} = \frac{1}{4}[h_{,\mu}h^{,\mu} - h_{\nu\rho,\mu}h^{\nu\rho,\mu} - 2h_{\mu\nu}{}^{,\mu}h^{,\nu} + 2h_{\nu\rho,\mu}h^{\mu\rho,\nu} - U]$$
$$U = \text{potential energy} = \mu^2(h_{\mu\nu}h^{\mu\nu} - h^2\eta_{\mu\nu})$$

with Euler-Lagrange equations $(\Box^2 - \mu^2)h_{\mu\nu} = 0$.
The required minimal energy conditions to measure the *dof* of this field are given by the solutions of

$$\frac{\partial U}{\partial h_{\mu\nu}} = 2\mu^2(h^{\mu\nu} - h\eta^{\mu\nu}) = 0 \Rightarrow \begin{cases} \mu \neq 0, h = 0, \text{short range field} \\ \mu = 0, h = 0, \text{long range field} \end{cases}$$

in the second case, the equations of motion become

$$\boxed{\Box^2 h_{\mu\nu} = 0}$$

Only this case coincide with the long range traceless -transverse -plane -polarized linear gravitational wave equation in the sense of Einstein.

Such coincidence suggests that the full non-linear relativistic gravitational field equation can be obtained from higher order perturbations of $\eta_{\mu\nu}$ by powers of $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^2_{\mu\nu} + \cdots$$

Indeed, it has been found that $g_{\mu\nu}$ satisfies Einstein's equations*

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \mathbf{k}\, T_{\mu\nu}}$$

Since $h_{\mu\nu}$ is a quantum field and **k** is completely unrelated to the Newtonian gravitational constant G, the main objection to the perturbative quantization of gravitation has just been removed.Does this means that the Gravitational field describe by the above equation is renormalizable?

In fairness such renormalization must be checked-out against the 't Hooft-Veltman previous result of 1974, and of the computer aided Goroff-Sagnotti verification of 1985.

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*Originally derived by R. Kraichnan and S. Gupta. See "Feynman Lectures on Gravitation".

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Falsifiability

If the answer of the previous question is positive, then the substitution of $8\pi G$ by **k** implies that the hierarchy of Einstein's gravitational field no longer applies. Consequently it becomes passive of being falsifiable in the following lines:

- 1. The detection of the linear gravitational wave $\Box^2 h_{\mu\nu} = 0$ resulting from the spin-2 field.
  For that purpose, a virtual spherical laser interferometer gravitational wave observatory (VSLIGO) capable of detecting quadrupole waves can be built as a detector of gravitational waves in the LHC 32km circumference.
- 2. The predicted production and detection of Hawking's quantum black holes and their subsequent evaporation in the laboratory, using a Kerr-Kruskal solution of Einstein's equations with **k**.

The use of a geometric language in quantum fundamental interactions may turn out to be valuable asset to the understanding of quantum mechanics processes as a whole.