



# Wargaming with Quadratic Forms and Brauer Configuration Algebras

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Dedicated to the Ukrainian People 🇺🇦



• Motivation





- This talk is another way of refusing Hardy's thoughts (G.H. Hardy, A Mathematician's Apology; 1940) regarding number theory applications.
- "Real mathematics has no effects on war. No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years.
- It is true that there are branches of applied mathematics, such as ballistics and aerodynamics, which have been developed deliberately for war and demand a quite elaborate technique; it is perhaps hard to call them "trivial", but none of them has any claim to rank "real". They are indeed repulsively ugly and intolerably dull.
- Mathematics is, as I said at Oxford, a **harmless and innocent** occupation.
- The trivial mathematics, on the other hand, has many applications in war. The gunnery experts and airplane designers, for example could not do their work without it. And the general effect of these applications is plain: mathematics facilitates (if not so obviously as physics or chemistry) modern, scientific, total war".





- In this talk, interactions between Brauer configuration algebras and quadratic forms with a problem proposed by Ramanujan in 1917 are used to define a wargame based on a missile defense system (MDS).





## The Road Map of the Talk

1. Quadratic forms and their relationships with the Gabriel's theorem.
2. The Bert Kostant's game.
3. Brauer configuration algebras.
4. Wargaming with admissible paths.
5. Cauchy's polygonal number theorem and a Ramanujan's problem regarding quadratic forms.



Quadratic forms and their relationships  
with the Gabriel's theorem.



- A quadratic form  $q = q(x_1, \dots, x_n)$  in  $n$  indeterminates  $x_1, \dots, x_n$  is said to be an **integral quadratic form**, if it is of the form:

$$q(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2 + \sum_{i < j} a_{ij} x_i x_j$$

- Where,  $a_{ij} \in \mathbb{Z}$  for all  $i, j$ .
- A vector  $x = [x_1, \dots, x_n]^t \in \mathbb{Z}^n$  is called **positive** if  $x \neq 0$  and  $x_j \geq 0$ , for all  $j$ ,  $1 \leq j \leq n$ . If a vector  $x$  is positive, then we write  $x > 0$ .
- An integral quadratic form  $q$  is called **weakly positive** if  $q(x) > 0$ , for any vector  $x > 0$ .
- $q$  is positive semidefinite if  $q(x) \geq 0$ , for any  $x \in \mathbb{Z}^n$ . It is **positive** if  $q(x) > 0$  for any  $x \neq 0$ .
- A vector  $x \in \mathbb{Z}^n$  such that  $q(x) = 1$  is called a **root** of  $q$ .



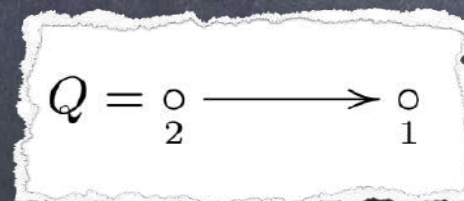
- The quadratic form  $q_Q(x)$  of a quiver  $Q$  has the form:

- $$q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}.$$

- If  $A = kQ$ , then the **Euler quadratic form** of  $A$ ,  $q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{i,j \in Q_0} a_{ij} x_i x_j$

where,  $a_{ij} = \dim_k \text{Ext}_A^1(s(i), s(j))$ . These quadratic forms coincide if  $Q$  is acyclic and connected.

- For instance, the quadratic form of



- is given by the equality  $q_Q(x) = x_1^2 + x_2^2 - x_1 x_2$ . Note that,  $(1,0)$ ,  $(0,1)$ ,  $(1,1)$  are positive roots of  $q_Q(x)$ .



- The reflection  $s_i : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$  at a vertex  $i$  of a finite, connected, and acyclic quiver  $Q$  is given by

- $$s_i(x) = x - 2(x, e_i)e_i$$

- In terms of the coordinates of  $x$ , we see that  $y = s_i(x)$  has coordinates

- $$y_j = x_j \text{ if } j \neq i, \quad y_j = -x_i + \sum_{k \rightarrow i} x_k.$$

- The **Weyl group**  $Q$  is the group of automorphisms of  $E = \mathbb{Q}^n$  generated by the set of reflections  $\{s_i\}_{i \in Q_0}$ .

- In our example,

- $$s_1(1,0) = (-1,0), \quad s_1(0,1) = (1,1), \quad s_2(1,0) = (1,1), \quad s_2(0,1) = (0,-1).$$



- The Coxeter transformation  $c$  of  $Q$  is given by the product  $c = s_{a_n} s_{a_{n-1}} \dots s_{a_2} s_{a_1}$ ,  $M(c) = \Phi_{kQ}$ .
- If  $Q$  is a quiver and  $c$  is its Coxeter transformation. Then,  $P_i = s_1 \dots s_{i-1}(e_i)$ ,  $1 \leq i \leq n$ .
- If  $m_i$  is the least integer such that  $c^{m_i} P_i = e_i$  then the set  $\{c^{-s} P_i \mid 1 \leq i \leq n, 0 \leq s \leq m_i\}$  equals the set of all positive roots of  $q_Q$ .
- Gabriel's Theorem. If  $A = kQ$  is a path  $k$ -algebra of a Dynkin graph  $\overline{Q}$ . Then, the mapping  $\dim : M \rightarrow \dim(M)$ , induces a bijection of indecomposable  $A$ -modules and the set of positive roots of the quadratic form  $q_Q$  of  $Q$ .
- An algebra  $A = kQ$  is representation-finite if its underlying diagram  $\overline{Q}$  is one of the Dynkin diagrams  $A_n, D_n, n \geq 4, E_6, E_7, E_8$ .







- In 2018 A. Postnikov gave a series of lectures at the MIT entitled Topics in Combinatorics.
- Among others, he introduced the Bert Kostant's game (finding the highest root) as follows:
- Let  $G = (V, E)$  be a simple graph, and set  $V = [n]$ .
- For  $i \in V$ , let  $N(i)$  denote the neighbors of  $i$ .
- For  $i \in V$ , we have  $c_i \geq 0$  chips; the vector  $(c_i)_{1 \leq i \leq n} = (c_1, c_2, \dots, c_n)$  is called a configuration.



- We say a vertex  $i$  is:

- 🐱 Happy if  $c_i = \frac{1}{2} \sum_{j \in N(i)} c_j$ .

- 🐱 Unhappy if  $c_i < \frac{1}{2} \sum_{j \in N(i)} c_j$ .

- 🐱 Excited if  $c_i > \frac{1}{2} \sum_{j \in N(i)} c_j$ .

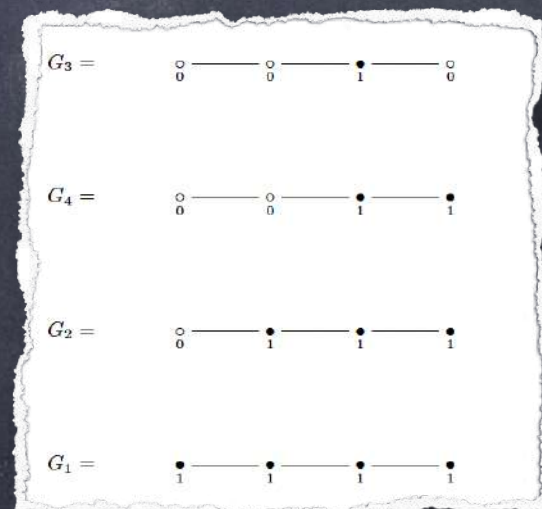
- Goal: Make everyone happy or excited.



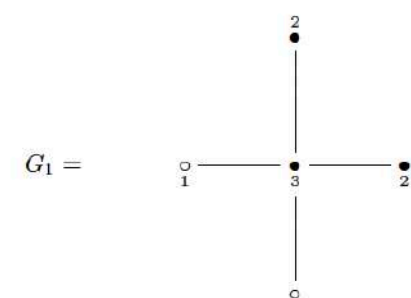
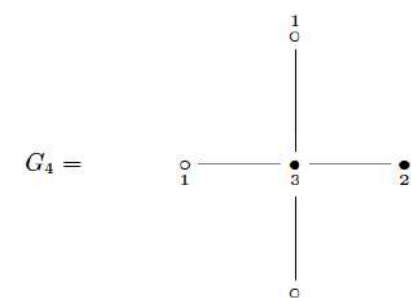
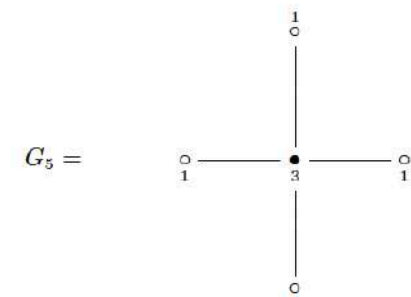
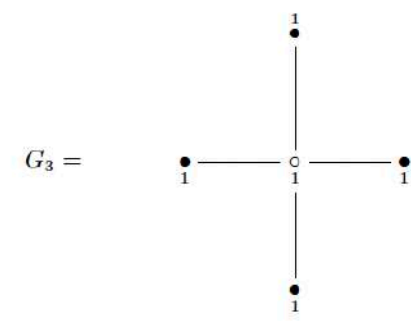
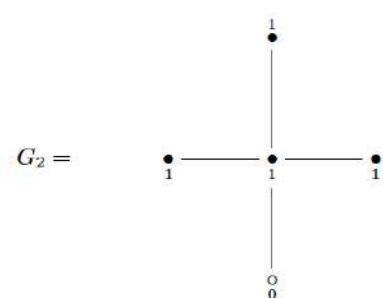
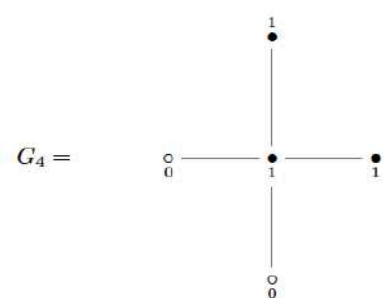
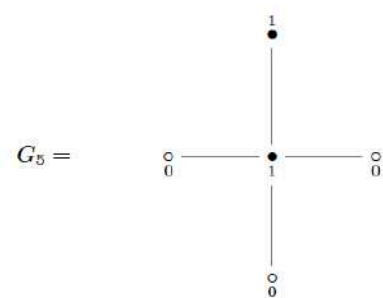
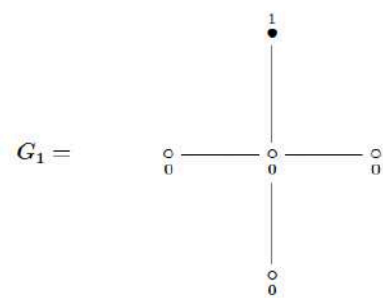
The game is played as follows:

- Initially no chips are present (hence  $c_i = 0$  for all  $i$  and all vertices are happy). Then we place a chip at vertex  $v_{i_0} = 1$ , so  $i_0$  is excited but neighbors of  $i_0$  are unhappy. Subsequently, do the following "reflection":
- Pick an unhappy vertex  $i$ , and replace  $c_i$  by

$$c_i \longrightarrow -c_i + \sum_{j \in N(i)} c_j.$$







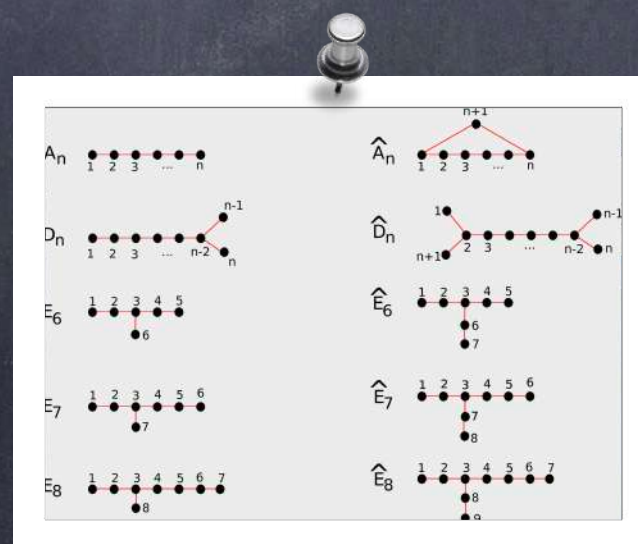


Definition. The graph  $G$  is of finite type if the game ends.

Proposition. If there is a way to play so that the game ends, then any sequence of moves eventually leads to a terminating state. Moreover, the final configuration vector does not depend on the choice of moves, nor the initial vertex we add a chip on.



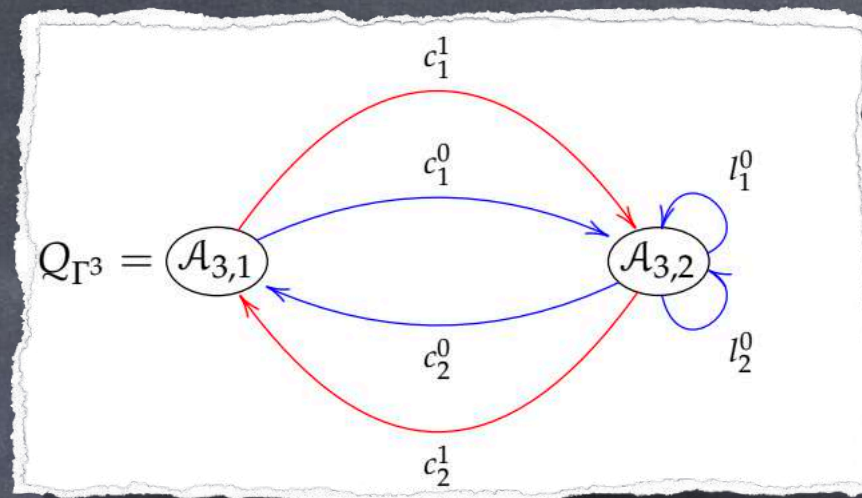
- Theorem. The following statements are equivalent for a graph  $G$ :
  1. Kostant's game is finite.
  2.  $G$  has no subgraph isomorphic to any extended Dynkin diagram.
  3.  $G$  is isomorphic to one of the laced Dynkin diagrams,  $A_n, D_n, E_6, E_7$ , and  $E_8$ .





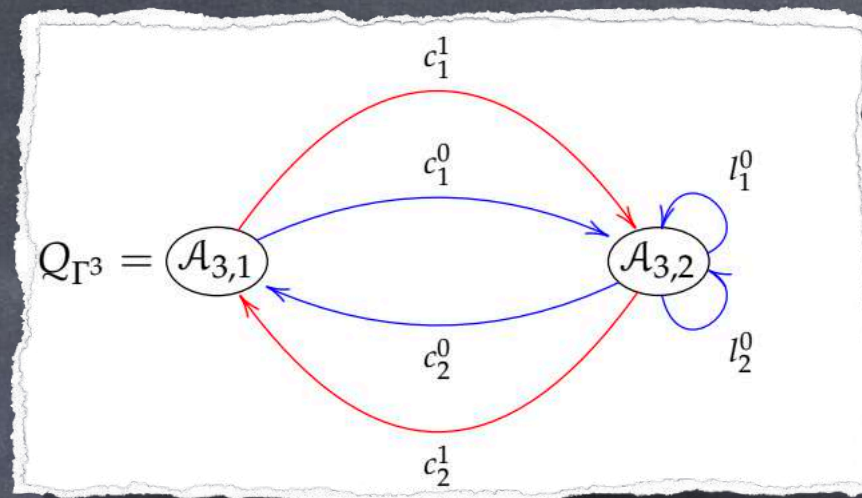






- Brauer configuration algebras (introduced by Green and Schroll in 2017) arise from some Brauer configurations which are quadruples of the form  $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ , where;
- $\Gamma_0$  Is a set of vertices,
- $\Gamma_1$  is collection multisets called polygons. Polygons consist of vertices (vertex repetition is allowed).
- $\mu$  is a multiplicity map,  $\mu : \Gamma_0 \rightarrow \mathbb{N}$  with  $\mu(\alpha) \geq 1$ , for any  $\alpha \in \Gamma_0$
- $\mathcal{O}$  is an orientation defined by way that a given vertex occurs in polygons on  $\Gamma_1$ . Actually, if  $S_\alpha = \{V_1, \dots, V_k\}$  is the maximal set of polygons where a non-truncated vertex  $\alpha$  occurs considering also repetitions then the orientation  $\mathcal{O}$  at  $\alpha$  is defined by endowing  $S_\alpha$  with a linear order  $<$  and adding a relation  $V_k < V_1$ .





- To each vertex  $\alpha \in \Gamma_0$  it is associated the **valency**  $\text{Val}(\alpha) = \sum_{\substack{U \in \Gamma_1 \\ \alpha \in U}} \text{Occ}(\alpha, U)$ .
- $\text{Occ}(\alpha, U)$  is the number of times that the vertex  $\alpha$  occurs in a polygon  $U$ .
- $\alpha$  is said to be truncated if  $\mu(\alpha)\text{Val}(\alpha) = 1$ . Otherwise  $\alpha$  is a non-truncated vertex.
- Fix a polygon  $V \in \Gamma_1$  and suppose that  $\text{occ}(\alpha, V) = t \geq 1$  then there are  $t$  indices  $i_1, \dots, i_t$  such that  $V = V_{i,j}$ . Then the **special  $\alpha$ -cycles at  $v$**  are the cycles  $S_{i_1}, \dots, S_{i_t}$  where  $v \in (Q_\Gamma)_1$  corresponds to the polygon  $V$ .
- If  $\alpha$  occurs only once in  $V$  and  $\mu(\alpha) = 1$  then there is only one special  $\alpha$ -cycle at  $v$ .



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**Algorithm 1:** Construction of a Brauer configuration algebra
 

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1. **Input** A reduced Brauer configuration  $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ .
2. **Output** The Brauer configuration algebra  $\Lambda_\Gamma = \mathbb{F}Q_\Gamma / I_\Gamma$
3. Construct the quiver  $Q_\Gamma = ((Q_\Gamma)_0, (Q_\Gamma)_1, s : (Q_\Gamma)_1 \rightarrow (Q_\Gamma)_0, t : (Q_\Gamma)_1 \rightarrow (Q_\Gamma)_0)$ 
  - (a)  $(Q_\Gamma)_0 = \Gamma_1$ ,
  - (b) For each cover  $V_i < V_{i+1} \in \Gamma_1$  define an arrow  $a \in (Q_\Gamma)_1$ , such that  $s(a) = V_i$  and  $t(a) = V_{i+1}$ ,
  - (c) Each relation  $V_i < V_i$  defines a loop in  $Q_\Gamma$ ,
  - (d) Each ordered set  $C_\alpha$  defines a cycle in  $Q_\Gamma$  called *special cycle*.
4. Define the path algebra  $\mathbb{F}Q_\Gamma$ ,
5. Construct  $I_\Gamma$ , which is generated by the following relations:
  - (a) If  $\alpha_i, \alpha_j \in U$ ,  $U \in \Gamma_1$  and  $C_{\alpha_i}, C_{\alpha_j}$  are corresponding special cycles then  $C_{\alpha_i}^{\mu(\alpha_i)} - C_{\alpha_j}^{\mu(\alpha_j)} = 0$ ,
  - (b) If  $C_{\alpha_i}$  is a special cycle associated to the vertex  $\alpha_i$  then  $C_{\alpha_i}^{\mu(\alpha_i)} a = 0$ , if  $a$  is the first arrow of  $C_{\alpha_i}$ ,
  - (c) If  $\alpha, \alpha' \in \Gamma_0$ ,  $\alpha \neq \alpha'$ ,  $a, b \in (Q_\Gamma)_1$ ,  $a \neq b$ ,  $ab \notin C_\alpha$  for any  $\alpha \in \Gamma_0$  then  $ab = 0$ , if  $a \in C_\alpha$ ,  $b \in C_{\alpha'}$  and  $ab \in \mathbb{F}Q_\Gamma$ ,
  - (d) If  $a$  is a loop associated to a vertex  $\alpha$  with  $val(\alpha) = 1$  and  $\mu(\alpha) > 1$  then  $a^{\mu(\alpha)+1} = 0$ .
6.  $\Lambda_\Gamma = \mathbb{F}Q_\Gamma / I_\Gamma$  is the Brauer configuration algebra.
7. For the construction of a basis of  $\Lambda_\Gamma$  follow the next steps:
  - (a) For each  $V \in \Gamma_1$  choose a non-truncated vertex  $\alpha_V$  and exactly one special  $\alpha$ -cycle  $C_{\alpha_V}$  at  $V$ ,
  - (b) Define:

$$A = \{\bar{p} \mid p \text{ is a proper prefix of some } C_\alpha^{\mu(\alpha)}\},$$

$$B = \{\overline{C_{\alpha_V}^{\mu(\alpha)}} \mid V \in \Gamma_1\}.$$

- (c)  $A \cup B$  is a  $\mathbb{F}$ -basis of  $\Lambda_\Gamma$ .
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## Some Properties of BCA's (Green and Schroll, 2017)

- There is a bijective correspondence between the set of indecomposable projective modules over  $kQ_\Gamma/I$  and polygons in  $\Gamma_1$ .
- The BCA  $kQ_\Gamma/I$  is a multiserial algebra.
- The number of summands in the heart of an indecomposable projective module  $P$  over  $kQ_\Gamma/I$  with radical square distinct of zero equals the number of non-truncated vertices of the polygons corresponding to  $P$  counting repetitions.
- If  $P$  is an indecomposable projective module over  $kQ_\Gamma/I$  corresponding to a polygon  $V$  then the radical of  $P$  is a sum of  $r$  uniserial modules, where  $r$  is the number of non-truncated vertices of  $V$  and where the intersection of any two of the uniserial modules is a simple module.
- Let  $\Gamma$  be a Brauer configuration algebra associated to the Brauer configuration  $\Lambda$  and let  $\mathcal{C} = \{C_1, \dots, C_t\}$  be a full set of equivalence class representatives of special cycles. Assume that for  $i = 1, \dots, t$ ,  $C_i$  is a special  $\alpha_i$ -cycle where  $\alpha_i$  is a non-truncated vertex in  $\Gamma$  then  $\dim_k(\Lambda) = 2|Q_0| + \sum_{C_i \in \mathcal{C}} |C_i|(n_i|C_i| - 1)$ , where  $|Q_0|$  denotes the number of vertices of  $Q$ ,  $|C_i|$  denotes the number of arrows in the  $\alpha_i$ -cycle  $C_i$  and  $n_i = \mu(\alpha_i)$ .



## On the center of a Brauer Configuration Algebra

- Let  $\Gamma$  be a reduced (i.e, without truncated vertices) and connected Brauer configuration and let  $Q$  be its induced quiver and let  $\Lambda$  be the induced Brauer configuration algebra such that  $\text{rad}^2\Lambda \neq 0$ , then the dimension of the center of  $\Lambda$  denoted  $\dim_k(Z(\Lambda))$  is given by the formula (Sierra, 2017)

- $$\dim_k(Z(\Lambda)) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#Loops(Q) - |\mathcal{C}_\Gamma|$$

- where  $|\mathcal{C}_\Gamma| = \{\alpha \in \Gamma_0 \mid \text{Val}(\alpha) = 1, \text{ and } \mu(\alpha) > 1\}$ .



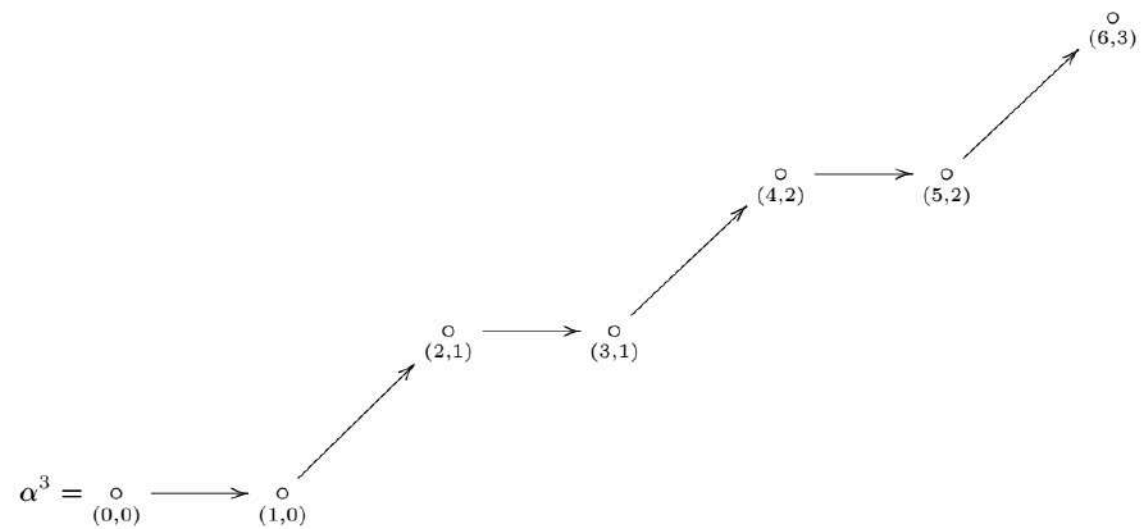
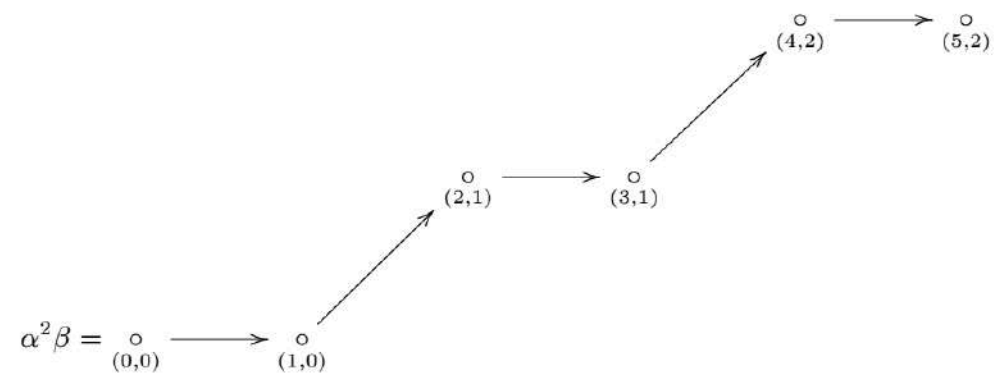
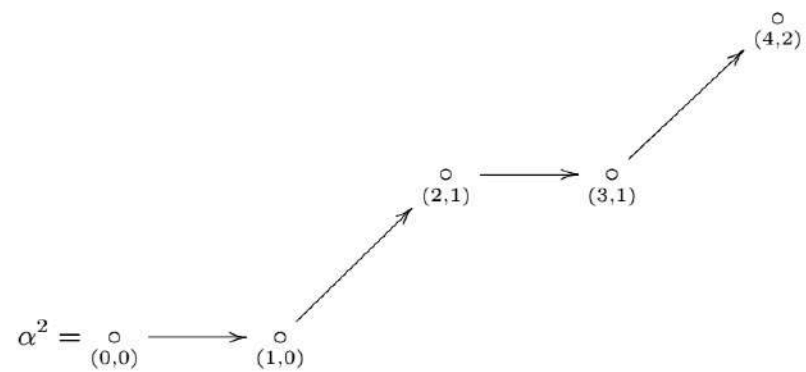
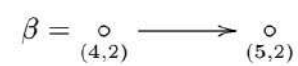
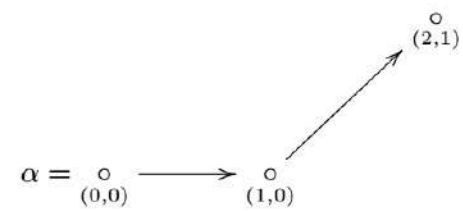
Wargaming with admissible paths.



### A Modified version of the Bert Kostant's Game (Firing Admissible Paths)

- Let us consider a digraph  $Q(a, b, c) = (Q_0, Q_1, s, t)$ , where:
- $Q_0 \subset \mathbb{N} \times \mathbb{N}$ .
- Any arrow  $\alpha \in Q_1$  belongs to a product of at most three **admissible paths**  $\mathcal{A}_1^a$  (the **left boundary path**, **l.b.p**),  $\mathcal{A}_2^b$  and  $\mathcal{A}_3^c$ , where  $a > 0$  and  $b, c$  are nonnegative integers. (the symbol  $\mathcal{A}_j^0$  for  $j \in \{2, 3\}$  means that the path  $\mathcal{A}_j$  does not appear in the product. Moreover,  $\mathcal{A}_j^0 \mathcal{A}_{j'}^{i'} = \mathcal{A}_{j'}^{i'} \mathcal{A}_j^0 = \mathcal{A}_{j'}^{i'}$ ),  $\mathcal{A}_j^i$  is a composition of  $i$  copies of  $\mathcal{A}_j$  whose set of arrows  $\{\alpha_{1,1}^a, \alpha_{1,2}^a, \dots, \alpha_{1,n_1}^a\}$ ,  $\{\beta_{2,1}^b, \beta_{2,2}^b, \dots, \beta_{2,n_2}^b\}$ , and  $\{\gamma_{3,1}^c, \gamma_{3,2}^c, \dots, \gamma_{3,n_3}^c\}$ , respectively satisfy the following conditions:
  1.  $n_3 \leq n_2$ .
  2. If  $m(\delta_{j,t}) \in \mathbb{N}$ ,  $\delta \in \{\alpha, \beta, \gamma\}$ ,  $1 \leq j \leq 3$  is the slope of an arrow  $\delta_{j,t} \in \mathcal{A}_j$  then  $m(\delta_{j,t}) = m(\delta_{j,t-1}) + 1$ , in particular,  $s(\alpha_{1,1}) = (0,0)$ , and  $m(\alpha_{1,1}) = m(\beta_{2,1}) = m(\gamma_{3,1}) = 0$ .
  3. For  $i \geq 2$ ,  $1 \leq r \leq n_j$ ,  $1 \leq j \leq 3$ ,  $\delta \in \{\alpha, \beta, \gamma\}$  it holds that the set of arrows  $\{\delta_{j,1}^i, \delta_{j,2}^i, \dots, \delta_{j,n_j}^i\}$  of  $\mathcal{A}_j^i$  are such that  $m(\delta_{j,r}^i) = m(\delta_{j,r})$ ,  $||\delta_{j,r}^i|| = ||\delta_{j,r}^i||$ ,  $t(\delta_{j,n_j}^{i-1}) = s(\delta_{j,1}^i)$  for all possible values of  $i, \delta$  and  $j$ . And  $t(\alpha_{1,n_1}^a) = s(\beta_{2,1}^b)$ ,  $t(\beta_{2,n_2}^b) = s(\gamma_{3,1}^c)$ .
  4.  $\alpha_{1,n_1}^a \beta_{2,s}^b = \beta_{2,n_2}^b \gamma_{3,s'}^c = \alpha_{1,t}^i \gamma_{3,t'}^{i'} = \beta_{2,u}^l \alpha_{1,u'}^{l'} = 0$ , for  $s, s' > 1$  and all the possible values of  $i, i', l, l', t, t', u$  and  $u'$ .
  5. Two admissible paths  $\mathcal{A}_j^i$  and  $\mathcal{A}_{j'}^{i'}$  are said to be **equivalent** if one is obtained from the other via slope permutations (e.g.  $\mathcal{A}_j = \{0,1,0,1,0\}$  is equivalent to  $\mathcal{A}_{j'} = \{0,0,1,0,1\}$ ).

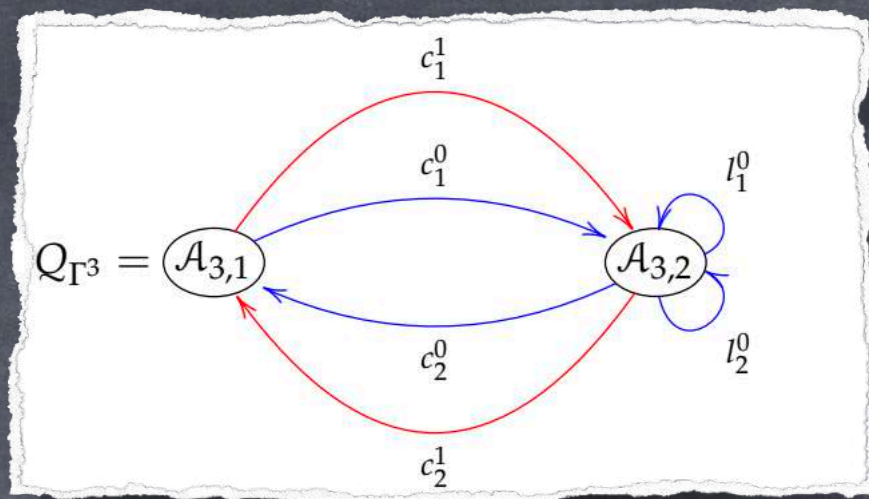






- Definition. For a fixed positive integer  $m$  and a nonnegative integer  $j$ . Let  $\mathcal{L}_j$  be a subset of  $\mathbb{N} \times \mathbb{N}$  such that
  - $$\mathcal{L}_j = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid y = -m(x - j)\}$$
  - We let  $P_{j,i}$  denote the points of  $\mathcal{L}_j$  whose coordinates have the form  $(i, -m(i - j))$ .
- Definition. If  $P_{j,i}, P_{j,i'} \in \mathcal{L}_j$  then they are equivalent. Thus, subsets  $\mathcal{L}_j$  constitute a partition of  $\mathbb{N} \times \mathbb{N}$ .

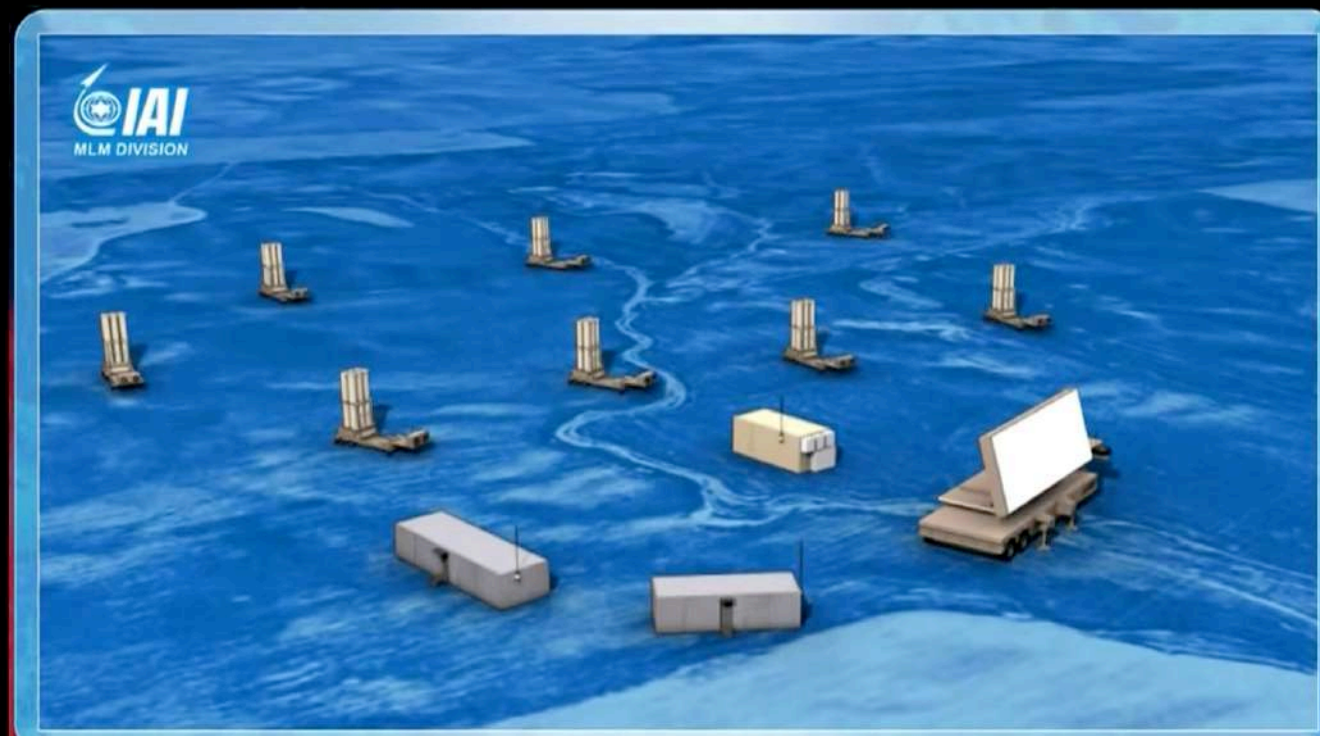




- For  $j > 1$  fixed, the set  $\mathfrak{A}_j$  consisting of all classes of admissible paths ending at points  $P_{j,i}$  defines a **Brauer configuration**  $\Gamma^j = (\Gamma_0^j, \Gamma_1^j, \mu^j, \mathcal{O}^j)$ , where
- $\Gamma_0^j = [\nu_j] = \{0, 1, 2, \dots, \nu_j\}$ ,  $\nu_j = \max\{m(\delta_j^i) \mid \delta_j^i \text{ is an arrow of } \mathcal{A}_j^i \in \mathfrak{A}_j\}$ .
- $\Gamma_1^j = \mathfrak{A}_j$ , i.e. polygons are representative of classes admissible paths, whose associated word  $w(\mathcal{A}_j^i)$  is given by the corresponding slope sequence.
- $\mu^j(s) = 3$  for any  $s \in [\nu_j]$ .
- If  $\mathfrak{A}_j = \{\mathcal{A}_{j,1}^{i_1}, \mathcal{A}_{j,2}^{i_2}, \dots, \mathcal{A}_{j,s}^{i_s}\}$ , where  $\mathcal{A}_{j,h}^{i_h}$  denotes a representative of a class of admissible paths. Thus, an ordering  $\mathcal{O}^j$  is defined in such a way that in successor sequences, it holds that  $\mathcal{A}_{j,h}^{i_h} < \mathcal{A}_{j,h+1}^{i_{h+1}}$ .



- The game we define is similar to the way of a missile defense system (MDS) works.
1. **Players:** Two adversary armies, **A** and **B**. Army B, launches missiles from a point  $(h,0)$ ,  $1 \leq h \leq j$  to a target  $T_B$ , located at a point  $(x,y)$  in a region  $R \subset \mathbb{N}^2$ , if the set of vertices of the left boundary path (*l.b.p*) is  $\{(\underline{x}_0, \underline{y}_0) = (0,0), (\underline{x}_1, \underline{y}_1), \dots, (\underline{x}_t, \underline{y}_t)\}$  then for some  $j$ ,  $0 \leq j \leq t$ , it holds that  $x = \underline{x}_j$ ,  $y > \underline{y}_j$ .
  2. **Gaming:** Army A protects a region  $Dome \subseteq \{(x,y) \in \mathbb{N}^2 \mid 0 \leq x \leq t_1 \geq j, 0 \leq y \leq t_2 \geq j\}$  with an MDS, which fires admissible paths. Missiles are endowed with a GPS device which defines a missile as a **missile-trajectory**, so we can say that army A launches admissible trajectories as a **Ground Based Interceptor (GBI)** does.
  3. Missiles launched by army B follows a linear trajectory with slope  $m$ . The launchers of army A are located at the point  $(0,0)$  their missiles have as goal intercepting those launched by B located at the points in the dome.
  4. **End of the Game:** The game is over once army A have launched all admissible paths with maximal slope associated with the class  $\mathcal{L}_j$  (the largest missile scope for which  $\pi_x(P_{x_{t-1},s}) \leq j \leq \pi_x(P_{x_t,s'})$ ,  $P_{x_{t-1},s}, P_{x_t,s'} \in l.b.p$ ).





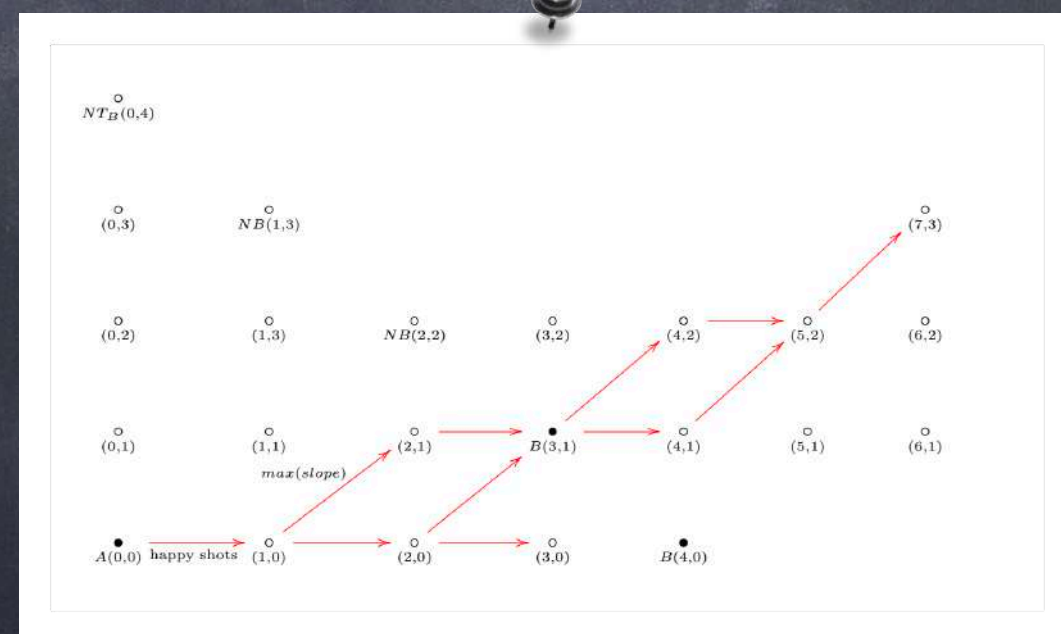
• If a missile launched by army B follows a trajectory determined by a class  $\mathcal{L}_j$  then a launch of the army A is said to be:

1. **Happy**, if exactly one class of admissible paths (only one shot) reaches  $\mathcal{L}_j$  (i.e.  $|\Gamma_1^j| = 1$ ).

2. **Unhappy**, if no class of admissible paths reaches  $\mathcal{L}_j$  ( $|\Gamma_1^j| = 0$ ).

3. **Excited**, if more than one class of admissible paths reaches  $\mathcal{L}_j$  (i.e.  $|\Gamma_1^j| > 1$ ).

• **Problem.** For which values of  $m, a, b$  and  $c$  any launch of the army A (to points of classes  $\mathcal{L}_j$ ) is happy or excited?





Cauchy's polygonal number theorem

And

A Ramanujan's problem regarding  
quadratic forms.



- In 1654 Fermat wrote the following letter to Pascal claiming that any number can be written as a sum of at most  $k$ ,  $k$ -gonal numbers:

"Ce que vous y trouverez de plus important regarde la proposition que tout nombre est compose d'un, de deux ou de trois triangles; d'un, de deux, de trois ou de quatre quarrés; d'un, de deux, de trois, de quatre ou de cinq pentagones; d'un, de deux, de trois, de quatre, de cinq ou de six hexagones, et à l'infiny.

Pour y parvenir, il faut demonstrier que tout nombre premier, qui surpasse de l'unité un multiple de 4, est composé de deux carrés comme 5, 13, 17, 29, 37, etc.

que 4 est composé de deux carrés comme 2, 13, 17, 29, 37, etc.

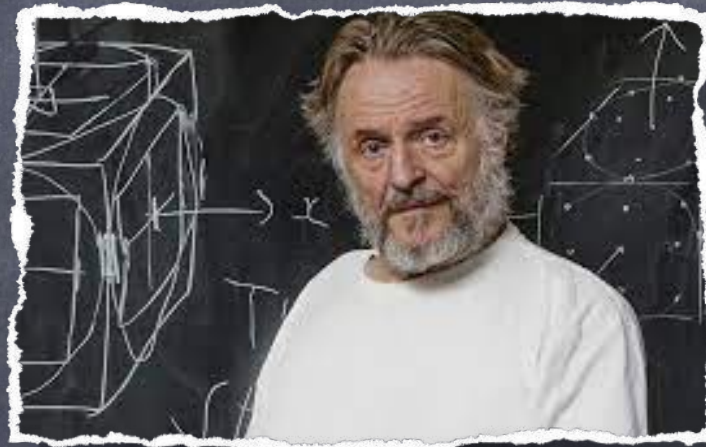
Pour y parvenir, il faut demonstrier que tout nombre premier, qui surpasse de l'unité un multiple



- Advances on this problem were reported by Liouville, Euler, Lagrange, Legendre, Gauss, etc.
- Gauss wrote in his 1796-07-10 diary entry:
- **EUREKA!  $num = \Delta + \Delta + \Delta$**
- Meaning that any number can be written as a sum of three triangular numbers.
- Lagrange in 1772 proved that any number is the sum of four square of numbers.
- In 1798 Legendre and Gauss in 1801 proved that **no number of the form  $4^a(8b + 7)$**  can be written as a sum of three square of numbers.



# Ramanujan-Conway-Barghava



- According to Duke, in 1917 Ramanujan published a paper which was to have a big impact on subsequent research on representations by quadratic forms. He considered the problem of finding all integers  $0 \leq a \leq b \leq c \leq d$  for which every positive integer is represented in the form  $ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2$ . Dickson observed that 54 forms out the 55 conjectured by Ramanujan were correct and that the quadruple  $(1, 2, 5, 5)$  does not represent the number 15.
- Conway and one of his students Schneeberger conjectured in 1993 the following result proved by Barghava in 2000:
- **Fifteen theorem.** If a positive integer-matrix quadratic form represents each of 1, 2, 3, 5, 6, 7, 10, 14, 15, then it represents all positive integers.



- New results regarding mixed sums of triangular and square numbers are given recently by Sun et al.

### Triangular numbers

Triangular numbers are those

$$T_n = \sum_{r=0}^n r = \frac{n(n+1)}{2} \quad (n \in \mathbb{N}).$$

Note that

$$T_{-n-1} = \frac{(-n-1)(-n)}{2} = T_n \quad \text{for all } n \in \mathbb{N}.$$

**Theorem** (conjectured by Fermat and proved by Gauss). Each  $n \in \mathbb{N}$  can be written as  $T_x + T_y + T_z$  with  $x, y, z \in \mathbb{N}$ .

**Liouville's Theorem** (Liouville, 1862). Let  $a, b, c \in \mathbb{Z}^+$  and  $a \leq b \leq c$ . Then any  $n \in \mathbb{N}$  can be written in the form  $aT_x + bT_y + cT_z$  if and only if  $(a, b, c)$  is among

$(1, 1, 1), (1, 1, 2), (1, 1, 4), (1, 1, 5), (1, 2, 2), (1, 2, 3), (1, 2, 4).$

### Sums of four squares

**Lagrange's Four-square Theorem** (1770). Each  $n \in \mathbb{N} = \{0, 1, 2, \dots\}$  can be written as the sum of four squares.

**S. Ramanujan's Observation** (confirmed by L.E. Dickson in 1927). There are totally 54 quadruples  $(a, b, c, d) \in (\mathbb{Z}^+)^4$  with  $a \leq b \leq c \leq d$  such that each  $n \in \mathbb{N}$  can be written as  $aw^2 + bx^2 + cy^2 + dz^2$  with  $w, x, y, z \in \mathbb{Z}$ . The 54 quadruples are  $(1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 2, 2, 2), (1, 1, 1, 3), (1, 1, 2, 3), (1, 2, 2, 3), (1, 1, 3, 3), (1, 2, 3, 3), (1, 1, 1, 4), (1, 1, 2, 4), (1, 2, 2, 4), (1, 1, 3, 4), (1, 2, 3, 4), (1, 2, 4, 4), (1, 1, 1, 5), (1, 1, 2, 5), (1, 2, 2, 5), (1, 1, 3, 5), (1, 2, 3, 5), (1, 2, 4, 5), (1, 1, 1, 6), (1, 1, 2, 6), (1, 2, 2, 6), (1, 1, 3, 6), (1, 2, 3, 6), (1, 2, 4, 6), (1, 2, 5, 6), (1, 1, 1, 7), (1, 1, 2, 7), (1, 2, 2, 7), (1, 2, 3, 7), (1, 2, 4, 7), (1, 2, 5, 7), (1, 1, 2, 8), (1, 2, 3, 8), (1, 2, 4, 8), (1, 2, 5, 8), (1, 1, 2, 9), (1, 2, 3, 9), (1, 2, 4, 9), (1, 1, 5, 9), (1, 1, 2, 10), (1, 2, 3, 10), (1, 2, 4, 10), (1, 2, 5, 10), (1, 1, 2, 11), (1, 2, 4, 11), (1, 1, 2, 12), (1, 2, 4, 12), (1, 1, 2, 13), (1, 2, 4, 13), (1, 1, 2, 14), (1, 2, 4, 14).$

### Mixed Sums of Squares and Triangular Numbers

In 2005 Z. W. Sun [Acta Arith. 2007] investigated what kind of mixed sums  $ax^2 + by^2 + cT_z$  or  $ax^2 + bT_y + cT_z$  (with  $a, b, c \in \mathbb{Z}^+$ ) are universal (i.e., all natural numbers can be so represented). This project was completed via three papers: Z. W. Sun [Acta Arith. 2007], S. Guo, H. Pan & Z. W. Sun [Integers, 2007], and B. K. Oh & Sun [JNT, 2009].

**List of all universal  $ax^2 + by^2 + cT_z$  or  $ax^2 + bT_y + cT_z$ :**

$T_x + T_y + z^2, T_x + T_y + 2z^2, T_x + T_y + 4z^2, T_x + 2T_y + z^2, T_x + 2T_y + 2z^2, T_x + 2T_y + 3z^2, T_x + 2T_y + 4z^2, 2T_x + T_y + z^2, 2T_x + 4T_y + z^2, 2T_x + 5T_y + z^2, T_x + 3T_y + z^2, T_x + 4T_y + z^2, T_x + 4T_y + 2z^2, T_x + 6T_y + z^2, T_x + 8T_y + z^2, T_x + y^2 + z^2, T_x + y^2 + 2z^2, T_x + y^2 + 3z^2, T_x + y^2 + 4z^2, T_x + y^2 + 8z^2, T_x + 2y^2 + 2z^2, T_x + 2y^2 + 4z^2, 2T_x + y^2 + z^2, 2T_x + y^2 + 2y^2, 4T_x + y^2 + 2z^2.$

### On $x(ax+1) + y(by+1) + z(cz+1)$ with $x, y, z \in \mathbb{Z}$

**Theorem** (Z.-W. Sun [JNT 171(2017)]) (i) Let  $a, b, c \in \mathbb{Z}^+$  with  $a \leq b \leq c$ . If  $f_{a,b,c}(x, y, z) := x(ax+1) + y(by+1) + z(cz+1)$  is universal over  $\mathbb{Z}$ , then  $(a, b, c)$  is among the following 17 triples:

$(1, 1, 2), (1, 2, 2), (1, 2, 3), (1, 2, 4), (1, 2, 5), (2, 2, 2), (2, 2, 3), (2, 2, 4), (2, 2, 5), (2, 2, 6), (2, 3, 3), (2, 3, 4), (2, 3, 5), (2, 3, 7), (2, 3, 8), (2, 3, 9), (2, 3, 10).$

(ii)  $f_{a,b,c}(x, y, z)$  is universal over  $\mathbb{Z}$  if  $(a, b, c)$  is among

$(1, 2, 3), (1, 2, 4), (1, 2, 5), (2, 2, 4), (2, 2, 5), (2, 3, 3), (2, 3, 4).$

**Conjecture** (Sun).  $f_{a,b,c}(x, y, z)$  is universal over  $\mathbb{Z}$  if  $(a, b, c)$  is among  $(2, 2, 6), (2, 3, 5), (2, 3, 7), (2, 3, 8), (2, 3, 9), (2, 3, 10).$

In 2017, Ju and Oh [arXiv:1701.02974] proved that

$$f_{2,2,6}(x, y, z) \text{ and } f_{2,3,c}(x, y, z) \quad (c = 5, 7)$$

are universal over  $\mathbb{Z}$ . The universality of  $f_{2,3,c}(x, y, z)$  over  $\mathbb{Z}$  for  $c = 8, 9, 10$  remains open.



**Theorem.** If  $m = 1$  and  $j \geq 1$  then

1.  $\dim_k \Lambda_{\Gamma^j} \leq 2p_{3\Delta}(j)(\nu_j p_{(a+b+c)\nu_j}^5 + 1)$ , where  $p_i^5$  denotes the  $i$ th pentagonal number and  $p_{3\Delta}(j)$  denotes the number of partitions of  $j$  into at most three triangular numbers.
2.  $\dim_k Z(\Lambda_{\Gamma^j}) \leq 1 + 3\nu_j + p_{3\Delta}(j)(1 + (a + b + c - 3)\nu_j)$ .
3. For corresponding Brauer configurations, it holds that  $|\Gamma_1^{27j+12}| = 3|\Gamma_1^{3j+1}|$ , for any  $j \geq 0$ .
4. Any GBI launch from  $(0,0)$  to a class  $\mathcal{L}_j$  is happy or excited if the triplet  $(a,b,c)$  with  $a \leq b \leq c$  is among the following list:

$(1,1,1), (1,1,2), (1,1,4), (1,1,5), (1,2,2), (1,2,3), (1,2,4)$ .

4. Any GBI launch from  $(0,0)$  to a class  $\mathcal{L}_j$  is happy or excited for a choice of  $a, b$ , and  $c$  if and only if it is happy or excited for  $j \in \{1,2,4,5,8\}$ .

**Proof.** Items 3 and 4 are consequences of the works of Liouville (1862) and Kane (2009), respectively.  $\square$



Definition. An extension  $(\mathcal{A}_j)_*$  of an admissible path  $\mathcal{A}_j = \{a_{j,1}, a_{j,2}, \dots, a_{j,t}\}$  is a product of the form  $(\mathcal{A}_j)_* = \{a_{j,1}, a_{j,2}, \dots, a_{j,t-1}, a_{j,1}, a_{j,2}, \dots, a_{j,t}\}$ . Extensions define new quivers  $Q_*(a, b, c)$  under the transformation  $Q(a, b, c) \longrightarrow Q_*(a, b, c)$  whose arrows belong to products of admissible paths one or two of them being extended.

For the sake of clarity, if it is necessary, we assume products of the form  $\mathcal{P}_1(a, b, c) = \mathcal{A}_1^a \mathcal{A}_2^b (\mathcal{A}_3^c)_*$  or  $\mathcal{P}_2 = \mathcal{A}_1^a (\mathcal{A}_2^b)_* (\mathcal{A}_3^b)_*$  to define arrows in extended quivers denoted  $Q_*^1(a, b, c)$  and  $Q_*^2(a, b, c)$ , respectively.







Thank You