$N = 2$ superconformal vertex algebras from Killing spinors

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(0,2) mirror symmetry on homogeneous Hopf surfaces, arXiv:2012.01851

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Background on VAs

Definition

A *vertex algebra* is a vector space V, a vacuum $|0\rangle \in V$, an operator $T\colon V \longrightarrow V$ and the state-field correspondence

$$\begin{aligned} Y(\cdot, z)\colon \quad V &\longrightarrow \mathcal{F}(V) \subseteq \mathrm{End}(V)\left[\left[z^{\pm 1}\right]\right] \\ a &\mapsto Y(a, z) := A(z) = \sum_{n\in\mathbb{Z}} z^{-1-n} a_{(n)} \end{aligned}$$

that satisfies certain axioms:

- **Translation covariance**. **Vacuum**. **Locality**.

V. G. Kac: University Lecture series, Vol. **10**. Providence, RI: Amer. Math. Soc. 1996. Second Edition, 1998.

The **OPE formula** (singular part) is given by

$$A(z)B(w) \sim \sum_{n\in\mathbb{N}} \frac{A(w)_{(n)}B(w)}{(z-w)^{n+1}};$$

$$[a_\lambda b] = \sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!} a_{(n)}(b),$$

and the **normally ordered product** (regular part) is defined by

$$:A(z)B(z): = A(z)_+B(z) + B(z)A(z)_-$$

$$:ab: = a_{(-1)}(b)$$

Remark

From the singular part we can construct "in a universal way" all the vertex algebra (**Reconstruction Theorem**).

Example

Given $c \in \mathbb{C}$ (*central charge*), the *Virasoro vertex algebra* is

$$Y(L,z) := L(z) = \sum_{n\in\mathbb{Z}} z^{-2-n} L_n,$$

where $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c$, $T := L_{-1}$.

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Example

Let $(\mathfrak{g}, (\cdot|\cdot))$ a quadratic Lie algebra. Given $k \in \mathbb{C}$ (*level*), we write

$$\hat{\mathfrak{g}} := \left(\mathfrak{g} \otimes \mathbb{C}\left[t^{\pm 1}\right]\right) \oplus \mathbb{C}k, \; T := -\partial_t.$$

The *currents* of $\mathfrak{g}$ are

$$a \in \mathfrak{g} \quad \mapsto \quad Y(a,z) := \sum_{n\in\mathbb{Z}} z^{-1-n}\left(at^n\right) \in \hat{\mathfrak{g}}\left[\left[z^{\pm 1}\right]\right].$$

Example

The *affinization* $V^k(\mathfrak{g})$ is the vertex algebra with OPE

$$[a_\lambda b] = [a, b] + \lambda\,(a|b)\,k.$$

Theorem (Sugawara '68)

If $\mathfrak{g}$ *simple or abelian, there is an embedding* $V\left(\{L, c\}\right) \hookrightarrow V^k\left(\mathfrak{g}\right)$.

H. Sugawara: Phys. Rev. **176** (1968) 1019–2025.

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Example

Let $(\mathfrak{g}, (\cdot|\cdot))$ a quadratic Lie algebra. Given $k \in \mathbb{C}$ (*level*), we write

$$\hat{\mathfrak{g}}_{\text{super}} := \left(\mathfrak{g} \otimes \mathbb{C}\left[t^{\pm 1}, \theta\right]\right) \oplus \mathbb{C}k,\ T := -\partial_t,\ S := \partial_\theta - \theta\partial_t,$$

being θ odd formal indeterminate such that $\theta^2 = 0$.

Example

The *supercurrents* of $\mathfrak{g}$ are

$$a \in \mathfrak{g} \quad \mapsto \quad Y(a,z) := \sum_{n\in\mathbb{Z}} z^{-1-n} \left(at^n\theta\right) \in \hat{\mathfrak{g}}_{\text{super}} \left[\left[z^{\pm 1}\right]\right],$$

The *superaffinization* $V^k\left(\mathfrak{g}_{\text{super}}\right)$ is the vertex algebra with OPE

$$\begin{aligned} [a_\lambda \Pi b] &= \Pi\,[a,b]\,, \\ [\Pi a_\lambda \Pi b] &= (b|a)\,k, \end{aligned}$$

being $\Pi\colon \mathfrak{g} \longrightarrow \Pi\mathfrak{g}$ the parity-reversing functor.

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Example

Given $c \in \mathbb{C}$ (*central charge*), the *Neveu-Schwarz vertex algebra* is

$$Y(G,z) := G(z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} z^{-n-\frac{3}{2}} G_n.$$

Example

We have $S = G_{-\frac{1}{2}}$ odd derivation such that $SG = 2L$ and

$$\begin{aligned} [L_m, G_n] &= \left(\frac{m}{2} - n\right) G_{m+n}, \\ [G_m, G_n] &= 2L_{m+n} + \frac{c}{3}\left(m^2 - \frac{1}{4}\right) \delta_{m,-n}. \end{aligned}$$

K. Barron: In: Representations and Quantizations (Shanghai, 1998). China High. Educ. Press, Beijing, 2000, pp. 9–35.

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Theorem (Kac, Todorov '85)

If $\mathfrak{g}$ *is simple or abelian, there is an embedding* $V(\{L, G, c\}) \hookrightarrow V^k(\mathfrak{g}_{super})$.

V. G. Kac, I. Todorov: Comm. Math. Phys. **102** (1985) 337–347.

Main Result(s)

Theorem (Álvarez-Cónsul, DADLH, García-Fernández '20)

Given a quadratic Lie algebra $(\mathfrak{g}, (\cdot|\cdot))$, provided that we have a solution of the **Killing spinor equations** *on it, we can construct*

$$V\left(\left\{J, L, G^{\pm}, c\right\}\right) \hookrightarrow V^{k}\left(\mathfrak{g}_{super}\right),$$

where the non-zero OPE formulas of the LHS are

$$[L_\lambda L] = (T + 2\lambda) + \frac{\lambda^3}{12} c, \qquad [J_\lambda J] = \frac{c}{3}\lambda,$$

$$[G^+{}_\lambda G^-] = L + \left(\frac{T}{2} + \lambda\right) J + \frac{c}{6}\lambda^2, \qquad [J_\lambda G^{\pm}] = \pm G^{\pm},$$

$$[L_\lambda G^{\pm}] = \left(T + \frac{3}{2}\lambda\right) G^{\pm} \qquad [L_\lambda J] = (T + \lambda) J,$$

being $G(z) = Y(G^+, z) + Y(G^-, z) := G^+(z) + G^-(z)$, where

$$G^{\pm}(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} z^{-\frac{3}{2} - n} G^{\pm}_n, \quad J(z) = \sum_{n \in \mathbb{Z}} z^{-1-n} J_n.$$

Background on SUSY VAs

We consider (complex) vector (super)spaces $V = V_0 \oplus V_1$, an odd $S\colon V \longrightarrow V$, and the *parity-reversing functor* $\Pi\colon V \longrightarrow \Pi V$.

The *translation algebra* $\mathcal{H} := \frac{\mathbb{C}[T,S]}{\left(\ [T,S]=0, [S,S]=2T\ \right)}$.

Notation: $\nabla \equiv (T, S)$.

The *parameter algebra* $\mathcal{L} := \frac{\mathbb{C}[\lambda,\chi]}{\left(\ [\lambda,\chi]=0, [\chi,\chi]=-2\lambda\ \right)}$.

Notation: $\Lambda \equiv (\lambda, \chi)$. Another pair: $\Gamma \equiv (\gamma, \eta)$.

Definition (Heluani, Kac '07)

A *SUSY LCA* is $(\mathcal{R}, [\cdot_\Lambda \cdot])$ where:

1. $\mathcal{R}$ is an $\mathcal{H}$-module.
2. $[\cdot_\Lambda \cdot] \colon \mathcal{R} \otimes \mathcal{R} \longrightarrow \mathcal{L} \otimes \mathcal{R}$ is a parity-reversing bilinear map satisfying the following axioms:

   2.1 $[Sa_\Lambda b] = \chi [a_\Lambda b], [a_\Lambda Sb] = -(-1)^{|a|} (S + \chi) [a_\Lambda b]$ (**sesquilinearity**).

   2.2 $[a_\Lambda b] = (-1)^{|a||b|} [b_{-\Lambda - \nabla} a]$ (**commutativity**).

   2.3 $[a_\Lambda [b_\Gamma c]] = (-1)^{|\Pi a|} \left[[a_\Lambda b]_{\Lambda + \Gamma} c\right] + (-1)^{|\Pi a||\Pi b|} [b_\Gamma [a_\Lambda c]]$ (**Jacobi identity**).

Definition

A *SUSY VA* is $(V, |0\rangle, ::, S, T, [\cdot_\Lambda \cdot])$ where:

1. $(V, [\cdot_\Lambda \cdot])$ is a SUSY LCA.
2. $((V, |0\rangle, ::), S, T)$ unital differential algebra satisfying:
   2.1. $: ab : -(-1)^{|a||b|} : ba := \int_{-\nabla}^{0} d\Lambda \, [a_\Lambda b]$ (**quasicommutativity**).
   2.2. $: (: ab :) c : - : a (: bc :) :=: \left(\int_0^\Lambda d\Lambda(a) \otimes [b_\Lambda c] \right) : + (-1)^{|a||b|} : \left(\int_0^\Lambda d\Lambda(b) \otimes [a_\Lambda c] \right) :$ (**quasiassociativity**).
3. $[a_\Lambda : bc :] =: [a_\Lambda b] c : + (-1)^{|\Pi a||b|} : b [a_\Lambda c] : + \int_0^\Lambda d\Gamma \, [[a_\Lambda b]_\Gamma c]$ (**Wick non-commutative formula**).

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Given $\mathcal{R}$ a SUSY LCA, there exists a unique SUSY VA denoted by $V(\mathcal{R})$ and called *universal enveloping SUSY VA* of $\mathcal{R}$.

R. Heluani, V. G. Kac: Commun. Math. Phys. **271** (2007) 103–178.

Remarks

- A SUSY VA has a canonical state-(super)field correspondence

$$Y^{\text{super}}(a,(z;\theta)) := \sum_{\substack{j\in\mathbb{Z}\\ J\in\{0,1\}}} \theta^{1-J} z^{-1-j} a_{(j|J)} = Y(a,z) + \theta Y(Sa,z).$$

- The OPE $[\cdot_\lambda\cdot] : V\otimes V \longrightarrow V$ and the odd endomorphism $S\colon V \longrightarrow V$ recover the SUSY structure, since

$$[a_\Lambda b] = \sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!} a_{(n|0)}(b) + \sum_{n\in\mathbb{N}} \frac{\chi\lambda^n}{n!} a_{(n|1)}(b) = [Sa_\lambda b] + \chi\,[a_\lambda b]\,.$$

Examples

- The *superaffinization* $V^k(\mathfrak{g}_{\text{super}})$ of level $k \in \mathbb{C}$ is defined from a quadratic Lie algebra $(\mathfrak{g}, (\cdot|\cdot))$ via the Λ-brackets

$$[\Pi a_\Lambda \Pi b] = \Pi [a, b] + \chi (a|b) k.$$

- The $N = 1$ *superconformal VA* is defined via the Λ-bracket

$$[H_\Lambda H] = (2T + \chi S + 3\lambda) H + \frac{\lambda^2 \chi}{3} c,$$

being $Y^{\text{super}}(H, (z; \theta)) := G(z) + 2\theta L(z)$.

- The $N = 2$ *superconformal VA* is defined via the Λ-brackets

$$[J_\Lambda J] = -\left(H + \frac{\lambda\chi}{3} c\right), \quad [H_\Lambda J] = (2T + 2\lambda + \chi S) J,$$

being $Y^{\text{super}}(J, (z; \theta)) := -iJ(z) - i\theta (G^-(z) - G^+(z))$.

Killing spinor equations

Definition (García-Fernández '19)

A *solution of the Killing spinor equations* on a real quadratic Lie algebra $(\mathfrak{g}, (\cdot|\cdot))$ is a triple (V_+, ε, η) such that:

- $\mathfrak{g} = V_+ \oplus V_+^\perp$.
- $\varepsilon \in V_+$ (divergence).
- $\eta \in S(V_+)$ is a non-vanishing spinor satisfying:

$$D_- \cdot \eta = \sum_{i,j} \left(\left[a_i, a^j\right] \middle| b \right) a^j a_i \cdot \eta = 0, \text{ for } b \in V_+^\perp \text{ (gravitino equation)}.$$

$$\not{D}^+ \cdot \eta = \frac{1}{6} \sum_{i,j,k} \left([a_k, a_i] \middle| a^j \right) a^k a^j a_i \cdot \eta - \varepsilon \cdot \eta = 0 \text{ (dilatino equation)}.$$

being $\{a_i\}_{i=1}^n, \{a^i\}_{i=1}^n \subseteq V_+$ two $(\cdot|\cdot)$-dual orthonormal basis.

M. Garcia-Fernandez: Adv. Math. **350** (2019) 1059–1108.

Let K be an **even-dimensional** compact Lie group. It admits left-invariant Hermitian structures compatible with the metric.

Proposition (Álvarez-Cónsul, DADLH, García-Fernández '20)

A left-invariant solution to the **Twisted Calabi-Yau equations**

$$d\Psi = \theta_\omega \wedge \Psi, \quad d\theta_\omega = 0, \quad dd^c\omega = 0$$

induces a solution of the Killing spinor equations on the space

$$\mathfrak{g} = \Gamma\left(TK \oplus T^*K, [\cdot,\cdot]_{-d^c\omega}\right)^K,$$

of left-invariant sections, which is a quadratic Lie algebra with the induced bracket and pairing.

Remark

$[\theta_\omega] = 0$ implies (Kähler-)Calabi-Yau condition.

Assume now $\dim V_+ = 2n$ and fix an orientation on V_+.

A pure spinor $\eta \in S(V_+)$ determines uniquely an almost complex structure J on V_+ compatible with $(\cdot|\cdot)|_{V_+}$ and the orientation. We can fix $\{x_1, Jx_1, \ldots, x_n, Jx_n\} \subseteq V_+^{\mathbb{C}} = V_+^{1,0} \oplus V_+^{0,1}$ an oriented orthonormal basis for $(\cdot|\cdot)|_{V_+}$ with associated isotropic basis

$$\epsilon_j = \frac{1}{\sqrt{2}}(x_j - iJx_j) \in V_+^{1,0}, \quad \bar{\epsilon}_j = \frac{1}{\sqrt{2}}(x_j + iJx_j) \in V_+^{0,1}.$$

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Proposition (Álvarez-Cónsul, DADLH, García-Fernández '20)

The triple (V_+, ε, J) is a solution of the Killing spinor equations if the following conditions are satisfied:

1) $\left[V_+^{1,0}, V_+^{1,0}\right] \subseteq V_+^{1,0}$, *(F-term equation)*

2) $\sum_{j=1}^{n} [\epsilon_j, \bar{\epsilon}_j] = 2iJ\varepsilon \in V_+^{\mathbb{C}}$. *(D-term equation)*

Example

Let $K = SU(2) \times U(1)$ with associated Lie algebra

$$\mathfrak{k} = \mathfrak{su}(2) \oplus \mathbb{R} = \langle v_1, v_2, v_3, v_4 \rangle \, ,$$

where $[v_2, v_3] = -v_1$, $[v_3, v_1] = -v_2$, $[v_1, v_2] = -v_3$, $[v_4, \cdot] = 0$.

For fixed $x, a, \ell > 0$, we have the pair $\left(V_+^{x,a}, \varepsilon^x \right)$ on

$$\mathfrak{g}_\ell = \Gamma \left(TK \oplus T^*K, [\cdot, \cdot]_{H_\ell} \right)^K \cong \mathfrak{k} \oplus \mathfrak{k}^*,$$

where $H_\ell = \ell v^{123}$ is a left-invariant three-form, defined by

$$\begin{aligned} V_\pm^{x,a} &= \left\{ v \pm g_{x,a}(v) \mid v \in \mathfrak{k} \right\} , \\ \varepsilon^x &= \pi_+ \left(-x v^4 \right) := -\frac{1}{2} \left(\frac{x}{a} v_4 + x v^4 \right) \in V_+^{x,a}, \end{aligned}$$

being the bi-invariant metric

$$g_{x,a} = \frac{a}{x} \left(v^1 \otimes v^1 + v^2 \otimes v^2 + v^3 \otimes v^3 + x^2 v^4 \otimes v^4 \right) .$$

Example

We set I_x as the almost complex structure defined by

$$I_x(v_4) = xv_1, \quad I_x(v_2) = v_3.$$

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Proposition (Álvarez-Cónsul, DADLH, García-Fernández '20)

The triple $(V_+^{x,a}, \varepsilon^x, I_x)$ is a solution of the Killing spinor equations on $\mathfrak{g}_\ell$ if and only if $\ell = a/x$.

Remark

In this example, we cannot have pairs of solutions for the Killing spinor equations. This is because we have left-invariant solutions.

$N = 2$ Supersymmetry from Killing Spinors

Let $(\mathfrak{g}, (\cdot|\cdot))$ be a finite-dimensional complex quadratic Lie algebra. Fix $V_+ := l \oplus \bar{l} \subseteq \mathfrak{g}$ non-degenerate isotropic subspaces, and set $V_- := V_+^\perp$.

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Consider $\{e_j, e^j\}_{j=1}^n \subseteq V_+$ odd isotropic basis, and fix $e \in V_+$ odd. Define the odd vectors

$$I_+ e := e_I - e_{\bar{I}}, \quad I_+ \left[e^j, e_j\right]_+ := \left[e^j, e_j\right]_I - \left[e^j, e_j\right]_{\bar{I}} \in V_+,$$

where the subscripts will denote the canonical projections, and $I_+ \colon V_+ \longrightarrow V_+$, $a \mapsto a_I - a_{\bar{I}}$.

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where the subscripts will denote the canonical projections, and $I_+ \colon V_+ \longrightarrow V_+$, $a \mapsto a_I - a_{\bar{I}}$.

Let $V^k(\mathfrak{g}_{\text{super}})$ with level $0 \neq k \in \mathbb{C}$. Define the even vectors

$$J_0 := \frac{i}{k} : e^j e_j :, \quad J := J_0 - \frac{S}{k} 2i I_+ e \in V^k(\mathfrak{g}_{\text{super}}).$$

Proposition (Álvarez-Cónsul, DADLH, García-Fernández '20)

Define $c_0 := 3\dim l$, $c := c_0 + \frac{12}{k}\left(e \middle| I_+\left[e^j, e_j\right]_+ - e\right)$ *and*

$$H' := H_0 + \tfrac{T}{k} I_+\left[e^j, e_j\right]_+,$$

$$H := H_0 + \tfrac{T}{k}\left(I_+\left[e^j, e_j\right]_+ - 2e\right) + \tfrac{1}{k^2} S\left(:\left[I_+ e, e^j\right] e_j : + : e^j\left[I_+ e, e_j\right] :\right).$$

where

$$\begin{aligned} H_0 := {} & \frac{1}{k}\left(: e_j\left(S e^j\right) : + : e^j\left(S e_j\right) :\right) \\ & + \frac{1}{k^2}\Big(: e_j\left(: e^k\left[e^j, e_k\right] :\right) : + : e^j\left(: e_k\left[e_j, e^k\right] :\right) : \\ & - : e_j\left(: e_k\left[e^j, e^k\right] :\right) : - : e^j\left(: e^k\left[e_j, e_k\right] :\right) :\Big). \end{aligned}$$

Then, one has

$$\left[J_{0\Lambda} J_0\right] = -\left(H' + \frac{\lambda\chi}{3} c_0\right), \quad \left[J_\Lambda J\right] = -\left(H + \frac{\lambda\chi}{3} c\right).$$

Lemma (Álvarez-Cónsul, DADLH, García-Fernández '20)

Assume that (V_+, e, iI_+) satisfies F-term equation. Then,

$$H_0 = \frac{1}{k}\left(: e_j \left(S e^j\right) : + : e^j \left(S e_j\right) :\right) + \frac{1}{k^2}\left(2 : e_j \left(: e^k \left[e^j, e_k\right]_- :\right) : \right. \\ \left. + : e^j \left(: e^k \left[e_j, e_k\right]_I :\right) : + : e_j \left(: e_k \left[e^j, e^k\right]_{\bar{I}} :\right) :\right),$$

and

$$\begin{aligned} \left[{H_0}_\Lambda a_I\right] &= \left(\lambda + 2T + \chi S\right) a_I - \tfrac{2}{k^2}\left(: e_j \left(: e_k \left[\left[a_I, e^j\right]_-, e^k\right]_+ :\right) : \right. \\ &+ : e_j \left(: e^k \left[a_I, \left[e^j, e_k\right]_-\right]_I :\right) : + : e^j \left(: e_k \left[\left[e^k, a_I\right]_-, e_j\right]_- :\right) : \\ &+ : \left[a_I, e^j\right]_- \left(: e_k \left[e_j, e^k\right]_- :\right) :\Big) + \tfrac{1}{k}\Big(\chi : e_j \left[a_I, e^j\right]_- : \\ &+ 2\left(T\left[e_j, \left[a_I, e^j\right]_-\right]_I - : e_j \left(S\left[a_I, e^j\right]_-\right) :\right) \\ &+ \left. \lambda\left(\left[e_j, \left[a_I, e^j\right]_-\right]_I + \left[\left[a_I, e^j\right]_-, e_j\right]_-\right)\right). \end{aligned}$$

Theorem (Álvarez-Cónsul, DADLH, García-Fernández '20)

A solution of the Killing spinor equations (V_+, e, il_+) *on* $(\mathfrak{g}, (\cdot|\cdot))$ *such that* $\dim V_+ = 2n$ *and*

$$\left[e^j, e_j\right] \in [l, l]^\perp \cap \left[\bar{l}, \bar{l}\right]^\perp$$

induces $V(\{J_0, H', c_0\}) \hookrightarrow V^k(\mathfrak{g}_{super})$ *embedding with* $c_0 = 3n$,

$$J_0 = \frac{i}{k} : e^j e_j :,$$

$$H' = \frac{1}{k}\left(: e_j \left(Se^j\right) : + : e^j \left(Se_j\right) :\right) + \frac{1}{k^2}\Big(2 : e_j \Big(: e^k \left[e^j, e_k\right]_- : \Big) :$$

$$+ : e^j \left(: e^k \left[e_j, e_k\right]_l : \right) : + : e_j \left(: e_k \left[e^j, e^k\right]_{\bar{l}} : \right) : \Big) + \frac{T}{k} l_+ \left[e^j, e_j\right].$$

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A solution of the Killing spinor equations (V_+, e, il_+) *on* $(\mathfrak{g}, (\cdot|\cdot))$ *such that* $\dim V_+ = 2n$ *and*

$$[e^j, e_j] \in [l, l]^\perp \cap [\bar{l}, \bar{l}]^\perp$$

induces $V(\{J_0, H', c_0\}) \hookrightarrow V^k(\mathfrak{g}_{super})$ *embedding with* $c_0 = 3n$,

$$J_0 = \frac{i}{k} : e^j e_j :,$$

$$H' = \frac{1}{k}\left(: e_j \left(S e^j\right) : + : e^j \left(S e_j\right) :\right) + \frac{1}{k^2}\left(2 : e_j \left(: e^k \left[e^j, e_k\right]_- :\right) :\right.$$

$$\left. + : e^j \left(: e^k [e_j, e_k]_l :\right) : + : e_j \left(: e_k \left[e^j, e^k\right]_{\bar{l}} :\right) :\right) + \frac{T}{k} l_+ \left[e^j, e_j\right].$$

Remark

Our result generalizes a construction by Getzler for Manin triples.

E. Getzler: Annals of Physics **237** (1995) 161–201.

Definition (Álvarez-Cónsul, DADLH, García-Fernández '20)

We say that $a \in \mathfrak{g}$ is *holomorphic* if $[a, l] \subset l$ and $\left[a, \bar{l}\right] \subset \bar{l}$.

Theorem (Álvarez-Cónsul, DADLH, García-Fernández '20)

A solution of the Killing spinor equations (V_+, e, il_+) *on* $(\mathfrak{g}, (\cdot|\cdot))$ *such that* $\dim V_+ = 2n$ *and* $e \in V_+$ *is holomorphic induces* $V(\{J, H, c\}) \hookrightarrow V^k(\mathfrak{g}_{super})$ *embedding with* $c = 3(n + 2(e|e))$,

$$J = \frac{i}{k} : e^j e_j : -\frac{S}{k} 2il_+ e,$$

$$H = \frac{1}{k}\left(: e_j \left(Se^j\right) : + : e^j \left(Se_j\right) :\right) + \frac{1}{k^2}\Big(2 : e_j \left(: e^k \left[e^j, e_k\right]_- :\right) :$$

$$+ : e^j \left(: e^k \left[e_j, e_k\right]_l :\right) : + : e_j \left(: e_k \left[e^j, e^k\right]_{\bar{l}} :\right) :\Big) = H_0.$$

Remark

Heluani-Zabzine constructed pairs of conmuting copies $\{J_\pm, H_\pm\}$ on the CDR of X from generalized Calabi-Yau metric manifolds. When X is compact, it is also Kähler. Their proof requires to have pairs of solutions for the Killing spinor equations.

R. Heluani, M. Zabzine: Comm. Math. Phys. **306** (2011) 333–364.

Remark

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Example (Álvarez-Cónsul, DADLH, García-Fernández '20)

Provided that the divergence ε^x is I_x-holomorphic, we have

$$V\left(\{J, H, c\}\right) \hookrightarrow V^2\left((\mathfrak{g}_\ell)_{\text{super}}\right) \hookrightarrow H^0\left(K, \Omega^{\text{ch}}_K\left(TK \oplus T^*K\right)\right)$$

with $c = 6 - 6/\ell$ as an application of the second Theorem.

More Applications and Open Problems

- Two solutions $(V_+^x, \varepsilon^x, I_x)$ and $(V_+^{\hat{x}}, \varepsilon^{\hat{x}}, -I_{\hat{x}})$ for the Killing spinor equations on $\mathfrak{g}_\ell$ are $(0,2)$ *mirror* for $\hat{x} = 1/(\ell x)$.

- We can construct an embedding for ('small') $N = 4$ *superconformal vertex algebras*, provided that we have hyperholomorphic solutions for the Killing spinor equations on $\mathfrak{g}_\ell$. We apply the first Theorem to obtain $c = 6$.

  Conjecture: This suggests a generalization of the presented construction for $N = 4$ superconformal vertex algebras when $\dim V_+ = 4n$ and $G_\eta = \mathrm{Sp}(n)$.

- **Conjecture**: We can construct an embbeding from the $N = 2$ superconformal vertex algebra into the CDR of a *Courant algebroid*, provided that it satisfies the Killing spinor equations with exact divergence.

Thank you!

¡Muchas gracias!

Muito obrigado!

Eskerrik asko!