On quasi Steinberg characters of complex reflection groups

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Outline

- Definition of quasi p-Steinberg characters
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- Murnaghan–Nakayama rule for the groups $G(r, 1, n)$
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Notation: p is a prime; G denotes a finite group.

Definition (p-regular element of a group G)

An element whose order is not divisible by p.

Let p divide the order of G.

Definition (p-Steinberg character)

An irreducible character χ of G such that $\chi(x) = \pm|C_G(x)|_p$, for every p-regular element x in G, where $C_G(x)$ denotes the centralizer of x in G.

Definition (quasi p-Steinberg character)

An irreducible character χ of G is called a quasi p-Steinberg character if $\chi(g) \neq 0$ for all p-regular elements g in G.

Example

- All irreducible characters of a p-group are quasi p-Steinberg characters.
- Linear characters of a finite group are quasi p-Steinberg characters.

So, we concentrate on nonlinear characters when talking about quasi p-Steinberg characters.

Theorem (Paul and Singla, 2021)

For $n \geq 3$, let λ be a partition of n such that $\lambda \neq (n), (1^n)$ and p be a prime. All triplets (n, λ, p) such that χ_λ is a quasi p-Steinberg character of S_n are given in Table below.

n	λ	p
3	$(2,1)$	2
4	$(2,2)$	2
4	$(3,1)$, $(2,1,1)$	3
5	$(4,1)$, $(2,1,1,1)$	2
5	$(3,2)$, $(2,2,1)$	5
6	$(3,2,1)$	2
6	$(4,2)$, $(2,2,1,1)$	3
8	$(5,2,1), (3,2,1,1,1)$	2

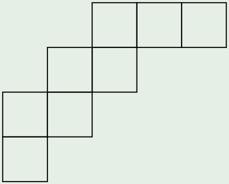
Murnaghan–Nakayama rule for S_n

Definition (Skew hook/Rim hook/Ribbon)

A skew diagram that is edgewise connected and contains no 2×2 subset of boxes.

The height of a ribbon is equal to one less than the number of rows in the ribbon.

Example



Its height is 3.

Definition

A ribbon tableau is a generalized tableau T with positive integral entries such that the entries in the rows and columns of T weakly increase, and all occurrences of a given entry lie in a single ribbon. The height of ribbon tableau T, denoted by $ht(T)$, is the sum of heights of all of its ribbons.

Theorem (Murnaghan–Nakayama rule)

For a partition λ of n and $\sigma \in S_n$, the character χ^λ is given by

$$\chi^\lambda(\sigma) = \sum_T (-1)^{ht(T)},$$

where the sum is over all ribbon tableaux T of shape λ and content given by the lengths of the cycles in σ.

Example

Let $\lambda = (4, 2)$ and $\sigma = (1, 2, 3)(4, 5)$. Then, the ribbon tableaux are

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 1 & 3 \\ \cline{1-2} \end{array}$$

So, $\chi^\lambda(\sigma) = 0$.

Representation theory of $G(r, 1, n)$

$$\begin{aligned} G(r,1,n) :&= \mathbb{Z}_r^n \rtimes S_n \\ &= \{(z_1, z_2, \ldots, z_n, \sigma) \mid z_i \in \mathbb{Z}_r \text{ for all } 1 \leq i \leq n, \sigma \in S_n\}. \end{aligned}$$

Various ways to study representation theory of $G(r, 1, n)$

- Theory of symmetric functions (Specht's Thesis/Macdonald's book on "Symmetric functions and Hall polynomials")
- Wigner–Mackey method of little groups
- The Okounkov–Vershik approach [Mishra and Srinivasan, 2016]

Theorem

The irreducible representations of $G(r, 1, n)$ are parametrized by r-partite partitions of n.

Murnaghan–Nakayama rule for $G(r, 1, n)$

It was first proved by Stembridge in 1989. The version we state here is by Adin, Postnikov and Roichman in 2010.

Sequence of ribbons

A **sequence of ribbons** $\mathbf{b} = (b_1, b_2, \ldots, b_t)$ corresponding to an r-partite Young diagram $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_{r-1})$ is obtained from a sequence of r-partite Young diagrams

$$\emptyset = \boldsymbol{\lambda}^{(0)} \subseteq \ldots \subseteq \boldsymbol{\lambda}^{(t)} = \boldsymbol{\lambda}$$

by defining $b_i := \boldsymbol{\lambda}^{(i)} \setminus \boldsymbol{\lambda}^{(i-1)}$ for $1 \leq i \leq t$ such that each b_i has $r-1$ empty components and the **nonempty component is a ribbon**.

r-partite ribbon tableau

An *r-partite ribbon tableau* T of shape $\boldsymbol{\lambda}$ is obtained by filling the boxes in the nonempty component of the ribbon b_i with entry i for each $1 \leq i \leq t$.

i-th index, i-th length and i-th height of T

$f_T(i) :=$ index in $\boldsymbol{\lambda}^{(i)}$ of the nonempty component in the r-tuple b_i;
$l_T(i) :=$ number of boxes in the nonempty component in b_i;
$ht_T(i) :=$ one less than the number of rows in the nonempty component in b_i.

Murnaghan–Nakayama rule for $G(r, 1, n)$

$$\chi^{\boldsymbol{\lambda}}(\pi) = \sum_{T \in RT_c(\boldsymbol{\lambda})} \prod_{i=1}^{t} (-1)^{ht_T(i)} \omega^{f_T(i).z(c_i)}$$

where

$\pi = (z_1, z_2, \ldots, z_n, \sigma)$,

cycle decomposition of σ is given by $c = (c_1, c_2, \ldots, c_t)$,

for $1 \leq i \leq t$, $l(c_i) =$ length of the cycle c_i, $z(c_i) =$ color of the cycle c_i,

ω is a primitive r-th root of unity,

$RT_c(\boldsymbol{\lambda})$ is the set of r-partite ribbon tableaux T of shape $\boldsymbol{\lambda}$ such that $l_T(i) = l(c_i)$ for all $1 \leq i \leq t$.

Complex reflection groups $G(r, q, n)$

Definition

For a positive integer q which divides r, we define a subgroup $G(r, q, n)$ of $G(r, 1, n)$ as follows:

$$G(r, q, n) := \{(z_1, z_2, \ldots, z_n, \sigma) \in G(r, 1, n) \mid \sum_{i=1}^{n} z_i \equiv 0 (\text{mod } q)\}.$$

By Shephard–Todd classification, the family $G(r, q, n)$ is the only infinite family of finite irreducible complex reflection groups.

Special subfamilies in the family $G(r, q, n)$

(a) Cyclic group of order r, $\mathbb{Z}/r\mathbb{Z} = G(r, 1, 1)$;
(b) Dihedral group of order $2r$, $D_{2r} = G(r, r, 2)$;
(c) Symmetric group $S_n = G(1, 1, n)$;
(d) Weyl group of type B_n is $G(2, 1, n)$;
(e) Weyl group of type D_n is $G(2, 2, n)$.

Notation: $m = \frac{r}{q}$

Representation theory of $G(r, q, n)$

The irreducible $G(r, q, n)$-modules are parametrized by the ordered pairs $(\tilde{\boldsymbol{\lambda}}, \delta)$, where $\tilde{\boldsymbol{\lambda}}$ is an (m, q)-necklace with total n boxes and $\delta \in C_{\boldsymbol{\lambda}}$, the stabilizer subgroup for the necklace $\tilde{\boldsymbol{\lambda}}$.

Main Results

Theorem (M., Paul and Singla)

Given a partition λ of n, define $\hat{\boldsymbol{\lambda}}^j = (\lambda_0, \lambda_1, \ldots, \lambda_j, \ldots, \lambda_{r-1})$, where $\lambda_j = \lambda$ for some $0 \leq j \leq r-1$, and $\lambda_k = \emptyset$ for $k \neq j$. Then, $\chi^{\hat{\boldsymbol{\lambda}}^j}$ is a quasi p-Steinberg character of $G(r,1,n)$ if and only if χ^λ is a quasi p-Steinberg character of S_n.

Proof of the easier part

Assuming $\chi^{\hat{\boldsymbol{\lambda}}^j}$ to be a quasi p-Steinberg character of $G(r,1,n)$, it follows that χ^λ is a quasi p-Steinberg character of S_n by the following identity:

$$\chi^\lambda(\sigma) = \chi^{\hat{\boldsymbol{\lambda}}^j}((0, \ldots, 0, \sigma)).$$

Theorem (M., Paul and Singla)

For an r-partite partition $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1})$ of n, the irreducible character $\chi^{\boldsymbol{\lambda}}$ is a quasi p-Steinberg character of $G(r, 1, n)$ in exactly the following cases:

General case

(i) $\lambda_j \vdash n$ *for some* j *and* $\lambda_k = \emptyset$ *for all* $j \neq i$, *and*

(ii) χ^{λ_j} *is a quasi p-Steinberg character of* S_n.

Additional cases for $n < 5$:

(a) *For* $n = 2$, *the character* $\chi^{\boldsymbol{\lambda}}$ *is a quasi* 2*-Steinberg character when* $\lambda_j = (1)$ *for some* $j, \lambda_k = (1)$ *for some* $k \neq j$, *and* $\lambda_l = \emptyset$ *for all* $l \notin \{j, k\}$.

(b) *For* $n = 3$, *the character* $\chi^{\boldsymbol{\lambda}}$ *is a quasi* 3*-Steinberg character when* $\lambda_j \vdash 2$ *for some* j, $\lambda_k = (1)$ *for some* $k \neq j$*and* $\lambda_l = \emptyset$ *for all* $l \notin \{j, k\}$.

(c) *For* $n = 4$, *the character* $\chi^{\boldsymbol{\lambda}}$ *is a quasi* 2*-Steinberg character when* $\lambda_j \vdash 3$ *for some* $j, \lambda_k = (1)$ *for* $k \neq j$, *and* $\lambda_l = \emptyset$ *for all* $l \notin \{j, k\}$.

Sketch of the proof

$p \nmid n$: Consider the element $\alpha = (0, \ldots, 0, (1, 2, \ldots, n))$ whose type is $((n), \emptyset, \ldots, \emptyset)$. Now $\chi^{\boldsymbol{\lambda}}(\alpha) \neq 0$ implies that $\lambda_j \vdash n$ for some j, $\lambda_k = \emptyset$ for $k \neq j$, and

$$\chi^{\boldsymbol{\lambda}}(\alpha) = \chi^{\lambda_j}((1, 2, \ldots, n)).$$

Thus, χ^{λ_j} is a quasi p-Steinberg character of S_n.

Why are there additional cases for $n < 5$?

$p \mid n$: $p \nmid n-1$. One of the subcases is $\lambda_j \vdash n-1$ for some j, $\lambda_k = (1)$ for some $k \neq j$ and $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$.

When $n \geq 5$, we have the following observations:

1. Either $\alpha_2 = (0, \ldots, 0, (1, 2, \ldots, n-2)(n-1, n))$ or $\alpha_3 = (0, \ldots, 0, (1, 2, \ldots, n-3)(n-2, n-1, n))$ is p-regular;
2. Also, $\chi^{\boldsymbol{\lambda}}(\alpha_2) = \chi^{\boldsymbol{\lambda}}(\alpha_3) = 0$.

Notation: $(\chi^{\boldsymbol{\lambda}})^*$ denotes an irreducible character of $G(r, q, n)$ which appears in $\mathrm{Res}^{G(r,1,n)}_{G(r,q,n)}\chi^{\boldsymbol{\lambda}}$. Note that $(\chi^{\boldsymbol{\lambda}})^*$ may not be unique.

Theorem (M., Paul and Singla)

The irreducible character $(\chi^{\boldsymbol{\lambda}})^*$ is a quasi p-Steinberg character of $G(r, q, n)$ in exactly the following cases:

General case

$\chi^{\boldsymbol{\lambda}}$ *is a quasi p-Steinberg character of $G(r, 1, n)$. In this case,*

$$(\chi^{\boldsymbol{\lambda}})^* = Res^{G(r,1,n)}_{G(r,q,n)}\chi^{\boldsymbol{\lambda}}.$$

Theorem (cont.)

Additional cases:

(a) *For* $n = 3, p = 2$, *the three two-dimensional characters* $(\chi^{\boldsymbol{\lambda}})^*$ *in* $Res_{G(r,q,3)}^{G(r,1,3)} \chi^{\boldsymbol{\lambda}}$ *are quasi 2-Steinberg characters. This case arises if and only if* r *and* q *are multiples of* 3, *and* $k = j + \frac{r}{3}, l = j + \frac{2r}{3}$ *for* $0 \leq j \leq \frac{r}{3} - 1$.

(b) *For* $n = 4, p = 3$, *the two three-dimensional characters* $(\chi^{\boldsymbol{\lambda}})^*$ *in* $Res_{G(r,q,4)}^{G(r,1,4)} \chi^{\boldsymbol{\lambda}}$ *are quasi 3-Steinberg characters. This case arises if and only if* r *and* q *are both even, and* $k = j + \frac{r}{2}$ *for* $0 \leq j \leq \frac{r}{2} - 1$.

Case 1: $p \nmid n$.
Subcase (1a): $p \nmid n-1$.

The element $\alpha_1 = (0, \ldots, 0, (1, 2, \ldots, n-1))$ is a p-regular element of $G(r, q, n)$. So, $(\chi^{\boldsymbol{\lambda}})^*(\alpha_1) \neq 0$. This implies that $\chi^{\boldsymbol{\lambda}}(\alpha_1) \neq 0$. Then, $\boldsymbol{\lambda}$ can be of one of the two forms:

(i) either $\lambda_j \vdash n$ for some j, $\lambda_k = \emptyset$ for all $k \neq j$:

$$(\chi^{\boldsymbol{\lambda}})^* = \mathrm{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}.$$

Also, $\boldsymbol{\lambda} = \hat{\boldsymbol{\lambda}}^j$. $\chi^{\hat{\boldsymbol{\lambda}}^j}$ is a quasi p-Steinberg character of $G(r, 1, n)$,

or

Sketch of the proof (cont.)

(ii) $\lambda_j \vdash n-1$ for some j, $\lambda_k = (1)$ for some $k \neq j$, $\lambda_l = \emptyset$ for all $l \notin \{j,k\}$: the corresponding irreducible character $(\chi^{\boldsymbol{\lambda}})^*$ is not a quasi p-Steinberg character of $G(r,q,n)$ because of the following observations when $n \geq 3$:

1. $(\chi^{\boldsymbol{\lambda}})^* = \mathrm{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^{\boldsymbol{\lambda}}$;
2. The element $\alpha = (0,\dots,0,(1,2,\dots,n))$ is p-regular;
3. $\chi^{\boldsymbol{\lambda}}(\alpha) = 0$.

For $n=2$, $(\chi^{\boldsymbol{\lambda}})^*$ is not a quasi p-Steinberg character if $\chi^{\boldsymbol{\lambda}}$ does not decompose as a representation of $G(r,q,n)$.

Sketch of the proof (cont.)

Subcase (1b): $p \mid n-1$. Then, $p \nmid n-2$ and $\alpha_2 = (0,\ldots,0,(1,2,\ldots,n-2))$ is p-regular. Then, $\chi^{\boldsymbol{\lambda}}(\alpha_2) \neq 0$. Then one of the following is true:

1. $\lambda_j \vdash n$ for some $j, \lambda_k = \emptyset$ for all $k \neq j$;
2. $\lambda_j \vdash n-1$ for some j, $\lambda_k = (1)$ for some $k \neq j$, $\lambda_l = \emptyset$ for all $l \notin \{j,k\}$;
3. $\lambda_j \vdash n-2$ for some j, $\lambda_k \vdash 2$ for some $k \neq j$, $\lambda_l = \emptyset$ for all $l \notin \{j,k\}$;
4. $\lambda_j \vdash n-2$ for some j, $\lambda_k = (1)$ for some $k \neq j$, $\lambda_l = (1)$ for some $l \notin \{j,k\}$, and $\lambda_u = \emptyset$ for all $u \notin \{j,k,l\}$;

Sketch of the proof (cont.)

For $n \geq 5$, when $\boldsymbol{\lambda}$ is of one of the forms (ii)-(iv), $(\chi^{\boldsymbol{\lambda}})^*$ is not a quasi p-Steinberg character of $G(r, q, n)$. And, if it is of form (i), then $\chi^{\hat{\boldsymbol{\lambda}}^j}$ is a quasi p-Steinberg character of $G(r, 1, n)$.
Here, $n \neq 2$ as $p \mid n-1$.

What happens when $n = 3, p = 2$ or $n = 4, p = 3$?

$n = 3, p = 2$. The only important form is $\lambda_j = (1)$ for some j, $\lambda_k = (1)$ for some $k \neq j$, $\lambda_l = (1)$ for some $l \notin \{j, k\}$, and $\lambda_u = \emptyset$ for all $u \notin \{j, k, l\}$. Also, $\chi^{\boldsymbol{\lambda}}$ decomposes into three two-dimensional irreducible characters of $G(r, q, n)$ if and only if r and q are multiples of 3, and $k = j + \frac{r}{3}, l = j + \frac{2r}{3}$ for $0 \leq j \leq \frac{r}{3} - 1$. And, in such a case, all these three two-dimensional irreducible characters of $G(r, q, n)$ are quasi 2-Steinberg characters.

Case 2: $p \mid n$ is studied using similar types of arguments.

Thank you