Holomorphic Integer Graded Vertex Superalgebras

Jethro van Ekeren (UFF)
Bely Rodriguez Morales (USP)

April 2022

Contents

Lie Superalgebras

Definition (Lie Superalgebras)

A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$[a,b] \in \mathfrak{g}_{p(a)+p(b)},$$

$$[a,b] = -p(a,b)\,[b,a]\,,$$

$$[[a,b]\,,c] = [a,[b,c]] - p(a,b)\,[b,[a,c]]\,,$$

where a, b have degrees $p(a), p(b)$ respectively and $p(a,b) = (-1)^{p(a)p(b)}$.

Lie Superalgebras

Definition (Killing Form)

For $\mathfrak{g}$, a finite dimensional Lie superalgebra, the Killing form is

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

$$\kappa(a, b) = STr_{\mathfrak{g}} ad(a) ad(b)$$

Lie Superalgebras

Definition (Killing Form)

For $\mathfrak{g}$, a finite dimensional Lie superalgebra, the Killing form is

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

$$\kappa(a, b) = STr_{\mathfrak{g}} ad(a) ad(b)$$

- Invariant: $\kappa([a, b], c) = \kappa(a, [b, c])$
- Supersymmetric: $\kappa(a, b) = p(a, b)\kappa(b, a)$
- Consistent: $\kappa(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}) = 0$

Lie Superalgebras

Definition (Killing Form)

For $\mathfrak{g}$, a finite dimensional Lie superalgebra, the Killing form is

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

$$\kappa(a,b) = STr_{\mathfrak{g}} ad(a) ad(b)$$

- Invariant: $\kappa([a,b],c) = \kappa(a,[b,c])$
- Supersymmetric: $\kappa(a,b) = p(a,b)\kappa(b,a)$
- Consistent: $\kappa(\mathfrak{g}_{\bar{0}},\mathfrak{g}_{\bar{1}}) = 0$

The Killing form of a simple Lie superalgebra might be degenerated.

Lie Superalgebras

Proposition

If $\mathfrak{g}$ be a finite dimensional Lie superalgebra whose Killing form is non-degenerated, then it decomposes into a direct sum of simple Lie superalgebras with non-degenerated Killing form.

Lie Superalgebras

Proposition

If $\mathfrak{g}$ be a finite dimensional Lie superalgebra whose Killing form is non-degenerated, then it decomposes into a direct sum of simple Lie superalgebras with non-degenerated Killing form.

Definition

A simple Lie superalgebra $\mathfrak{g}$ is said to be of classical type if $\mathfrak{g}_{\bar{1}}$ is a completely reducible $\mathfrak{g}_{\bar{0}}$-module.

Lie Superalgebras

Proposition

If $\mathfrak{g}$ be a finite dimensional Lie superalgebra whose Killing form is non-degenerated, then it decomposes into a direct sum of simple Lie superalgebras with non-degenerated Killing form.

Definition

A simple Lie superalgebra $\mathfrak{g}$ is said to be of classical type if $\mathfrak{g}_{\bar{1}}$ is a completely reducible $\mathfrak{g}_{\bar{0}}$-module.

Proposition

If $\mathfrak{g}$ is a finite dimensional simple Lie superalgebra with non-degenerated Killing form then $\mathfrak{g}$ is of classical type.

Kac's Lie Superalgebras Classification

The finite dimensional simple Lie superalgebras fall into the following classes

Classical type with non degenerated Killing form: $A(m,n)$ with $m \neq n$, $B(m,n)$, $C(n)$, $D(m,n)$ with $m \neq n-1$, $F(4)$, $G(3)$.

Kac's Lie Superalgebras Classification

The finite dimensional simple Lie superalgebras fall into the following classes

Classical type with non degenerated Killing form: $A(m,n)$ with $m \neq n$, $B(m,n)$, $C(n)$, $D(m,n)$ with $m \neq n-1$, $F(4)$, $G(3)$.

Classical type with vanishing Killing form: $A(n,n)$, $D(n+1,n)$, $D(2,1;\alpha)$, $P(n)$, $Q(n)$.

Kac's Lie Superalgebras Classification

The finite dimensional simple Lie superalgebras fall into the following classes

Classical type with non degenerated Killing form: $A(m,n)$ with $m \neq n$, $B(m,n)$, $C(n)$, $D(m,n)$ with $m \neq n-1$, $F(4)$, $G(3)$.

Classical type with vanishing Killing form: $A(n,n)$, $D(n+1,n)$, $D(2,1;\alpha)$, $P(n)$, $Q(n)$.

Non classical type with vanishing Killing form: $W(n)$, $S(n)$, $H(n)$, $\widetilde{S}(n)$.

Kac's Lie Superalgebras Classification

The finite dimensional simple Lie superalgebras fall into the following classes

Classical type with non degenerated Killing form: $A(m,n)$ with $m \neq n$, $B(m,n)$, $C(n)$, $D(m,n)$ with $m \neq n-1$, $F(4)$, $G(3)$.

Classical type with vanishing Killing form: $A(n,n)$, $D(n+1,n)$, $D(2,1;\alpha)$, $P(n)$, $Q(n)$.

Non classical type with vanishing Killing form: $W(n)$, $S(n)$, $H(n)$, $\widetilde{S}(n)$.

Where $B(m,n) := \mathbf{osp}(2m+1,2n), m \geq 0, n \geq 1$ and

$$\mathbf{osp}_{\bar{0}}(2m+1,2n) := \left\{X \in \mathfrak{gl}_{\bar{0}}(2m+1,2n) : (Xu,v) = -(u,Xv)\right\}$$

$$\mathbf{osp}_{\bar{1}}(2m+1,2n) := \left\{X \in \mathfrak{gl}_{\bar{1}}(2m+1,2n) : (Xu,v) = -(-1)^{p(u)}(u,Xv)\right\}$$

Vertex superalgebras

Definition

A vertex superalgebra is a super vector space V, a distinguished vector $\mathbb{1} \in V_{\bar{0}}$ and linear map

$$Y_z : V \to \mathbf{Field}(V), \quad v \mapsto Y(v,z),$$

such that the following axioms are satisfied:

(vacuum axiom) $Y(\mathbb{1},z) = \mathbf{id}$, $Y(v,z)\mathbb{1} \in V[\![z]\!]$, $Y(v,z)\mathbb{1}|_{z=0} = v$;

(translation invariance) $[T, Y(v,z)] = \partial_z Y(v,z)$;

(locality axiom) $\left[Y_{z_1}(v_1), Y_{z_2}(v_2)\right](z_1 - z_2)^N = 0$ for $N >> 0$.

Where the translation endomorphism $T \in \mathbf{End}(V)$ is defined as $Tv = \partial_z Y(v,z)\mathbb{1}|_{z=0}$.

Conformal Field Theory Type Algebras

Definition

A conformal structure on V of central charge c consist of a vector $\omega \in V$ whose modes $L_n = \omega_{(n+1)}$ satisfy the Virasoro relations,

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c,$$

the action of L_0 on V is semisimple with finite dimensional eigenspaces (write V_n for the eigenspace with eigenvalue n) and $L_{-1} = T$.

Conformal Field Theory Type Algebras

Definition

A conformal structure on V of central charge c consist of a vector $\omega \in V$ whose modes $L_n = \omega_{(n+1)}$ satisfy the Virasoro relations,

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c,$$

the action of L_0 on V is semisimple with finite dimensional eigenspaces (write V_n for the eigenspace with eigenvalue n) and $L_{-1} = T$.

Definition

A conformal vertex V algebra is said to be of CFT type (conformal field theory type) if

$V_n = 0$ for $n < 0$,

$V_0 = \mathbb{C}\mathbb{1}$,

Conformal Field Theory Type Algebras

Proposition

Let $V = \bigoplus_{n \in \mathbb{Z}_+} V_n$ be a conformal vertex super algebra of CFT type then the product

$$\cdot_{(0)} \cdot : V_1 \otimes V_1 \to V_1$$
$$a \otimes b \mapsto a_{(0)} b$$

equips V_1 with a Lie superalgebra structure.

Definition

A vertex algebra is C_2-cofinite if the subspace $V_{(-2)}V$ has finite codimension.

Definition

A vertex algebra is C_2-cofinite if the subspace $V_{(-2)}V$ has finite codimension.

Proposition

Let V be a $\mathbb{Z}$-graded conformal vertex superalgebra. If V is simple, of CFT type and $L_1 V_1 = 0$ and

$$\langle u, v\rangle = p(u,v)u_{(1)}v$$

defines a nondegenerated invariant bilinear form.

In this context invariant means that for all $a, u, v \in V$ we have

$$(Y(a,z)u, v) = p(a,u)\left(u, Y(e^{zL_1}(-z^{-2})^{L_0})v\right).$$

Holomorphic $\mathbb{Z}$-graded vertex superalgebras

A vertex superalgebra is said to be self-contragredient if it possesses a nondegenerated invariant bilinear form.

Holomorphic $\mathbb{Z}$-graded vertex superalgebras

A vertex superalgebra is said to be self-contragredient if it possesses a nondegenerated invariant bilinear form.

Definition

Holomorphic $\mathbb{Z}$-graded vertex superalgebras Let V be a $\mathbb{Z}$-graded conformal vertex superalgebra. We call V *holomorphic* if it is *self-contragredient*, *C_2-cofinite* and *rational* and if, moreover *the unique irreducible ordinary V-module is the adjoint module V itself* (in particular V is simple).

Problem

Classify holomorphic integer graded vertex superalgebras at small central charge.

Problem

Classify holomorphic integer graded vertex superalgebras at small central charge.

Compute the possible Lie superalgebras that might appear as the V_1 part.

Problem

Classify holomorphic integer graded vertex superalgebras at small central charge.

Compute the possible Lie superalgebras that might appear as the V_1 part.

Proposition

The central charge must be a multiple of 8.

Main Result. Case $c = 24$

Theorem

Let V be a holomorphic $\mathbb{Z}$-graded vertex superalgebra of central charge 24 for which $sdim(V_1) \neq 24$. Then V_1 is either zero or is isomorphic to a finite sum

$$\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$$

of simple Lie superalgebras, each even or of type $B(0, n) = osp(1, 2n)$. Moreover there are at most 1332 possible Lie superalgebras for V_1.

Sketch of the proof

Sketch of the proof

- Prove that if $sdim(V_1) \neq 24$ then the Killing form on V_1 is nondegenerated and therefore V_1 decomposes into a sum of simple Lie superalgebras.

Sketch of the proof

- Prove that if $sdim(V_1) \neq 24$ then the Killing form on V_1 is nondegenerated and therefore V_1 decomposes into a sum of simple Lie superalgebras.
- Prove the formula $\frac{h_i^\vee}{k_i} = \frac{sdim(V_1)}{24}$ for all $i = 1, \ldots, r$.

Sketch of the proof

- Prove that if $sdim(V_1) \neq 24$ then the Killing form on V_1 is nondegenerated and therefore V_1 decomposes into a sum of simple Lie superalgebras.
- Prove the formula $\frac{h_i^\vee}{k_i} = \frac{sdim(V_1)}{24}$ for all $i = 1, \ldots, r$.
- Find a bounds for the superdimension of V_1, for instance, we found $0 \leq sdim(V_1) \leq 1344$

Sketch of the proof

- Prove that if $sdim(V_1) \neq 24$ then the Killing form on V_1 is nondegenerated and therefore V_1 decomposes into a sum of simple Lie superalgebras.
- Prove the formula $\dfrac{h_i^\vee}{k_i} = \dfrac{sdim(V_1)}{24}$ for all $i = 1, \ldots, r$.
- Find a bounds for the superdimension of V_1, for instance, we found $0 \leq sdim(V_1) \leq 1344$
- Develop a script using a computer (for instance using Python) to find a list of all possible decompositions for V_1 as a direct sum of simple Lie superalgebras.

Cases $c = 8$, $c = 16$

Theorem

Let V be a holomorphic $\mathbb{Z}$-graded vertex superalgebra of central charge $c = 8$ or $c = 16$ then V is purely even.

Next steps...

Reduce the list of those 1332 Lie superalgebras for $V1$.

Next steps...

Reduce the list of those 1332 Lie superalgebras for $V1$.

Analyze the case of super dimension 24.

Next steps...

Reduce the list of those 1332 Lie superalgebras for $V1$.

Analyze the case of super dimension 24.

Produce non trivial examples (non purely even) of holomorphic integer graded vertex superalgebras.

Next steps...

Reduce the list of those 1332 Lie superalgebras for $V1$.

Analyze the case of super dimension 24.

Produce non trivial examples (non purely even) of holomorphic integer graded vertex superalgebras.

Very ambitious goal

Classify holomorphic integer graded vertex superalgebras at a certain central charge.

End of the presentation

Thank you

PROJECT SUPPORTED BY

CAPES FAPERJ CNPq serrapilheira