On the representation theory of affine vertex algebras on conformal and collapsing levels

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Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

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Motivation

- Study representation theory of affine VOA $L_k(\mathfrak{g})$.
- Idea: use affine W-algebras to study $L_k(\mathfrak{g})$.

  New concepts/constructions
- Collapsing levels
- Semi-simplicity of KL_k for k beyond admissible
- Free-field realizations motiviated by inverses of QHR (Rio talk)

Affine vertex and W–algebras

- $\mathfrak{g}$ simple Lie (super)algebra over $\mathbb{C}$.
- $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$ the affine Kac–Moody Lie algebra.
- $V^k(\mathfrak{g})$ universal affine VOA of level k (k is not critical).
- As $\hat{\mathfrak{g}}$–module $V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0} + \mathbb{C}K)} \mathbb{C}.1$.
- $L_k(\mathfrak{g})$ simple quotient of $V^k(\mathfrak{g})$
- $L^{\mathfrak{g}}$ Sugawara Virasoro vector in $L_k(\mathfrak{g})$ of central charge

$$c(sug) = \frac{k \mathrm{sdim} \mathfrak{g}}{k + h^{\vee}}.$$

- Let V are VOA with conformal vector ω_V, U subVOA with conformal vector ω_U. U is conformally embedded into V if

$$\omega_U = \omega_V.$$

Affine vertex and W–algebras

- For f nilpotent element in $\mathfrak{g}$ and $k \in \mathbb{C}$ one associate the universal affine W-algebra $W^k(\mathfrak{g}, f)$ as $H_f(V^k(\mathfrak{g}))$ where H_f is quantum Hamiltonian reduction function
- $W_k(\mathfrak{g}, f)$ simple quotient of $W^k(\mathfrak{g}, f)$.
- Let $\mathcal{V}(\mathfrak{g}^\natural)$ be the affine vertex subalgebra of $W_k(\mathfrak{g}, f)$.
- If $W_k(\mathfrak{g}, f)$ collapses to its affine subalgebra $\mathcal{V}(\mathfrak{g}^\natural)$, we say that k is a collapsing level.
- If $\mathcal{V}(\mathfrak{g}^\natural)$ is conformally embedded in $W_k(\mathfrak{g}, f)$ we say that k is a conformal level .
- Note: each collapsing level is conformal.

Construction and classification of collapsing levels

- The case of minimal nilpotent $f = f_\theta$ [D.A, Kac, Moseneder, Papi, Perše '18] (Method: using KW λ–bracket for $W^k(\mathfrak{g}, f_\theta)$)
- In general, OPE formulas for $W^k(\mathfrak{g}, f)$ are not completely known. Studying general cases requires different approaches:
- f general, k admissible [Arakawa, van Ekeren, Moreau '21]
- k general, f of hook and rectangular type (case A) [D.A, Moseneder, Papi '22]
- Some cases based on explicit OPE [D.A, Perše, Vukorepa '21], [Fasquel '21]
- More to be done

Conformal and collapsing levels: the case $f = f_\theta$

The central charge of minimal affine W–algebra $W_k(\mathfrak{g}, f_\theta)$ is
$c(\mathfrak{g}, k, f_\theta) = \frac{\text{sdim}\,\mathfrak{g}}{k+h^\vee} - 6k + h^\vee - 4.$

Theorem (D.A, Kac, Moseneder, Papi, Perše '18)

- *The embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, f_\theta)$ is conformal if and only if $c_{\mathfrak{g}^\natural} = c(\mathfrak{g}, k, f_\theta)$ where $c_{\mathfrak{g}^\natural}$ is the Sugawara central charge of $\mathcal{V}(\mathfrak{g}^\natural)$.*
- *Assume that k is conformal and non-collapsing, then*

$$k = -\frac{2}{3}h^\vee \quad \text{or} \quad k = -\frac{h^\vee - 1}{2}.$$

- *k is collapsing if and only if $p(k) = 0$ for certain quadratic polynomial p.*

Collapsing levels: the case $f = f_\theta$

The polynomial $p(k)$

$\mathfrak{g}$	$p(k)$	$\mathfrak{g}$	$p(k)$
$sl(m\|n)$, $n \neq m$	$(k+1)(k+(m-n)/2)$	E_6	$(k+3)(k+4)$
$psl(m\|m)$	$k(k+1)$	E_7	$(k+4)(k+6)$
$osp(m\|n)$	$(k+2)(k+(m-n-4)/2)$	E_8	$(k+6)(k+10)$
$spo(n\|m)$	$(k+1/2)(k+(n-m+4)/4)$	F_4	$(k+5/2)(k+3)$
$D(2,1;a)$	$(k-a)(k+1+a)$	G_2	$(k+4/3)(k+5/3)$
$F(4)$, $\mathfrak{g}^\natural = so(7)$	$(k+2/3)(k-2/3)$	$G(3)$, $\mathfrak{g}^\natural = G_2$	$(k-1/2)(k+3/4)$
$F(4)$, $\mathfrak{g}^\natural = D(2,1;2)$	$(k+3/2)(k+1)$	$G(3)$, $\mathfrak{g}^\natural = osp(3\|2)$	$(k+2/3)(k+4/3)$

Classification of conformal levels: the general case

Let $c(\mathfrak{g}, k, f)$ denotes the central charge of $W^k(\mathfrak{g}, f)$. Let L be conformal vector in $W^k(\mathfrak{g}, f)$ and $L^{\mathfrak{g}^\natural}$ Sugawara conformal vector in its affine vertex subalgebra.

Theorem (D.A, Moseneder, Papi '22)

Assume that $W^k(\mathfrak{g}, f)$ is generated by $\mathfrak{g}^\natural$ and by

$$(L - L^{\mathfrak{g}^\natural}) \bigcup S$$

with S homogeneous such that

$$(L - L^{\mathfrak{g}^\natural})(2)X = 0, \quad \textit{if } X \in S \textit{ and } L(0)X = 2X.$$

Then $\mathcal{V}(\mathfrak{g}^\natural)$ is conformally embedded into $W_k(\mathfrak{g}, f)$ if and only if

$$c_{\mathfrak{g}^\natural} = c(\mathfrak{g}, k, f).$$

Example: Hook affine W-algebras of type A

- Consider W-algebra $W_k(\mathfrak{g}, f_{m,n})$ for $\mathfrak{g} = sl(m+n)$,
- The partition representing the nilpotent element $f_{m,n}$ is the hook $(m, 1^n)$
- $\mathfrak{g}^\natural = gl(n)$. $\mathcal{V}(\mathfrak{g}^\natural)$ is certain quotient of $V^{k+m-1}(gl(n))$.
- In [D.A, Moseneder, Papi '22] we prove

(1) The embedding $\mathcal{V}(\mathfrak{g}^\natural) \hookrightarrow W_k(\mathfrak{g}, f_{m,n})$ is conformal if and only if

$$k = k^{(i)}_{m,n}, \quad 1 \le i \le 4,$$

where $k^{(1)}_{m,n} = -\frac{m}{m+1} h^\vee$ $(n > 1)$, $k^{(2)}_{m,n} = -\frac{(m-1)h^\vee - 1}{m}$ $(n \ge 1)$, $k^{(3)}_{m,n} = -\frac{(m-2)h^\vee + 1}{m-1}$ $(n \ge 1,\ m > 1)$, $k^{(4)}_{m,n} = -\frac{(m-1)h^\vee}{m}$.

(2) Levels $k^{(3)}_{m,n}$ $(m \neq n-1)$ and $k^{(4)}_{m,n}$ are collapsing.

Collapsing vs conformal levels

- Collapsing level is always conformal.
- Problem: Determine when certain conformal level is collapsing or non-collapsing.
- In the case $m = 2$ (minimal nilpotent case) we know that $k_{m,n}^{(i)}$ is non-collapsing iff $i = 1, 2$.
- In the hook case $m \geq 3$, we can prove that $k = k_{m,n}^{(i)}$, for $i = 1, 2$ is non-collapsing only if k is admissible.
- We conjecture that $k = k_{m,n}^{(1)}$ is always non-collapsing.
- $k = k_{m,n}^{(2)}$ is sometimes collapsing, sometimes non-collapsing.

Level $k^{(1)}_{p-1,2}$ and $\mathcal{R}^{(p)}$–algebra.

- In [D.A. '16] we introduce logarithmic vertex algebra $\mathcal{R}^{(p)}$, which is an infinite-direct sum of $L_{-2+\frac{1}{p}}(gl(2))$–modules.
- It was proved in [D.A, Creutzig, Genra, Yang '21] that

$$W_{k^{(1)}_{p-1,2}}(sl(p+2), f_{p-1,2}) \cong \mathcal{R}^{(p)}.$$

- $\implies$ $k^{(1)}_{p-1,2}$ is non-collapsing.
- We proved that $k^{(2)}_{3p,2} = -3p - 1 + \frac{1}{p}$ is collapsing by using a tensor category/ fusion rules argument.
- Note that main difference is that $k^{(1)}_{p-1,2} = -\frac{p^2-1}{p}$ is admissible for $sl(p+1)$, while $k^{(2)}_{3p,2} = -3p - 1 + \frac{1}{p}$ is not admissible for $sl(3p+2)$.

Theorem

Assume that $k = k_{m,n}^{(i)}$ *for* $i \in \{1, 2\}$ *is admissible for* $sl(m+n)$, $n \geq 3$. *Then*

$$W_k = W_k(\mathfrak{g}, f_{m,n}) = \bigoplus_{i \in \mathbb{Z}} W_k^{(i)},$$

and each $W_k^{(i)} = \{v \in W_k \mid J(0)v = iv\}$ *is an irreducible* $W_k^{(0)}$*–module:*

- $W_k^{(i)} = L_{k_1}^{sl(n)}(i\omega_1) \otimes M(k_0, i)$ *if* $i \geq 0$,
- $W_k^{(i)} = L_{k_1}^{sl(n)}(-i\omega_{n-1}) \otimes M(k_0, i)$ *if* $i < 0$.

In particular, $\mathcal{V}(\mathfrak{g}^\natural) \cong W_k(\mathfrak{g}, f_{m,n})^{(0)} = V(sl(n)) \otimes V^{k_0}(\mathbb{C}J)$ *is a simple vertex algebra which is conformally embedded in* $W_k(\mathfrak{g}, f_{m,n})$.

Remark.

Note that level k_1 *is not admissible for* $sl(n)$, *and that the above theorem implies that* $L_{k_1}^{sl(n)}(i\omega_1)$, $L_{k_1}^{sl(n)}(-i\omega_{n-1})$ *are* $L_{k_1}(sl(n))$*–modules. We believe that these modules provide a complete list of* $L_{k_1}(sl(n))$*–modules in the category of ordinary modules.*

The category KL_k

- A $V^k(\mathfrak{g})$–module M is in KL^k if

(1) M is locally finite as a $\mathfrak{g}$–module;

(2) M admits decomposition into generalized eigenspaces for $L^{\mathfrak{g}}(0)$ whose eigenvalues are bounded below.

- Category KL_k: $L_k(\mathfrak{g})$–modules which are in KL^k.
- For $\mathfrak{g}$ Lie superalgebra, we introduce KL_k^{fin}, subcategory of KL_k consists of weight modules.
- Semi-simpilicty of KL_k and KL_k^{fin} [D.A-Kac-Moseneder-Papi-Perše '18]
- Tensor category of KL_k modules [Creutzig-Yang '21]
- But $L_k(\mathfrak{g})$ usually has weak modules outside KL_k (Tomoyuki talk)

Semi-simplicity of KL_k

We prove the following results on complete reducibility result in KL_k

Theorem (AKMPP, 2018)

Assume that $\mathfrak{g}$ is a simple Lie algebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Then KL_k is a semi-simple category in the following cases:

- *k is a collapsing level.*
- *$W_k(\mathfrak{g}, f_\theta)$ is a rational vertex operator algebra.*
- *$W_k(\mathfrak{g}, f_\theta)$ has semi-simple category of ordinary modules.*

Theorem (AMP, 2021)

Assume that $\mathfrak{g}$ is a simple Lie superalgebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Then KL_k^{fin} is a semi-simple category in the following cases:

- *k is a collapsing level.*
- *$W_k(\mathfrak{g}, f_\theta)$ is a rational vertex operator superalgebra.*

When $KL_k = KL_k^{fin}$?

Example $L_1(\mathfrak{gl}(1|1))$ shows that in general $KL_k \neq KL_k^{fin}$.

Theorem

Assume that KL_k^{fin} is semi-simple and that for any irreducible $L_k(\mathfrak{g})$–module M in KL_k we have

$$Ext^1(M_{top}, M_{top}) = \{0\} \tag{1}$$

in the category of finite-dimensional $\mathfrak{g}$–modules. Then KL_k is semisimple and $KL_k = KL_k^{fin}$.

Applications of theorem require:

- Classification of irreducible modules in KL_k
- Identification of top components as irreducible, highest weight $\mathfrak{g}$–modules.
- Study extensions of irreducible, finite-dimensional modules for Lie superalgebras.

The category KL_{-1} for $\mathfrak{g} = sl(m|1)$

- Level $k = -1$ is collapsing $\implies$ KL_k^{fin} is semisimple.
- We need to show $Ext^1(M, M) = \{0\}$ for any irreducible module M in KL_k.
- So we need to exclude non-split self extensions

$$0 \to M \to M^{ext} \to M \to 0$$

  such that M^{ext} is non-weight and/or logarithmic module in KL_k.
- We prove that top components of irreducible modules in KL_k are singly atypical $\mathfrak{g}$–modules.
- It was proved in [Germoni '98] that singly atypical modules don't have non-trivial self-extensions in the category of finite-dimensional $\mathfrak{g}$–modules.
- $\implies$ KL_k is semi-simple.

$\mathfrak{g} = sl(2|1)$

- Let $k = -(m+1)/(m+2)$, $m \in \mathbb{Z}_{\geq 0}$.
- $W_k(\mathfrak{g}, f_\theta)$ is a rational $N = 2$ superconformal algebra [D.A, 2001].
- $\implies KL_k^{fin}$ is semisimple.
- $\mathfrak{g}_{\bar{0}} = sl(2) \times \mathbb{C}z$, where z is center of $\mathfrak{g}_{\bar{0}}$.
- We prove that the center of $\mathfrak{g}_{\bar{0}}$ belongs to a rational vertex algebra $D_{m+1,2} \subset L_k(\mathfrak{g})$, and we have conformal embedding

$$\mathcal{V}(sl(2)) \otimes D_{m+1,2} \hookrightarrow L_k(\mathfrak{g}).$$

- $\implies KL_k$ is semi-simple.

Examples when KL_k is not semisimple

- Let $\mathfrak{g} = sl(m|1)$, $k = 1$.
- Kac-Wakimoto realization of $L_k(\mathfrak{g}) \hookrightarrow \mathcal{S} \otimes F_m$,

  $\mathcal{S}$ is Weyl vertex algebra of rank 1 ($\beta\gamma$ system), which is generated by fields $a^{\pm}$ such that
- $[a^{\pm}_{\lambda} a^{\pm}] = 0, \quad [a^{+}_{\lambda} a^{-}] = \mathbf{1}$.
- F_m the Clifford vertex algebra of rank m (bc–system), generated by fermionic fields $\Psi_i^{\pm}$, $i = 1, \dots, m$.
- The Weyl vertex algebra $\mathcal{S}$ can be embedded into a lattice type vertex algebra $\Pi(0)$ such that negative powers $(a^+)^{-m}$ of a^+ belong to $\Pi(0)$ (localisation).

Examples when KL_k is not semisimple

Theorem

Define $\widetilde{w} := (a^+)^{-m} \otimes : \Psi_1^+ \cdots \Psi_m^+ :\in \Pi(0) \otimes F_m$. *Then:*

- $\widetilde{W} = L_1(\mathfrak{g})\widetilde{w}$ *is a highest weight* $L_1(\mathfrak{g})$*–module in the category* KL_k^{fin}.
- $\widetilde{W}$ *is reducible and it contains a proper submodule isomorphic to* $L_1(\mathfrak{g})$.

In particular, the category KL_k^{fin} *is not semisimple for* $k = 1$.

Examples when KL_k is not semisimple

- Let now $k \in \mathbb{Z}_{>0}$ is arbitrary.
- In [Gorelik-Serganova '18] the authors proved that $L_k(\mathfrak{g}) = V^k(\mathfrak{g})/I$, where I is the ideal in $V^k(\mathfrak{g})$ generated by the singular vector $e_\theta(-1)^{k+1}\mathbf{1}$.
- Applying this together with previous theorem we get:

Theorem

The category KL_k^{fin} is not semisimple for any $k \in \mathbb{Z}_{>0}$.

Conjecture

Let $\mathfrak{g} = sl(2|1)$. The category KL_k is semisimple if and only if $k \in \{-1, -\frac{m+1}{m+2} \mid m \in \mathbb{Z}_{\geq 0}\}$.

Main literature

- D. A. V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, An application of collapsing levels to the representation theory of affine vertex algebras, IMRN (2020)
- D. A., P. Möseneder Frajria, P. Papi, On the semisimplicity of the category KL_k for affine Lie superalgebras, arXiv:2107.12105 [math.RT]
- D. A., P. Möseneder Frajria, P. Papi, New approaches for studying conformal embeddings and collapsing levels for W–algebras arXiv:2203.08497[math.RT]

Thank you