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$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie (super)algebras and detectable parastatististics

Talk given at:
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(logo of the CNPq group: **Algebraic Structures in Field Theory**)

Talk based on two papers:

o-o-o

F. T., *$\mathbb{Z}_2 \times \mathbb{Z}_2$-parastatistics in multiparticle quantum Hamiltonians,*
J. Phys. A: Math. Theor. 54, 115203 (2021); arXiv:2008.11554[hep-th].

Comment: **$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded paraparticles are theoretically observable**

o-o-o

F. T., *Inequivalent quantizations from gradings and $\mathbb{Z}_2 \times \mathbb{Z}_2$-parabosons,*
J. Phys. A: Math. Theor. 54, 355202 (2021); arXiv:2104.09692[hep-th].

Comment: **Fundamental ambiguity in quantization: a given single particle quantum Hamiltonian implies inequivalent multiparticle sectors induced by gradings.**

Framework:

"Color" Lie algebras and superalgebras introduced in

- V. Rittenberg and D. Wyler, *Generalized Superalgebras*, Nucl. Phys. **B 139**, 189 (1978).
- V. Rittenberg and D. Wyler, *Sequences of $Z_2 \otimes Z_2$ graded Lie algebras and superalgebras*, J. Math. Phys. **19**, 2193 (1978).
- M. Scheunert, *Generalized Lie algebras*, J. Math. Phys. **20**, 712 (1979).

Parastatistics recovered from graded Hopf algebras endowed with a braided tensor product:

- S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge (1995).

- In the '80s, $\mathbb{Z}_2 \times \mathbb{Z}_2$-superalgebras received some attention from physicists (but no systematic investigation of their properties, e.g. $\mathbb{Z}_2 \times \mathbb{Z}_2$-fields, ...):
Lukierski, Vasiliev, Tolstoy, Jarvis, Yang, Wybourne, Zheltukhin, Wills-Toro, ...

- Mathematicians continued to investigate them (Scheunert, ...).

- From 2000 their connection with parastatistics started being investigated:
Yang, Jing, Kanakoglou, Daskaloyannis, Tolstoy, Stoilova, Van der Jeugt

- Renewed attention to Rittenberg-Wyler $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras since

N. Aizawa, Z. Kuznetsova, H. Tanaka and F. Toppan, *$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie symmetries of the Lévy-Leblond equations*, Prog. Theor. Exp. Phys. **2016**, 123A01 (2016); arXiv:1609.08224[math-ph].

Comment: $\mathbb{Z}_2 \times \mathbb{Z}_2$–graded superalgebras are dynamical symmetries of a well-known system of PDEs, describing the nonrelativistic Lévy-Leblond spinors.

- Current wave:
N. Aizawa, K. Amakawa, S. Doi, Z. Kuznetsova, J. Segar, A. J. Bruce, S. Duplij, J. Grabowski, N. Poncin, E. Ibarguengoytia, J. Van der Jeugt, N. Stoilova, P. S. Isaac, C. Quesne, ...

$\mathbb{Z}_2 \times \mathbb{Z}_2$-**graded superalgebras**

versus

$\mathbb{Z}_2 \times \mathbb{Z}_2$-**graded algebras:**

the difference is in the assignment of

commutators, $[A, B] = AB - BA$**,**

and

anticommutators, $\{A, B\} = AB + BA$**.**

Ordinary physics

	0	1
0	$[\cdot,\cdot]$	$[\cdot,\cdot]$
1	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$

In ordinary physics we deal with two types of particles, bosons and fermions, which are accommodated in 1 bit of information:

- bosons (0),
- fermions (1).

Comment: the anticommutator $\{\cdot,\cdot\}$ encodes the Pauli exclusion principle for fermions.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras

	00	10	01	11
00	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$[\cdot,\cdot]$
10	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$
01	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$
11	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$	$[\cdot,\cdot]$

Comment. In $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra physics the particles are accommodated in 2 bits of information:

- **ordinary bosons (00),**
- **exotic bosons (11),**
- **parafermions of (10) type,**
- **parafermions of (01) type.**

Comment

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra physics
is an extension of ordinary physics:

- ordinary bosons are accommodated in the 00-graded sector,

- ordinary fermions are accommodated in the 10-graded sector,

- the graded sectors 01- and 11- are empty.

Remark: the 10- and 01- sectors are on equal footing.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebras

	00	10	01	11
00	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$[\cdot,\cdot]$
10	$[\cdot,\cdot]$	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$
01	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$
11	$[\cdot,\cdot]$	$\{\cdot,\cdot\}$	$\{\cdot,\cdot\}$	$[\cdot,\cdot]$

Comment. In $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra physics the particles are accommodated in 2 bits of information:

- ordinary bosons (00),
- parabosons of (11) type,
- parabosons of (10) type,
- parabosons of (01) type.

Comment

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebra physics applies
to models which do not contain fermions:

- the ordinary bosons are accommodated in the 00-graded sector,
- the three types of parabosons are accommodated in the remaining 11-, 10-, 01- graded sectors.

Remark 1: if the 10- and 01- graded sectors are empty, 00-bosons and 11-parabosons are indistinguishable.

Remark 2: the 10- , 01- and 11- sectors are on equal footing.

Important Question

Even if the 2-bit "colored" particles (00,10,01,11) are an extension of ordinary physics, can we observe their color?

Stated otherwise: is there a quantum measurement that cannot be mimicked by black-white (0,1) pictures of composite bosons/fermions?

A positive answer would prove that

- **Rittenberg-Wyler (super)algebras can play a role in physics**
- $\mathbb{Z}_2 \times \mathbb{Z}_2$**–graded paraparticles can in principle be detected**

The answer is given by investigating a toy model case

This question became relevant when Bruce-Duplij in 2020 produced a 4×4 matrix quantum Hamiltonian which is both an example of supersymmetric quantum mechanics and invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$-one-dimensional Poincaré superalgebra:

$$H = \frac{1}{2}\begin{pmatrix} -\partial_x^2 + W^2 + W' & 0 & 0 & 0 \\ 0 & -\partial_x^2 + W^2 + W' & 0 & 0 \\ 0 & 0 & -\partial_x^2 + W^2 - W' & 0 \\ 0 & 0 & 0 & -\partial_x^2 + W^2 - W' \end{pmatrix}$$

with $W \equiv W(x)$ and $W' \equiv \frac{d}{dx} W(x)$.

The systematic construction of classical $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant model was given in Aizawa-Kuznetsova-F.T. in 2020 and the quantization of these models in 2021.

The question was formulated as such: is there a new physics implied by the $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariance or is it just a nice redundant feature with no observationally measurable consequences?

Connection with parastatistics

For a single-particle quantum Hamiltonian the grading is just a conventional label void of any physical significance.
For a multi-particle quantum Hamiltonians the statistics of the (para)particles plays a physical role: the multi-particle wave functions possess mixed symmetry.

One approach to parastatistics is based on 1953 Green's trilinear relations. Palev and Ganchev proved that trilinear relations are recovered from graded Jacobi identies of certain Lie superalgebras.

Yang-Jing, followed by Tolstoy and by Stoilova-Van der Jeugt investigated $\mathbb{Z}_2 \times \mathbb{Z}_2$-parastatistics in the context of trilinear relations.

An alternative (more flexible) approach derives the parastatistics in the mixed-symmetry properties encoded in a Hopf algebra braided tensor products.

Kanakoglou and Daskaloyannis investigated $\mathbb{Z}_2 \times \mathbb{Z}_2$-parastatistics in the Hopf algebra context.

The connection between the Hopf algebras' and trilinear relations' approaches to parastatistics is discussed by Aneva-Popov and Kanakoglou-Daskaloyannis.

Basic properties of $\mathbb{Z}_2$-graded Lie algebras and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded color Lie (super)algebras

(comment: unified treatment with unified symbols)

The three (super)algebras under considerations are:

i) the $\mathbb{Z}_2$-graded Lie algebras,
ii) the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and
iii) the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras.

Each one of the above classes of Lie (super)algebras will be defined over the field of either real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers.

The round bracket (A, B) denotes:

either a commutator $[A, B] = AB - BA$
or an anticommutator $\{A, B\} = AB + BA$,

depending on the grading of the Lie (super)algebra generators A, B.

Gradings

Let A, B, C be three Lie (super)algebra generators. Their respective gradings are the vectors $\vec{\alpha} = deg(A)$, $\vec{\beta} = deg(B)$, $\vec{\gamma} = deg(C)$, where

case *i*:
$\vec{\alpha}^T = \vec{\alpha}$ is a 1-component vector, such that $\vec{\alpha} = (\alpha)$, with $\alpha \in \{0,1\}$;
cases *ii* and *iii*:
$\vec{\alpha}$ is a 2-component vector $\vec{\alpha}^T = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \in \{0,1\}$.

A inner product ($\vec{\alpha} \cdot \vec{\beta} \in \{0,1\}$) is defined for a given pair of $\vec{\alpha}$, $\vec{\beta}$ gradings. It is respectively given, for each of the three classes above, as

$$\begin{array}{rcl} \textbf{case } i): & \vec{\alpha}\cdot\vec{\beta} & := & \alpha\beta \in \{0,1\}, \\ \textbf{case } ii): & \vec{\alpha}\cdot\vec{\beta} & := & \alpha_1\beta_2 - \alpha_2\beta_1 \in \{0,1\}, \\ \textbf{case } iii): & \vec{\alpha}\cdot\vec{\beta} & := & \alpha_1\beta_1 + \alpha_2\beta_2 \in \{0,1\}, \end{array}$$

where the additions on the right hand sides are taken $\bmod 2$.

Definition of brackets (A, B)

$$\boxed{(A, B) := AB - (-1)^{\vec{\alpha}\cdot\vec{\beta}}BA,}$$

so that

$$(B, A) \quad = \quad (-1)^{\vec{\alpha}\cdot\vec{\beta}+1}(A, B).$$

The grading $deg((A, B))$ of the Lie (super)algebra generator (A, B) is

$$deg((A, B)) \quad = \quad \vec{\alpha} + \vec{\beta},$$

where, in each of the vector components, the sums are taken mod 2.

A graded Lie (super)algebra $\mathcal{G}$ is endowed with a $(\cdot, \cdot) : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ bracket defined for each A, B pair of generators in $\mathcal{G}$ ($A, B \in \mathcal{G}$).

A graded Lie (super)algebra satisfies, for any A, B, C triple of generators of $\mathcal{G}$, the graded Jacobi identity:

$$\boxed{(-1)^{\vec{\gamma}\cdot\vec{\alpha}}(A, (B, C)) + (-1)^{\vec{\alpha}\cdot\vec{\beta}}(B, (C, A)) + (-1)^{\vec{\beta}\cdot\vec{\gamma}}(C, (A, B)) = 0.}$$

To properly (anti)symmetrize bosons and fermions in the language of the coproduct, the notion of braided tensor (which naturally incorporates a braid statistics) has to be used.

The $\mathbb{Z}_2$-grading is the simplest non-trivial example of braiding.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading is the next simplest case.

In a braided tensor product,

$$(U_A \otimes U_B)(U_C \otimes U_D) \quad = \quad U_A \Psi(U_B \otimes U_C) U_D,$$

U_B and U_C are braided by an operator Ψ acting on their tensor product; Ψ is called the "braiding operator".

In applications to the $\mathbb{Z}_2^{[p]}$, $p = 1, 2$ gradings, the braiding reads

$$\boxed{(U_A \otimes U_B)(U_C \otimes U_D) = (-1)^{\epsilon_B \cdot \epsilon_C} (U_A U_C) \otimes (U_B U_D)}$$

and corresponds to a sign.

For the creation operator $f^\dagger$, with $(f^\dagger)^2 = 0$:
in the bosonic interpretation (for $\epsilon_{f^\dagger} = 0$) the coproduct gives

$$\boxed{\Delta((f^\dagger)^2) = (\mathbf{1} \otimes f^\dagger + f^\dagger \otimes \mathbf{1})(\mathbf{1} \otimes f^\dagger + f^\dagger \otimes \mathbf{1}) = 2f^\dagger \otimes f^\dagger \neq 0.}$$

In the fermionic interpretation (for $\epsilon_{f^\dagger} = 1$) the coproduct gives

$$\begin{aligned} \Delta((f^\dagger)^2) &= (\mathbf{1} \otimes f^\dagger + f^\dagger \otimes \mathbf{1})(\mathbf{1} \otimes f^\dagger + f^\dagger \otimes \mathbf{1}) = f^\dagger \otimes \mathbf{1} \cdot \mathbf{1} \otimes f^\dagger + \mathbf{1} \otimes f^\dagger \cdot f^\dagger \otimes \mathbf{1} = \\ &= \boxed{f^\dagger \otimes f^\dagger - f^\dagger \otimes f^\dagger = 0.} \end{aligned}$$

The physical consequence is that the coproduct, combined with the braided tensor, encodes the Pauli exclusion principle for fermions.

The permutation of spaces for the tensor products $\mathcal{U} \otimes \ldots \otimes \mathcal{U}$ of a graded Universal Enveloping Lie superalgebra $\mathcal{U}(\mathfrak{g})$ are defined as:

$$S_{12}^{(2)} \quad : \quad U_A \otimes U_B \mapsto (-1)^{\epsilon_A \cdot \epsilon_B} U_B \otimes U_A, \quad (U_{A,B} \in \mathcal{U} \quad \text{and} \quad S_{12}^{(2)} \cdot S_{12}^{(2)} = \mathbf{1}).$$

For the abstract Universal Enveloping Algebra $\mathcal{U}$, represented on a vector space V under the R representation, a hat denotes the action of the operators induced by the coproduct:

$$\text{for } R : \mathcal{U} \to V, \qquad \widehat{\Delta} := \Delta|_R \in End(V \otimes V), \qquad \text{with } \widehat{\Delta(U)} \in V \otimes V.$$

Multiparticle sectors

The $M > 1$ multiparticle Hilbert space $\mathcal{H}^{(M)}$ is a subset of tensor products of M single-particle Hilbert spaces:

$$\boxed{\mathcal{H}^{(M)} \subset \mathcal{H}^{(1)} \otimes \ldots \otimes \mathcal{H}^{(1)}, \qquad (\text{tensor product of } M \text{ spaces})}.$$

The coassociativity property allows to recursively determine $\Delta^{(M+1)}$ as

$$\Delta^{(M+1)} \quad = \quad (\mathbf{1} \otimes \Delta)\Delta^{(M)} = (\Delta \otimes \mathbf{1})\Delta^{(M)} \qquad (\text{with } \Delta^{(1)} \equiv \Delta),$$

where $\Delta^{(M)}$ maps $\mathcal{U}$ in the tensor product of $M+1$ spaces:

$$\boxed{\Delta^{(M)} \ : \ \mathcal{U} \to \mathcal{U}^{\otimes M+1}}.$$

Example: an M-particle bosonic vacuum $|vac\rangle^{(M)}$ is determined by the Fock conditions for the annihilation operator f:

$$\widehat{\Delta^{(M-1)}(f)}|vac\rangle^{(M)} = 0,$$
$$|vac\rangle^{(M)} = |vac\rangle \otimes \ldots \otimes |vac\rangle \in \mathcal{H}^{(M)}.$$

An excited state is created through

$$\boxed{\widehat{\Delta^{(M-1)}((f^{\dagger})^{r})}|vac\rangle^{(M)}}.$$

Comment:

In a non-interacting, first-quantized, multi-particle quantum theory, an additive observable like energy is encoded in the coproduct.

The coassociativity of the coproduct ensures the construction of the n-particle states:

$$1 \to 2 \to 3 \to \ldots \to n \to \ldots$$

$\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings induce inequivalent multiparticle quantizations

$6+3=9$ inequivalent <u>multiparticle</u> Hilbert spaces

induced by **6 standard** and **3 non-standard** gradings of a single-particle quantum Hamiltonian.

Toy-model example: a 4×4 matrix quantum oscillator

Once more about the 4×4 matrix Hamiltonian

$$\boxed{H_{osc} = \frac{1}{2}\begin{pmatrix} -\partial_x^2 + x^2 - 1 & 0 & 0 & 0 \\ 0 & -\partial_x^2 + x^2 - 1 & 0 & 0 \\ 0 & 0 & -\partial_x^2 + x^2 + 1 & 0 \\ 0 & 0 & 0 & -\partial_x^2 + x^2 + 1 \end{pmatrix}}.$$

It is also invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$-Lie algebra:

$$Q_{10} = \frac{-i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \partial_x - x & 0 \\ 0 & 0 & 0 & \partial_x - x \\ \partial_x + x & 0 & 0 & 0 \\ 0 & \partial_x + x & 0 & 0 \end{pmatrix},$$

$$Q_{01} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & \partial_x - x \\ 0 & 0 & \partial_x - x & 0 \\ 0 & -\partial_x - x & 0 & 0 \\ -\partial_x - x & 0 & 0 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The Hamiltonian H is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$-abelian Lie algebra $\mathfrak{a}$ defined by the following set of (all vanishing) 6 (anti)commutators:

$$[H, Q_{10}] = [H, Q_{01}] = [H, Z] = 0, \qquad \{Q_{10}, Q_{01}\} = \{Z, Q_{10}\} = \{Z, Q_{01}\} = 0.$$

The grading assignment is

$$H \in \mathfrak{a}_{00}, \quad Q_{10} \in \mathfrak{a}_{10}, \quad Q_{01} \in \mathfrak{a}_{01}, \quad Z \in \mathfrak{a}_{11}.$$

The creation/annihilation oscillators $a^\dagger, a$, given by

$$a = \frac{i}{\sqrt{2}}(\partial_x + x), \qquad a^\dagger = \frac{i}{\sqrt{2}}(\partial_x - x),$$

satisfy the commutator

$$[a, a^\dagger] = 1.$$

The matrix raising (lowering) operators $f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ (f_{11}, f_{10}, f_{01}) are

$$f_{11}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{10}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{01}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$f_{11} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_{01} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In terms of these operators the Hamiltonian H_{osc} can be re-expressed as

$$H_{osc} \quad = \quad a^\dagger a \cdot \mathbb{I}_4 + f_{10}^\dagger f_{10} + f_{01}^\dagger f_{01} = a^\dagger a \cdot \mathbb{I}_4 + \Lambda, \quad \text{with} \quad \Lambda = diag(0,0,1,1).$$

We denote a $m \times m$ identity matrix as $\mathbb{I}_m$.
The normalized lowest weight vector $|0;00\rangle$ satisfies the conditions

$$a|0;00\rangle = f_{11}|0;00\rangle = f_{10}|0;00\rangle = f_{01}|0;00\rangle = 0.$$

$$\boxed{|0;00\rangle = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.}$$

The single-particle Hilbert space $\mathcal{H}$ is spanned by the orthonormal vectors $|n;00\rangle, |n;11\rangle, |n;10\rangle, |n;01\rangle$:

$$\begin{aligned} |n;00\rangle &= \frac{(a^\dagger)^n}{\sqrt{n!}}|0;00\rangle, & |n;10\rangle &= \frac{(a^\dagger)^n}{\sqrt{n!}} f_{10}^\dagger|0;00\rangle, \\ |n;11\rangle &= \frac{(a^\dagger)^n}{\sqrt{n!}} f_{11}^\dagger|0;00\rangle, & |n;01\rangle &= \frac{(a^\dagger)^n}{\sqrt{n!}} f_{01}^\dagger|0;00\rangle. \end{aligned}$$

At most a single power of $f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger$ enters the spanning vectors since we have, for any pair of such operators,

$$f_\sharp^\dagger f_\flat^\dagger \quad = \quad 0, \qquad \text{with} \quad \sharp, \flat \in \{11, 10, 01\}.$$

Due to the commutators

$$[H_{osc}, a^\dagger] = a^\dagger, \quad [H_{osc}, f_{10}^\dagger] = f_{10}^\dagger, \quad [H_{osc}, f_{01}^\dagger] = f_{01}^\dagger, \quad [H_{osc}, f_{11}^\dagger] = 0,$$

the states are energy eigenstates whose eigenvalues are read from

$$\begin{array}{llll} H_{osc}|n;00\rangle = n|n;00\rangle, & & H_{osc}|n;10\rangle = (n+1)|n;10\rangle, \\ H_{osc}|n;11\rangle = n|n;11\rangle, & & H_{osc}|n;01\rangle = (n+1)|n;01\rangle. \end{array}$$

One should note that the vacuum state is doubly degenerate:

$$H_{osc}|0;00\rangle = H_{osc}|0;11\rangle \quad = \quad 0.$$

We introduce the exchange matrices X_{11}, X_{10}, X_{01}. They are hermitian operators which mutually interchange the 11, 10 and 01 sectors:

$$X_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, X_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, X_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices X_{11}, X_{10}, X_{01} are building blocks in the construction of the observables that we will discuss later.

The 6 standard $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings

the $\mathbb{Z}_2$-grading assignment the 4×4 matrix Hamiltonian H_{osc} corresponds to a block-diagonal supermatrix of $(4-p|p)$ type, with $p = 0, 1, 2, 3$. The $(4|0)$ case for $p = 0$ coincides with the ordinary bosonic matrix. The $p = 4$ case is excluded if we require the vacuum state to be even (bosonic).
The six assignments are:

1) $\{f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\} \in 0, \quad \{\emptyset\} \in 1 \quad$ for $(4|0)$;
2) $\{f_{11}^\dagger, f_{10}^\dagger\} \in 0, \quad \{f_{01}^\dagger\} \in 1 \quad$ for $(3|1)$;
3) $\{f_{11}^\dagger\} \in 0, \quad \{f_{10}^\dagger, f_{01}^\dagger\} \in 1 \quad$ for $(2|2)$;
4) $\{\emptyset\} \in 0, \quad \{f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\} \in 1 \quad$ for $(1|3)$;
5) $\{f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\} \in \mathbb{Z}_2^2 \cdot LSA$;
6) $\{f_{11}^\dagger, f_{10}^\dagger, f_{01}^\dagger\} \in \mathbb{Z}_2^2 \cdot LA$.

The corresponding vanishing (anti)commutators defining the graded abelian algebras $\mathfrak{a}_j$, where $j = 1, 2, \ldots, 6$, are

$$
\begin{array}{ll}
\mathfrak{a}_1 : & [f_{11}^\dagger, f_{10}^\dagger] = [f_{10}^\dagger, f_{01}^\dagger] = [f_{01}^\dagger, f_{11}^\dagger] = 0; \\
\mathfrak{a}_2 : & [f_{11}^\dagger, f_{10}^\dagger] = [f_{10}^\dagger, f_{01}^\dagger] = [f_{01}^\dagger, f_{11}^\dagger] = \{f_{01}^\dagger, f_{01}^\dagger\} = 0; \\
\mathfrak{a}_3 : & [f_{11}^\dagger, f_{10}^\dagger] = \{f_{10}^\dagger, f_{01}^\dagger\} = [f_{01}^\dagger, f_{11}^\dagger] = \{f_{10}^\dagger, f_{10}^\dagger\} = \{f_{01}^\dagger, f_{01}^\dagger\} = 0; \\
\mathfrak{a}_4 : & \{f_{11}^\dagger, f_{10}^\dagger\} = \{f_{10}^\dagger, f_{01}^\dagger\} = \{f_{01}^\dagger, f_{11}^\dagger\} = \{f_{11}^\dagger, f_{11}^\dagger\} = \{f_{10}^\dagger, f_{10}^\dagger\} = \{f_{01}^\dagger, f_{01}^\dagger\} = 0; \\
\mathfrak{a}_5 : & \{f_{11}^\dagger, f_{10}^\dagger\} = [f_{10}^\dagger, f_{01}^\dagger] = \{f_{01}^\dagger, f_{11}^\dagger\} = \{f_{10}^\dagger, f_{10}^\dagger\} = \{f_{01}^\dagger, f_{01}^\dagger\} = 0; \\
\mathfrak{a}_6 : & \{f_{11}^\dagger, f_{10}^\dagger\} = \{f_{10}^\dagger, f_{01}^\dagger\} = \{f_{01}^\dagger, f_{11}^\dagger\} = 0.
\end{array}
$$

The 6 standard multi-particle quantizations, associated to the respective gradings, are denoted as follows:

$$
\begin{array}{llllll}
\mathbf{(4|0)}: & \mathfrak{a}_1, & \mathbf{(2|2)}: & \mathfrak{a}_3, & \mathbb{Z}_2^2\text{-}\mathbf{PF}: & \mathfrak{a}_5, \\
\mathbf{(3|1)}: & \mathfrak{a}_2, & \mathbf{(1|3)}: & \mathfrak{a}_4, & \mathbb{Z}_2^2\text{-}\mathbf{PB}: & \mathfrak{a}_6.
\end{array}
$$

In the last column **PF** and **PB** stand for, respectively, parafermions and parabosons.

The 3 non-standard gradings

The non-standard cases are obtained by applying decompositions of the supermatrices which do not coincide with the ordinary block-diagonal ones.
See: F. Delduc, F. Gieres, S. Gourmelen and S. Theisen, *Non-standard matrix formats of Lie superalgebras*, Int. J. Mod. Phys. **A 14**, 4043 (1999); arXiv:math-ph/9901017.

In a $\mathbb{Z}_2$-grading, the standard decomposition of a vector $v^T = (B, B, F, F)$ with 2 bosons and 2 fermions can be replaced by $v^T = (B, F, B, F)$. The entries of the fermionic supermatrices are respectively accommodated as

$$\text{standard case:} \quad \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \quad \text{non-standard case:} \quad \begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \end{pmatrix}.$$

For 3 bosons and 1 fermion we pass from $v^T = (B, B, B, F)$ to $v^T = (B, F, B, B)$:

$$\text{standard case:} \quad \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & 0 \end{pmatrix}, \quad \text{non-standard case:} \quad \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & * & * \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}.$$

The key issue to notice is that the raising operator $f_{11}^\dagger$ becomes fermionic in the non-standard decompositions above. This implies that the Pauli exclusion principle applies to the 0-energy particles created by $f_{11}^\dagger$.

The three non-standard decompositions for the Hamiltonian H_{osc} are not equivalent to the standard ones.

Nevertheless, in all three cases these decompositions can be recovered from their corresponding standard ones after changing the Hamiltonian $H_{osc} = a^\dagger a \cdot \mathbb{I}_4 + \Lambda$, with $\Lambda = diag(0,0,1,1)$, into the permuted Hamiltonian $\overline{H}_{osc}$ given by

$$\overline{H}_{osc} \quad = \quad a^\dagger a \cdot \mathbb{I}_4 + \overline{\Lambda}, \quad \text{with} \quad \overline{\Lambda} = diag(0,1,1,0).$$

These three non-standard multi-particle quantizations are denoted as $\mathbf{(3|1)}_{ns}$, $\mathbf{(2|2)}_{ns}$, $\mathbb{Z}_2^2$-$\mathbf{PF}_{ns}$. Their corresponding graded algebras are

$$\begin{array}{lcl}
\mathbf{(3|1)}_{ns}: & \mathfrak{a}_2 \quad \text{for} & H_{osc} \mapsto \overline{H}_{osc}, \\
\mathbf{(2|2)}_{ns}: & \mathfrak{a}_3 \quad \text{for} & H_{osc} \mapsto \overline{H}_{osc}, \\
\mathbb{Z}_2^2\text{-}\mathbf{PF}_{ns}: & \mathfrak{a}_5 \quad \text{for} & H_{osc} \mapsto \overline{H}_{osc}.
\end{array}$$

The 2-particle Hilbert spaces

The orthonormal vectors spanning the 2-particle Hilbert spaces have the form

$$\boxed{|m; I\rangle = \frac{1}{\sqrt{m!}}\left(\frac{i}{\sqrt{2}}(\partial_x + \partial_y - x - y)\right)^m \cdot (\pi^{-\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2)}) \cdot V_I},$$

where V_I are 16-component constant orthonormal vectors which can be expressed in the v_j basis (v_j has entry 1 in the j-th position and 0 otherwise).

The 2-particle Hilbert spaces induced by the 6 standard gradings are denoted as $\mathcal{H}_k^{(2)}$; the suffix $k = 1, 2, \ldots, 6$ denotes the respective graded algebras.

The finite dimensional Hilbert spaces $\overline{\mathcal{H}}_k^{(2)} \subset \mathcal{H}_k^{(2)}$ are spanned by the V_I vectors by taking $m = 0$.

	(4|0)	**(3|1)**	**(2|2)**	**(1|3)**	$\mathbb{Z}_2^2$-**PF**	$\mathbb{Z}_2^2$-**PB**
$V_1 = v_1$	X	X	X	X	X	X
$V_2 = v_6$	X	X	X		X	X
$V_3 = v_{11}$	X	X				X
$V_4 = v_{16}$	X					X
$V_5 = \frac{1}{\sqrt{2}}(v_2 + v_5)$	X	X	X	X	X	X
$V_6 = \frac{1}{\sqrt{2}}(v_3 + v_9)$	X	X	X	X	X	X
$V_7 = \frac{1}{\sqrt{2}}(v_4 + v_{13})$	X	X	X	X	X	X
$V_8 = \frac{1}{\sqrt{2}}(v_7 + v_{10})$	X	X	X			
$V_9 = \frac{1}{\sqrt{2}}(v_7 - v_{10})$				X	X	X
$V_{10} = \frac{1}{\sqrt{2}}(v_8 + v_{14})$	X	X	X			
$V_{11} = \frac{1}{\sqrt{2}}(v_8 - v_{14})$				X	X	X
$V_{12} = \frac{1}{\sqrt{2}}(v_{12} + v_{15})$	X	X			X	
$V_{13} = \frac{1}{\sqrt{2}}(v_{12} - v_{15})$			X	X		X

Spanning vectors of the standard finite dimensional 2-particle Hilbert spaces of the 4×4 matrix oscillator. The first four columns correspond to supermatrices: **(4|0)**, i.e. the bosonic case, **(3|1)**, **(2|2)**, i.e. the supersymmetric case, and **(1|3)**. The last two columns present the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Hilbert spaces for parafermions ($\mathbb{Z}_2^2$-**PF**) and parabosons ($\mathbb{Z}_2^2$-**PB**). The "X" denotes the presence of the vector.

The absence, in certain cases, of the vectors V_2, V_3, V_4. It is a consequence of the Pauli exclusion principle for (para)fermions.

Degeneracy of the energy levels

The degeneracy of a energy level depends on the given quantization. The results are summarized in the table below which presents the nine cases (1 to 6 corresponding to the standard decompositions, 7, 8 and 9 to the non-standard ones). For any given quantization the degeneracy of its energy levels $n = 2, 3, 4, \ldots$ is the same:

	$E = 0$	$E = 1$	$E = n \geq 2$
1^*- $\mathbf{(4\|0)}$	3	7	10
2 - $\mathbf{(3\|1)}$	3	7	9
$3^\dagger$- $\mathbf{(2\|2)}$	3	7	8
4 - $\mathbf{(1\|3)}$	2	6	7
$5^\dagger$- $\mathbb{Z}_2^2$-**PF**	3	7	8
6^*- $\mathbb{Z}_2^2$-**PB**	3	7	10
7 - $\mathbf{(3\|1)}_{ns}$	2	6	9
$8^\ddagger$ - $\mathbf{(2\|2)}_{ns}$	2	6	8
$9^\ddagger$ - $\mathbb{Z}_2^2$-$\mathbf{PF}_{ns}$	2	6	8

The inequivalence of the quantizations 1 versus 6, 3 versus 5 and 8 versus 9 cannot be read from this table; it requires a subtler analysis of other observables.

Discriminating 2-particle observables: how to discriminate $\mathbb{Z}_2 \times \mathbb{Z}_2$-parabosons from bosons

The 2-particle observables discriminating parabosons from bosons should satisfy the following requirements:

i) they should apply to both bosonic and parabosonic Hilbert spaces,

ii) they should be hermitian and

iii) they should belong to the 00-graded sector of the parabosonic theory in order to have real (00-graded) eigenvalues.

The following set of 2-particle observables, constructed in terms of the exchange operators X_{11}, X_{10}, X_{01}, satisfy the above three criteria:

$$X_s = X_{10} \otimes X_{10}, \quad X_t = X_{01} \otimes X_{01}, \quad X_u = X_{11} \otimes X_{11}, \quad X_* = X_s + X_t + X_u.$$

Under the $\mathbf{S}_3$ permutations which interchange the parabosonic sectors $11, 10, 01$, the operators X_s, X_t are mapped into X_u, while X_* is $\mathbf{S}_3$-invariant. Without loss of generality we can therefore consider the two operators X_u, X_*.

In order to make easier the comparison of the bosonic versus parabosonic Hilbert spaces it is convenient to rename the respective vectors. They will be expressed in terms of a sign ε ($\varepsilon = +1$ for bosons, $\varepsilon = -1$ for parabosons).

The ε sign encodes the property that the bosonic wave functions are totally symmetric, while the parabosonic wave functions have mixed symmetry:

$$
\begin{array}{lll}
U_{00,A} = v_1, & & \\
U_{00,B} = v_6, & U_{11} = \frac{1}{\sqrt{2}}(v_2 + v_5), & \boxed{W_{11,\varepsilon} = \frac{1}{\sqrt{2}}(v_{12} + \varepsilon v_{15})}, \\
U_{00,C} = v_{11}, & U_{10} = \frac{1}{\sqrt{2}}(v_3 + v_9), & \boxed{W_{10,\varepsilon} = \frac{1}{\sqrt{2}}(v_8 + \varepsilon v_{14})}, \\
U_{00,D} = v_{16}, & U_{01} = \frac{1}{\sqrt{2}}(v_4 + v_{13}), & \boxed{W_{01,\varepsilon} = \frac{1}{\sqrt{2}}(v_7 + \varepsilon v_{10})}.
\end{array}
$$

The suffix denotes the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of the vector in the parabosonic case.

The difference between the two Hilbert spaces should be spotted by measuring the subspaces spanned by $W_{11,\varepsilon}, W_{10,\varepsilon}, W_{01,\varepsilon}$ and should appear as ε-dependent eigenvalues.

The eigenvectors of X_u with nonvanishing eigenvalues are $U_\pm$ and $W_{11,\varepsilon}$:

$$X_u U_\pm = \pm U_\pm \quad \text{for} \quad U_\pm = U_{00,C} \pm U_{00,D}, \qquad \boxed{X_u W_{11,\varepsilon} = \varepsilon W_{11,\varepsilon}}.$$

The eigenvectors of X_* with their respective nonvanishing eigenvalues are

$$\begin{aligned}
X_*(U_{00,B} - U_{00,C}) &= -(U_{00,B} - U_{00,C}), \qquad & \boxed{X_* W_{11,\varepsilon} = \varepsilon W_{11,\varepsilon}},\\
X_*(U_{00,C} - U_{00,D}) &= -(U_{00,C} - U_{00,D}), \qquad & \boxed{X_* W_{10,\varepsilon} = \varepsilon W_{10,\varepsilon}},\\
X_*(U_{00,D} - U_{00,B}) &= -(U_{00,D} - U_{00,B}), \qquad & \boxed{X_* W_{01,\varepsilon} = \varepsilon W_{01,\varepsilon}}.
\end{aligned}$$

The presence of the ε eigenvalues proves that, by performing X_u, X_* measurements, one can determine whether a system under consideration is composed by ordinary bosons or by $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded parabosons.

Comment

Besides the observables discriminating bosons versus $\mathbb{Z}_2 \times \mathbb{Z}_2$-parabosons, an analogous construction produces:

observables discriminating the standard supersymmetric versus the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-parafermionic quantization (cases 3 versus 5) and

observables discriminating the nonstandard supersymmetric versus the nonstandard $\mathbb{Z}_2 \times \mathbb{Z}_2$-parafermionic quantization (cases 8 versus 9).

$\Rightarrow$ **9 inequivalent quantizations are obtained from gradings.**

Conclusions

Paraparticles recovered from Rittenberg-Wyler 2-bit physics are theoretically observable.

Question: where can we expect to find them?

Different possibilities:

- **Fundamental physics (relativistic QFTs, effects of quantum gravity at Planck scale, dark matter, . . .),**

- **Laboratory physics as emergent structures (collective modes) in condensed matter,**

- **Direct construction either via lego (metamaterials) or by manipulating qubits.**

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Thanks for the attention!