# Faces of polyhedra associated with Relation modules[1]

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[1]Joint work with Luis Enrique Ramirez (UFABC)

# Relations graph

Any subset $\mathcal{C} \subseteq \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+ \subset \mathfrak{V} \times \mathfrak{V}$ will be called a *set of relations*.

By $G(\mathcal{C})$ we denote the directed graph with set of vertices $\mathfrak{V}$, and arrow from vertex $(i,j)$ to $(r,s)$ if and only if $((i,j);(r,s)) \in \mathcal{C}$.

Set

$$\mathfrak{V} := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq j \leq i \leq n\}.$$

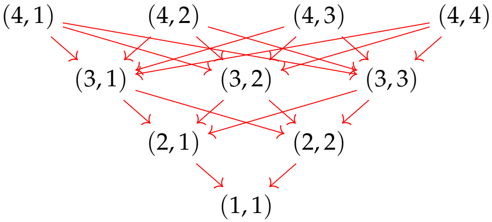
For $n = 5$

$$\begin{array}{ccccccccc}
(5,1) & & (5,2) & & (5,3) & & (5,4) & & (5,5) \\
& (4,1) & & (4,2) & & (4,3) & & (4,4) & \\
& & (3,1) & & (3,2) & & (3,3) & & \\
& & & (2,1) & & (2,2) & & & \\
& & & & (1,1) & & & &
\end{array}$$

$$\mathcal{R}^+ := \{((i,j);(i-1,t)) \mid 2 \leq j \leq i \leq n,\ 1 \leq t \leq i-1\}$$

For $n = 4$

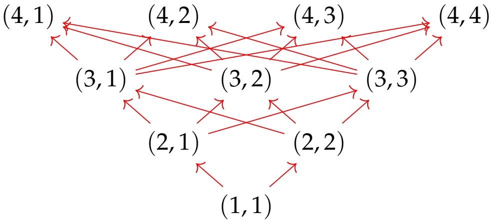
$\mathcal{R}^+ =$



$$\mathcal{R}^- := \{((i,j);(i+1,s)) \mid 1 \le j \le i \le n-1,\ 1 \le s \le i+1\}$$

For $n = 4$

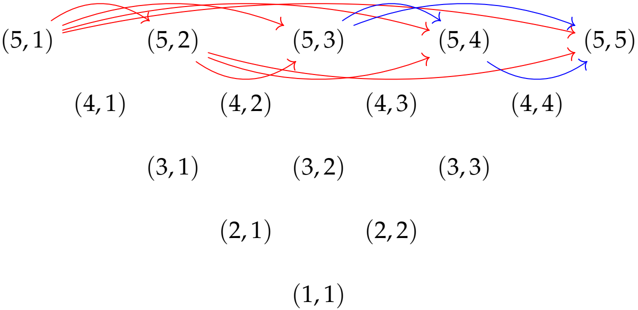
$\mathcal{R}^- =$



$$\mathcal{R}^0 := \{((n,i);(n,j)) \mid 1 \leq i \neq j \leq n\}$$

For $n = 5$

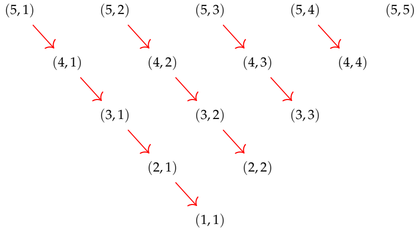
$\mathcal{R}^0 =$



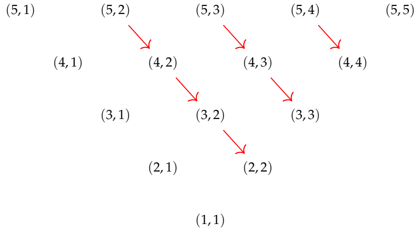
# Examples

For any $1 \leq k \leq n$, we consider $\mathcal{C}_k^+ := \{((i+1,j);(i,j)) \mid k \leq j \leq i \leq n-1\}$.

$G(\mathcal{C}_1^+)=$

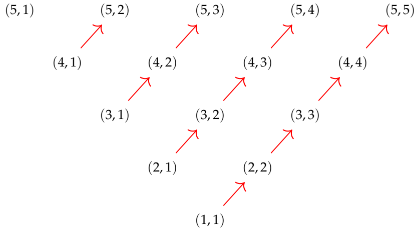



$G(\mathcal{C}_2^+)=$

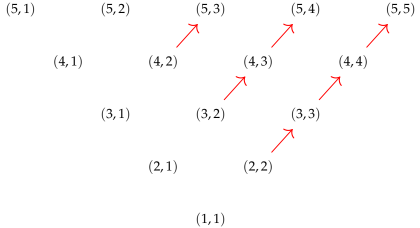


For $1 \leq k \leq n$, we consider $\mathcal{C}_k^- := \{((i,j);(i+1,j+1)) \mid k \leq j \leq i \leq n-1\}$.

$G(\mathcal{C}_1^-)=$



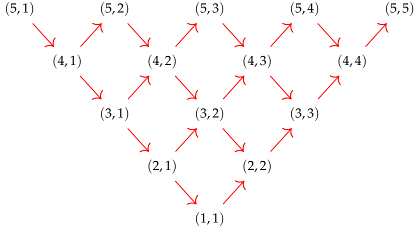

$G(\mathcal{C}_2^-)=$



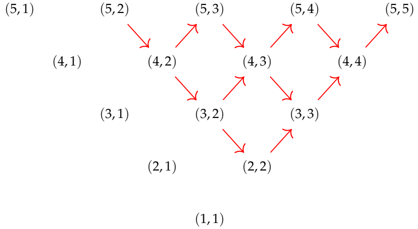
# Examples

For any $1 \leq k \leq n$, we consider $\mathcal{C}_k := \mathcal{C}_k^+ \cup \mathcal{C}_k^-$

$G(\mathcal{C}_1)=$




$G(\mathcal{C}_2)=$



# Gelfand-Tsetlin tableaux

For $M \in \mathbb{C}^{\frac{n(n+1)}{2}}$ denote by $T(M)$ the image of $M$ via the natural isomorphism between $\mathbb{C}^{\frac{n(n+1)}{2}}$ and $\mathbb{C}^n \times \cdots \times \mathbb{C}^1$ and its entries by $m_{ij}$. Hence, we can picture $T(M)$ as a triangular tableau height $n$, such tableaux will be called *Gelfand–Tsetlin tableaux*.

$$
\begin{array}{ccccccccccc}
\boxed{m_{n1}} & & \boxed{m_{n2}} & & & \cdots & & & \boxed{m_{n,n-1}} & & \boxed{m_{nn}} \\
& \boxed{m_{n-1,1}} & & & & \cdots & & & & \boxed{m_{n-1,n-1}} & \\
& & & \cdots & & \cdots & & \cdots & & & \\
& & & & \boxed{m_{21}} & & \boxed{m_{22}} & & & & \\
& & & & & \boxed{m_{11}} & & & & &
\end{array}
$$

For any $\mathbb{A} \subseteq \mathbb{C}^{n(n+1)/2}$, we denote by $\mathbf{T}(\mathbb{A})$ the set of all Gelfand–Tsetlin tableaux $T(L)$, with $L \in \mathbb{A}$.

## $\mathcal{C}$-realizations

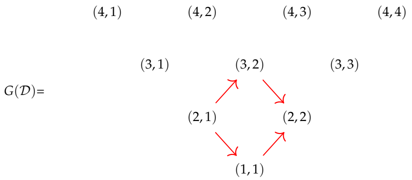
Let $\mathcal{C}$ be a set of relations and $T(L)$ any Gelfand-Tsetlin tableau, we say that:

- $T(L)$ *satisfies* $\mathcal{C}$, if

$$l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}, \quad \text{for any arrow } (i,j) \longrightarrow (r,s) \text{ in } G(\mathcal{C}).$$

- $T(L)$ is a *$\mathcal{C}$-realization*, if $T(L)$ satisfies $\mathcal{C}$ and for any $1 \leq k \leq n-1$ we have, $l_{ki} - l_{kj} \in \mathbb{Z}$ if only if $(k,i)$ and $(k,j)$ are in the same connected component of $G(\mathcal{C})$.

Consider $\mathcal{D}$ to be the set of relations with associated graph




$$T(X)= \begin{array}{cccc} \sqrt{2} & 2 & 3 & 4 \\ 1 & 0 & 2 & \\ 1 & -2 & & \\ 0 & & & \end{array} \qquad T(Y)= \begin{array}{cccc} \sqrt{2} & 2 & 3 & 4 \\ \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & & \\ 0 & & & \end{array}$$

# $\mathcal{C}$-realizations

$\mathbb{Z}_0^{n(n+1)/2}$ will denote the set of vectors $M$ in $\mathbf{T}(\mathbb{Z}_0^{n(n+1)/2})$ such that $m^{(n)} = \mathbf{0}$.

Suppose that $T(L)$ satisfies $\mathcal{C}$. We will denote by:

- $\mathcal{B}_{\mathcal{C}}(T(L))$ the set of all Gelfand-Tsetlin tableaux in $\mathbf{T}\left(L + \mathbb{Z}_0^{n(n+1)/2}\right)$ satisfying $\mathcal{C}$.
- $V_{\mathcal{C}}(T(L))$ the vector space with basis $\mathcal{B}_{\mathcal{C}}(T(L))$.

A set of relations $\mathcal{C}$ is called *admissible* if for any $\mathcal{C}$-realization $T(L)$, $V_{\mathcal{C}}(T(L))$ is a $\mathfrak{gl}_n$-module, with the following action on any $T(M) \in \mathcal{B}_{\mathcal{C}}(T(L))$,

$$E_{k,k+1}(T(M)) = -\sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k+1}(m_{ki} - m_{k+1,j} + j - i)}{\prod_{j \neq i}^{k}(m_{ki} - m_{kj} + j - i)} \right) T(M + \delta^{ki}), \qquad (1)$$

$$E_{k+1,k}(T(M)) = \sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k-1}(m_{ki} - m_{k-1,j} + j - i)}{\prod_{j \neq i}^{k}(m_{ki} - m_{kj} + j - i)} \right) T(M - \delta^{ki}), \qquad (2)$$

$$E_{kk}(T(M)) = \left( \sum_{i=1}^{k} m_{ki} - \sum_{i=1}^{k-1} m_{k-1,i} \right) T(M), \qquad (3)$$

where $\delta^{ki}$ stands for the vector in $T_n(\mathbb{Z})$ such that $(\delta^{ki})_{rs} = \delta_{kr}\delta_{is}$.

If $\mathcal{C}$ is admissible, we will call $V_{\mathcal{C}}(T(L))$ a *relations module*.

**Futorny–Ramirez–Zhang**, 2019

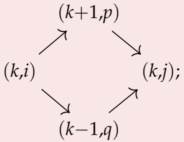
Suppose that $\mathcal{C}$ is a noncritical set of relations whose associated graph $G = G(\mathcal{C})$ satisfies the following conditions:

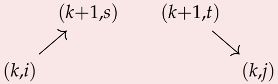
(i) $G$ is reduced;

(ii) $G$ does not contain loops, and $(k, i) \succeq (k, j)$ implies $i \leq j$;

(iii) If $G$ contains an arrow connecting $(k, i)$ and $(k+1, t)$, then $(k+1, s)$ and $(k, j)$ with $i < j$, $s < t$ are not connected in $G$.

$\mathcal{C}$ is an admissible set of relations if and only if, for any connected component $G(\mathcal{E})$ of $G(\mathcal{C})$ and any adjoining pair $((k, i); (k, j))$ in $G(\mathcal{E})$, there exist $p, q$ such that $\mathcal{E}_1 \subseteq \mathcal{E}$ or, there exist $s < t$ such that $\mathcal{E}_2 \subseteq \mathcal{E}$, where the graphs associated to $\mathcal{E}_1$ and $\mathcal{E}_2$ are as follows

$G(\mathcal{E}_1)=$ 


$G(\mathcal{E}_2)=$ 

# Gelfand–Tsetlin Construction

**Remark**

*The sets of relations* $\mathcal{C}_k$, $\mathcal{C}_k^+$ *and* $\mathcal{C}_k^-$ *are admissible set of relations for any* $1 \leq k \leq n$.

**Theorem (Gelfand–Tsetlin, 1950)**

*If* $\lambda := (\lambda_1, \ldots, \lambda_n)$ *is an integral dominant* $\mathfrak{gl}_n$*-weight and* $T(\Lambda)$ *is the Gelfand–Tsetlin tableau of height n with entries* $\lambda_{ki} := \lambda_i$, *then* $V_{\mathcal{C}_1}(T(\Lambda))$ *is isomorphic to the simple finite dimensional module* $L(\lambda)$. *Moreover,*

(i) $\mathcal{B}_{\mathcal{C}_1}(T(\Lambda))$ *is a basis of* $V_{\mathcal{C}_1}(T(\Lambda))$.

(ii) *For any* $\mu = (\mu_1, \ldots, \mu_n) \in \mathfrak{h}^*$, *the weight space* $L(\lambda)_\mu$ *has a basis*

$$\left\{ T(X) \in \mathcal{B}_{\mathcal{C}_1}(T(\Lambda)) \mid w_k(X) = \mu_k, \textit{ for all } k = 1, \ldots, n \right\}.$$

**Definition**

A subset $P$ of a $\mathbb{R}$-vector space $V$ is called a *polyhedron* if it is the intersection of finitely many closed halfspaces. The *dimension* of $P$ is given by $\dim(\mathrm{aff}(P))$. A *polytope* is a bounded polyhedron.

Associated with a set of relations $\mathcal{C}$ we will define polyhedra in $\mathbb{R}^{n(n+1)/2}$.

**Definition**

Let $\mathcal{C}$ be any set of relations, $X \in \mathbb{R}^{n(n+1)/2}$ is called a *$\mathcal{C}$-pattern*, if

$$x_{ij} \geq x_{rs} \text{ for any arrow } (i,j) \longrightarrow (r,s) \text{ in } G(\mathcal{C}).$$

The *kth weight linear map* $w_k : \mathbb{C}^{n(n+1)/2} \longrightarrow \mathbb{C}$ is defined by

$$w_k(X) := \begin{cases} \sum_{i=1}^{k} x_{ki} - \sum_{i=1}^{k-1} x_{k-1,i}, & \text{if } 2 \leq k \leq n; \\ x_{11}, & \text{if } k = 1. \end{cases}$$

**Definition**

Fix $\lambda, \mu \in \mathbb{R}^n$ and $\mathcal{C}$ a set of relations. We consider the following polyhedra in $\mathbb{R}^{n(n+1)/2}$ associated with $\mathcal{C}$

$$P_{\mathcal{C}} := \left\{ X \in \mathbb{R}^{n(n+1)/2} \mid X \text{ is a } \mathcal{C}\text{-pattern} \right\},$$

$$P_{\mathcal{C}}(\lambda) := \left\{ X \in P_{\mathcal{C}} \mid x_{nj} = \lambda_j \text{ for all } 1 \leq j \leq n \right\},$$

$$P_{\mathcal{C}}(\lambda, \mu) := \{ X \in P_{\mathcal{C}}(\lambda) \mid w_i(X) = \mu_i \text{ for all } 1 \leq i \leq n \}.$$

- $P_{\mathcal{C}}$ is always unbounded.
- $P_{\mathcal{C}}(\lambda)$ is a polytope if and only if the maximal and minimal points of $G(\mathcal{C})$ belong to $\{(n,1), \ldots, (n,n)\}$.

Given a Gelfand-Tsetlin tableau $T(L)$, elements in $L + \mathbb{Z}_0^{n(n+1)/2}$ are called *L-integral points*.

**Theorem (B., Ramirez, 2022)**

*Let $\mathcal{C}$ be any admissible set of relations, $T(L)$ a $\mathcal{C}$-realization, and $V = V_{\mathcal{C}}(T(L))$ the corresponding relation $\mathfrak{gl}_n$-module. Set $\lambda = (l_{n1}, \dots, l_{nn})$, and $\mu = (w_1(L), w_2(L), \dots, w_n(L)) \in \mathfrak{h}^*$.*

(i) *The polyhedra $P_{\mathcal{C}}$ and $P_{\mathcal{C}}(\lambda)$ have the same number of L-integral points, and this number is equal to $\dim(V)$.*

(ii) *The number of L-integral points in $P_{\mathcal{C}}(\lambda, \mu)$ is equal to $\dim(V_\mu)$.*

# Polyhedra associated with sets of relations

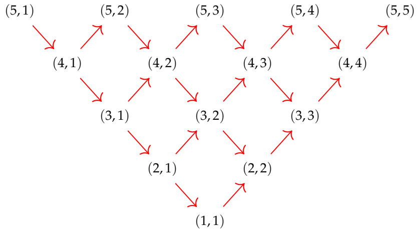
## Corollary

*Let $T(L)$ be a $\mathcal{C}_1$-realization, $\lambda = (l_{n1}, \ldots, l_{nn})$, and $\mu$ a weight of $V_{\mathcal{C}_1}(T(L))$.*

*$V_{\mathcal{C}_1}(T(L))$ is isomorphic to the **simple finite-dimensional module** $L(\lambda)$.*

- *The number of L-integral points in $P_{\mathcal{C}_1}$ and $P_{\mathcal{C}_1}(\lambda)$ is finite and equal to $\dim(L(\lambda))$.*
- *The number of L-integral points in $P_{\mathcal{C}_1}(\lambda, \mu)$ is finite and equal to $\dim(L(\lambda)_\mu)$.*

$G(\mathcal{C}_1)$=



# Polyhedra associated with sets of relations

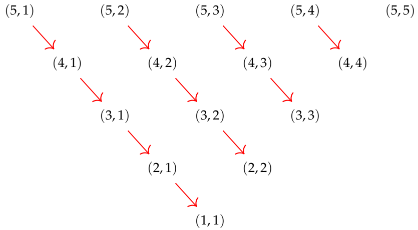
## Corollary

*Let $T(L)$ be a $\mathcal{C}_1^+$-realization, $\lambda = (l_{n1}, \ldots, l_{nn})$, and $\mu$ a weight of $V_{\mathcal{C}_1^+}(T(L))$.*

*The module $V_{\mathcal{C}_1^+}(T(L))$ is isomorphic to the* ***generic Verma module*** *$M(\lambda)$ and*

- *$P_{\mathcal{C}_1^+}$ and $P_{\mathcal{C}_1^+}(\lambda)$ contains infinitely many L-integral points.*
- *The number of L-integral points in $P_{\mathcal{C}_1^+}(\lambda, \mu)$ is $\dim M(\lambda)_\mu < \infty$.*

$G(\mathcal{C}_1^+)$=



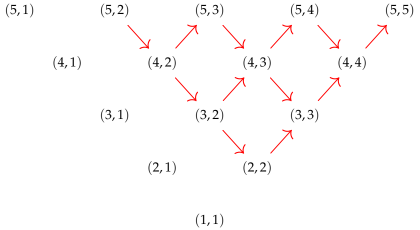
## Corollary

*Let $T(L)$ be a $\mathcal{C}_2$-realization, $\lambda = (l_{n1}, \dots, l_{nn})$, and $\mu$ a weight of $V_{\mathcal{C}_2}(T(L))$.*

*If $\tilde{\lambda} := (l_{n2}, \dots, l_{nn})$ is a dominant $\mathfrak{gl}_{n-1}$-weight, then $V_{\mathcal{C}_2}(T(L))$ is isomorphic to the **cuspidal module** $L(\tilde{\lambda})$, which is an infinite-dimensional module with finite weight spaces of dimension $\dim(L(\tilde{\lambda}))$*

- *$P_{\mathcal{C}_2}$ and $P_{\mathcal{C}_2}(\lambda)$ contains infinitely many L-integral points.*
- *If $\mu$ is a weight of $V_{\mathcal{C}_2}(T(L))$, the number of L-integral points in $P_{\mathcal{C}_2}(\lambda, \mu)$ is equal to $\dim(L(\tilde{\lambda}))$ .*

$G(\mathcal{C}_2)$=



A hyperplane $H$ is called a *support hyperplane* of the polyhedron $P$, if $H \cap P \neq \emptyset$, and $P$ is contained in one of the two closed halfspaces bounded by $H$, and such intersection $F = H \cap P$ is a *face* of $P$.

Faces of dimension 0 are called *vertices*, and an *edge* is a face of dimension 1. In general, a face $F$ of dimension $k$ is called a *k-face*.

For any point $x$ in a polyhedron $P$, there exists a unique face $F$ such that $x \in \text{int}(F)$. This face is the unique minimal element in the set of faces of $P$ containing $x$ and will be called *minimal face* for $x$.

# Tiling of a $\mathcal{C}$-pattern

Let $\mathcal{C}$ be a set of relations, and $X \in \mathbb{R}^{n(n+1)/2}$ be a $\mathcal{C}$-pattern.

## Definition

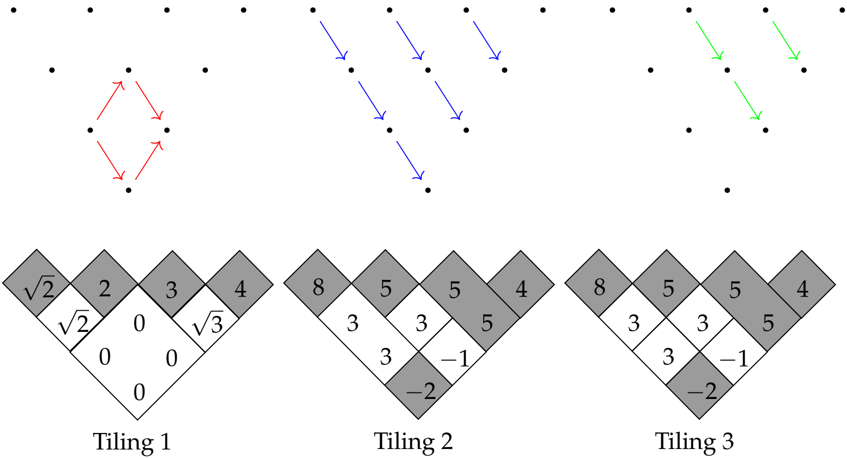
Set $\sim \subseteq \mathfrak{V} \times \mathfrak{V}$ given by $(i,j) \sim (r,s)$ if and only if there exists a path in $G(\mathcal{C})$ connecting $(i,j)$ and $(r,s)$ with the entries of $X$ associated with the vertices in the walk being equal.

The partition of $\mathfrak{V}$ induced by the relation will be called tiling, and is denoted by $\mathcal{M}_{\mathcal{C}}(X)$. The equivalence classes will be called tiles.

A tile $\mathcal{M}$ is called *top-free* if $\mathcal{M} \cap \{(n,1),(n,2),\ldots,(n,n)\} = \emptyset$.

A tile $\mathcal{M}$ is called *top/bottom-free* if $\mathcal{M} \cap \{(1,1),(n,1),(n,2),\ldots,(n,n)\} = \emptyset$.

# Examples of tiling



### Definition

Given a $\mathcal{C}$-pattern $X$, with tiling $\mathcal{M}_{\mathcal{C}}(X)$ and set of top-free tales $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_s$, we define a *tiling matrix* to be the matrix

$$A_{\mathcal{M}_{\mathcal{C}}(X)} := (a_{ik}) \in \mathbb{M}_{n-1\times s}(\mathbb{Z}_{\geq 0}) \text{ where } a_{ik} = |\{j \mid (i,j) \in \mathcal{M}_k\}|\,.$$

For $\mathcal{C}$-patterns without top-free tiles we consider $A_{\mathcal{M}}$ to be the identity matrix of order $n-1$.

**Remark**

*The matrix $A_{\mathcal{M}_c(X)}$ depends of the chosen order of the top-free tales, but the dimension of the kernel of $A_{\mathcal{M}}$ does not depend.*

We will enumerate the tales from left to right and from bottom to top.

Let us consider the tilings $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ from latter Example. The corresponding tiling matrizes are:

$$A_{\mathcal{T}_1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_{\mathcal{T}_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, A_{\mathcal{T}_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

**Proposition [B., Ramirez, 2022]**

Let $\mathcal{C}$ be a set of relations, $X$ a $\mathcal{C}$-pattern, $\lambda = (x_{n1}, \ldots, x_{nn})$, and $\mu = (w_1(X), w_2(X), \ldots, w_n(X))$.

(i) If $\mathcal{M}_{\mathcal{C}}(X)$ does not have top-free tiles, then $X$ is a vertex of $P_{\mathcal{C}}(\lambda)$.

(ii) If $\mathcal{M}_{\mathcal{C}}(X)$ does not have top/bottom-free tiles, then $X$ is a vertex of $P_{\mathcal{C}}(\lambda, \mu)$.

**Theorem [B., Ramirez, 2022]**

Let $\mathcal{C}$ be a set of relations, $X$ a $\mathcal{C}$-pattern, and $\mathcal{M}_{\mathcal{C}}(X)$ its associated tiling. Set $\lambda = (x_{n1}, \ldots, x_{nn})$, and $\mu = (w_1(X), w_2(X), \ldots, w_n(X))$. Then,

(i) The dimension of the minimal face of $P_{\mathcal{C}}$ containing $X$ is equal to the number of tales in $\mathcal{M}_{\mathcal{C}}(X)$.

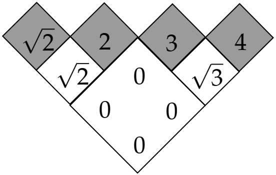
(ii) The dimension of the minimal face of $P_{\mathcal{C}}(\lambda)$ containing $X$ is equal to the number of top-free tales in $\mathcal{M}_{\mathcal{C}}(X)$.

(iii) The dimension of the minimal face of $P_{\mathcal{C}}(\lambda, \mu)$ containing $X$ is equal to the dimension of the kernel of $A_{\mathcal{M}_{\mathcal{C}}(X)}$.

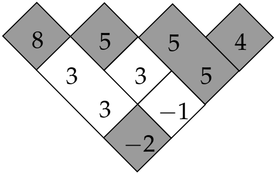
# Example

Therefore, the dimension of minimal faces containing the tableaux from previous Examples are given by

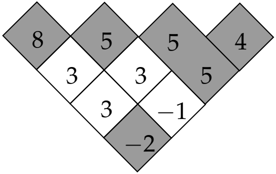
| | $\lambda$ | $\mu$ | $P_{\mathcal{C}}$ | $P_{\mathcal{C}}(\lambda)$ | $P_{\mathcal{C}}(\lambda,\mu)$ |
|---|---|---|---|---|---|
| Tiling 1 | $(\sqrt{2},2,3,4)$ | $(0,0,\sqrt{2}+\sqrt{3},9-\sqrt{3})$ | 7 | 3 | 1 |
| Tiling 2 | $(8,5,5,4)$ | $(-2,4,9,11)$ | 8 | 4 | 1 |
| Tiling 3 | $(8,5,5,4)$ | $(-2,4,9,11)$ | 9 | 5 | 2 |



Tiling 1



Tiling 2



Tiling 3

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Thank you for your attention.