Non-injective homomorphisms between certain Verma modules

or

N=1 super Heisenberg–Virasoro algebra at level 0

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Joint work with B. Jandrić and D. Adamović:

- *On the* $N = 1$ *super Heisenberg–Virasoro vertex algebra*, in Lie Groups, Number Theory, and Vertex Algebras, Contemporary Mathematics, Vol. 768 (AMS 2021)
- *The* $N = 1$ *super Heisenberg–Virasoro vertex algebra at level zero*, J. of Algebra and Applications, 171-JAA (2021)

Heisenberg–Virasoro algebra

The Heisenberg–Virasoro algebra $\mathcal{H}$ (Arbarello, De Concini, Kac, Procesi '88):

$$L(z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2} \qquad \alpha(z) = \sum_{n\in\mathbb{Z}} \alpha(n) z^{-n-1}$$

$$\begin{aligned}
[L(m), L(n)] &= (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}c_L \\
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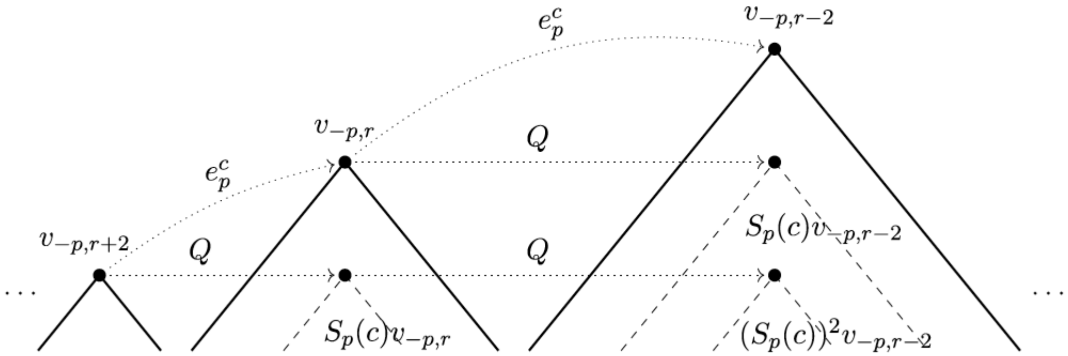
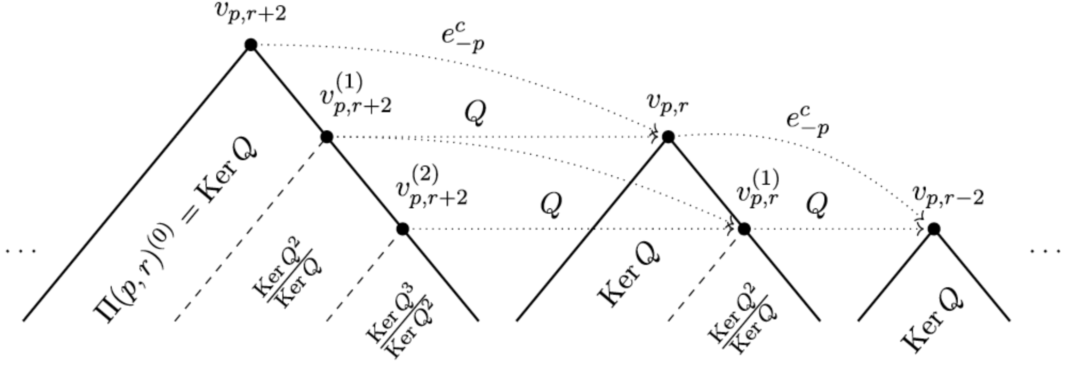
$$\begin{aligned}
c_\alpha \neq 0 &\implies \mathcal{H} \cong M(1) \otimes \mathrm{Vir} \\
c_\alpha = 0 &\implies \mathcal{H} \text{ is a simple, irrational VOA}
\end{aligned}$$

Level zero

Highest weight reps in case $c_\alpha = 0$ studied by Y. Billig ('03), D. Adamović – GR ('15, '19)

- irreducibles classified, structure of Verma modules [B]
- free field realisation [AR]
- screening operator $\longrightarrow$ Verma module embeddings, irreducibles as kernels, formulas for singular vectors [AR]
- fusion rules [AR]
- deformed action of $L(z)$ $\longrightarrow$ logarithmic modules, non-split self-extensions [AR]


$v_{-p,r-2}$
e_p^c
$v_{-p,r}$
Q
e_p^c
$S_p(c)v_{-p,r-2}$
$v_{-p,r+2}$
Q
Q
$S_p(c)v_{-p,r}$
$(S_p(c))^2 v_{-p,r-2}$
$\cdots$
$\cdots$


$v_{p,r+2}$
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Q
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$\Pi(p,r)(0) = \operatorname{Ker} Q$
$\frac{\operatorname{Ker} Q^2}{\operatorname{Ker} Q}$
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$N = 1$ Super Heisenberg–Virasoro

$\mathcal{SH}$ (D. Adamović, B. Jandrić, GR)

$$L(z), \qquad \alpha(z), \qquad G(z) = \sum_{n \in \mathbb{Z}} G(n + \tfrac{1}{2}) z^{-n-2}, \qquad \Psi(z) = \sum_{n \in \mathbb{Z}} \Psi(n + \tfrac{1}{2}) z^{-n-1}$$

$\langle L(z), \alpha(z) \rangle$ – Heisenberg–Virasoro $\mathcal{H}$ $\qquad$ $\langle L(z), G(z) \rangle$ – the $N = 1$ Neveu–Schwarz $\mathcal{N}$

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$$\left[\Psi(m+\frac{1}{2}), \Psi(n+\frac{1}{2})\right]_+ = \delta_{m+n+1,0}c_\alpha; \qquad \left[\alpha(m), \Psi(n+\frac{1}{2})\right] = 0;$$

$$\left[L(n), \Psi(m+\frac{1}{2})\right] = -\frac{2m+n+1}{2}\Psi(m+n+\frac{1}{2});$$

$$\left[G(n+\frac{1}{2}), \alpha(m)\right] = -m\Psi(m+n+\frac{1}{2});$$

$$\left[G(n+\frac{1}{2}), \Psi(m+\frac{1}{2})\right]_+ = \alpha(m+n+1) + 2m\delta_{m+n+1,0}c_{L,\alpha}.$$

Level zero

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- **The $\mathrm{N} = 1$ super Heisenberg–Virasoro algebra at level zero** $\mathcal{SH}_c$ or just $\mathcal{SH}$.

We shall present

- Free field realisation of $\mathcal{SH}_c$ and its highest weight modules
- Several screening operators among them
- Characters of hw modules, simplicity of $\mathcal{SH}_c$
- Structure of Verma modules
- Explicit formulas for (sub)singular vectors
- Morphisms between Verma modules (some of them non-injective)

Highest weight reps

Let $c, h, h_\alpha \in \mathbb{C}$. Verma module:

$$V(c, h, h_\alpha),$$

Irreducible h.w. module:

$$L(c, h, h_\alpha).$$

Determinant formula gives a classification of irreducibles:

Irreps

Theorem

Let $c \neq 0$. The Verma module $V(c, h, h_\alpha)$ is irreducible if and only if

$$p = \frac{h_\alpha}{c} - 1 \notin \mathbb{Z}^*.$$

Assume that $p \in \mathbb{Z}^*$. Then there is a singular vector in $V(c, h, h_\alpha)$ of conformal weight

- $|p|$ *if* p *is even,*
- $|\frac{p}{2}|$ *if* p *is odd.*

Highest weight reps

Parametrisation:

$$\begin{aligned} h[p,r] &= (1-p^2)\frac{c-3}{24} - rp, \\ h_\alpha[p,r] &= (1+p)c, \\ M[p,r] &= M(c, h[p,r], h_\alpha[p,r]) \qquad \text{for any h.w. module.} \end{aligned}$$

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The Verma module $V[p,r]$ is reducible iff $p \in \mathbb{Z}^*$.

Free field realisation - VOSA

- $M(1)$ rank 2 Heisenberg algebra generated by lattice $L = \mathbb{Z}x \oplus \mathbb{Z}y$ such that

$$\langle x, x\rangle = \langle y, y\rangle = 0, \qquad \langle x, y\rangle = 2$$

- $\mathcal{H}$ generated by

$$\begin{aligned} \alpha(z) &= -c\ x(z) \\ \omega_{\mathcal{H}}(z) &= \frac{1}{2}x(z)y(z) + \frac{c-3}{24}\partial x(z) - \frac{1}{2}\partial y(z) \end{aligned}$$

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- $F^{(2)}$ fermionic algebra generated by $\Psi^{\pm}(z) = \sum_{n\in\mathbb{Z}} \Psi^{\pm}(n+\frac{1}{2})z^{-n-1}$

$$\begin{aligned} \left[\Psi^{+}(r), \Psi^{-}(s)\right]_{+} &= \delta_{r+s,0}, \qquad \left[\Psi^{\pm}(r), \Psi^{\pm}(s)\right]_{+} = 0 \\ \omega_{fer}(z) &= \frac{1}{2}\partial\left(\Psi^{+}(z)\Psi^{-}(z)\right) \end{aligned}$$

Free field realisation - VOSA

$\mathcal{SH}$ as subalgebra of Heisenberg-Clifford algebra $SM(1) := M(1) \otimes F^{(2)}$:

$$
\begin{aligned}
L(z) &= \omega_{\mathcal{H}}(z) + \omega_{fer}(z) \\
\alpha(z) &= -c\ x(z) \\
G(z) &= \sqrt{2}\left(\frac{1}{2}x(z)\Psi^{+}(z) + \frac{1}{2}y(z)\Psi^{-}(z) + \frac{c-3}{12}\partial\Psi^{-}(z) - \partial\Psi^{+}(z)\right) \\
\Psi(z) &= -\sqrt{2}c\ \Psi^{-}(z)
\end{aligned}
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Free field realisation - modules

- Lattice VOSA $V_L = \mathbb{C}[L] \otimes SM(1)$.

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$$v_{p,r} = e^{-\frac{p+1}{2}\left(y - \frac{c-3}{12}x\right) + rx}, \qquad p, r \in \mathbb{C}$$

of conformal weight $(h[p, r], h_\alpha[p, r])$.

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- Fock module

$$\mathcal{F}_{p,r} = SM(1) \cdot v_{p,r}$$

- Contragredient module

$$\mathcal{F}^*_{p,r} \cong \mathcal{F}_{-p,-r}$$

Free field realisation - modules

Theorem

If $p \in \mathbb{C} \setminus \mathbb{Z}_{<0}$ *then*

$$\mathcal{SH} \cdot v_{p,r} = \mathcal{F}_{p,r} \cong V[p,r].$$

Proof by equality of characters and $\mathcal{SH} \cdot v_{p,r} = \mathcal{F}_{p,r}$.

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Theorem

If $p \in \mathbb{Z}_{<0}$ *then*

$$L[p,r] \cong \mathcal{SH} \cdot v_{p,r} < \mathcal{F}_{p,r}.$$

Proof by showing there are no $\mathcal{SH}$-singular vectors in $\mathcal{F}_{p,r}$.

Screening operators

Define

$$a = \Psi^- \left(-\frac{1}{2}\right) e^{\frac{1}{2}x} = \Psi^- \left(-\frac{1}{2}\right) v_{-1,\frac{1}{2}}, \qquad a(z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}$$

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$Q = a(0)$ is a **screening operator** i.e. (anti)commutes with $\mathcal{SH}$.

$\mathcal{SH}^{+} \cdot Qv_{p,r} = 0$ so either $Qv_{p,r} = 0$ or $Qv_{p,r}$ is a singular vector.

Schur polynomials

Schur polynomials $S_r(\alpha) := S_r(\alpha(-1), \alpha(-2), \cdots)$ defined by:

$$\exp\left(\sum_{n=1}^{\infty} \alpha(-n)\frac{z^n}{n}\right) = \sum_{r=0}^{\infty} S_r(\alpha) z^r.$$

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$$a(n)v_{p,r-\frac{1}{2}} = -\frac{1}{\sqrt{2}c_{L,\alpha}} \sum_{i=0}^{\frac{p-1}{2}-n} \Psi(-i-\tfrac{1}{2}) S_{\frac{p-1}{2}-n-i}\left(-\frac{\alpha}{2c_{L,\alpha}}\right) v_{p,r}.$$

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Note that $S_r(\alpha) = 0$ if $r \notin \mathbb{Z}_{\geq 0}$.

Singular vector $p > 0$ odd

Theorem

Assume that $p \in \mathbb{Z}_{>0}$ is odd. Then

$$u_{p,r} = Qv_{p,r-\frac{1}{2}} = \sum_{i=0}^{\frac{p-1}{2}} \Psi(-i-\tfrac{1}{2}) S_{\frac{p-1}{2}-i}\left(-\frac{\alpha}{2c_{L,\alpha}}\right) v_{p,r}$$

is a singular vector of weight $h[p,r] + \frac{p}{2} = h[p, r - \frac{1}{2}]$ in $V[p,r] = \mathcal{F}_{p,r}$.

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- Is it isomorphic to the Verma module?

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- $p > 0$ even?

Simple current extensions and twisted modules

Construction from Adamović, Milas, *On W-Algebras Associated to (2,p) Minimal Models and Their Representations*, IMRN 2010 (2010) 20:

Lattice VOSA

$$\Pi(0)^{\frac{1}{2}} = \mathbb{C}\left[\mathbb{Z}\frac{x}{2}\right] \otimes SM(1)$$

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σ–twisted $\Pi(0)^{\frac{1}{2}}$–module

$$\Pi(0)^{\frac{1}{2}} \cdot v_{p,r} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_{p,r+\frac{n}{2}}$$

Second screening

Recall $a = \Psi^- \left(-\frac{1}{2}\right) e^{\frac{1}{2}x} \in \Pi(0)^{\frac{1}{2}}$, $\qquad a(z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}$. Then

$$a^{tw}(z) = \sum_{n\in\mathbb{Z}} a\left(n + \frac{1}{2}\right) z^{-n-\frac{1}{2}}$$

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Even derivation:

$$S^{tw} = \sum_{i=0}^{\infty} \frac{1}{i+\frac{1}{2}} a\left(-i - \frac{1}{2}\right) a\left(i + \frac{1}{2}\right).$$

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Then

$$\mathcal{G}^{tw} = e_0^{x} - S^{tw}$$

is a screening operator on σ–twisted $\Pi(0)^{\frac{1}{2}}$–modules.

Singular vector $p > 0$ even

Theorem

Assume that $p \in \mathbb{Z}_{>0}$ *is even. Then* $u_{p,r} = \mathcal{G}^{tw} v_{p,r-1}$ *is a singular vector of weight* $h[p,r] + p = h[p, r-1]$ *in* $V[p,r] = \mathcal{F}_{p,r}$.

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Explicit formula:

$$u_{p,r} = (e_0^x - S^{tw}) v_{p,r-1} = S_p\left(-\frac{\alpha}{c_{L,\alpha}}\right) v_{p,r} +$$
$$\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{k+\frac{1}{2}} \left(\sum_{i \geq 0} \Psi(i + k - \tfrac{p-1}{2}) S_i(-\tfrac{\alpha}{2c_{L,\alpha}}) \right) \left(\sum_{j \geq 0} \Psi(j - k - \tfrac{p+1}{2}) S_j(-\tfrac{\alpha}{2c_{L,\alpha}}) \right) v_{p,r}$$

Structure of $V[p, r]$, for $p > 0$ even

Theorem

Assume that $p \in \mathbb{Z}_{>0}$ is even.

(i) $\mathcal{G}^{tw}$ *is injective on* $\mathcal{F}_{p,r}$*, therefore it restricts to inclusion of the Verma module* $V[p, r-1] = \mathcal{F}_{p,r-1}$ *into* $V[p, r]$.

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(ii) *We have the following family of singular vectors in $V[p, r] = \mathcal{F}_{p,r}$:*

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(iii) *$L[p,r] = V[p,r]/V[p,r-1]$.*

(iii) will follow from the character formula.

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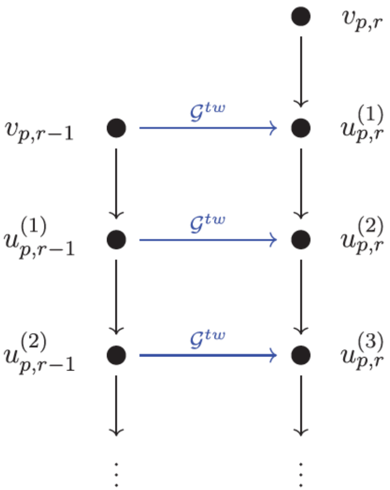

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We have $Q^2 = 0$. Consider a subalgebra

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$$S = \sum_{n \in \mathbb{Z}} \frac{1}{i} a(-i) a(i)$$

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Screening operator on $\overline{\Pi(0)}^{\frac{1}{2}}$

$$\mathcal{G} = e_0^x - S$$

also on modules

$$\overline{\mathcal{F}_{p,r}} = \mathrm{Ker}_{\mathcal{F}_{p,r}} Q$$

Subsingular vectors $p > 0$ odd

Theorem

Assume that $p \in \mathbb{Z}_{>0}$ *is odd. Then* $w_{p,r} = \mathcal{G} v_{p,r-1}$ *is a singular vector in* $\mathcal{F}_{p,r}/\overline{\mathcal{F}_{p,r}}$*, hence a subsingular vector in* $\mathcal{F}_{p,r}$ *of weight* $h[p,r] + p = h[p,r-1]$ *in* $V[p,r] = \mathcal{F}_{p,r}$*.*

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Explicit formula:

$$w_{p,r} = (e_0^x - S)v_{p,r-1} = S_p\left(-\frac{\alpha}{c_{L,\alpha}}\right) v_{p,r} + \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k}\left(\sum_{i\geq 0} \Psi(i+k-\tfrac{p}{2}) S_i(-\tfrac{\alpha}{2c_{L,\alpha}})\right)\left(\sum_{j\geq 0} \Psi(j-k-\tfrac{p}{2}) S_j(-\tfrac{\alpha}{2c_{L,\alpha}})\right) v_{p,r}$$

Structure of $V[p, r]$, for $p > 0$ odd

Theorem

Assume that $p \in \mathbb{Z}_{>0}$ is odd.

(i) *We have the following family of (sub)singular vectors in $V[p, r] = \mathcal{F}_{p,r}$:*

- *Singular vectors* $u_{p,r}^{(n)} = \mathcal{G}^n Q v_{p,r-n-1/2}$, $n \in \mathbb{Z}_{\geq 0}$.
- *Subsingular vectors* $w_{p,r}^{(n)} = \mathcal{G}^n v_{p,r-n}$, $n \in \mathbb{Z}_{>0}$.

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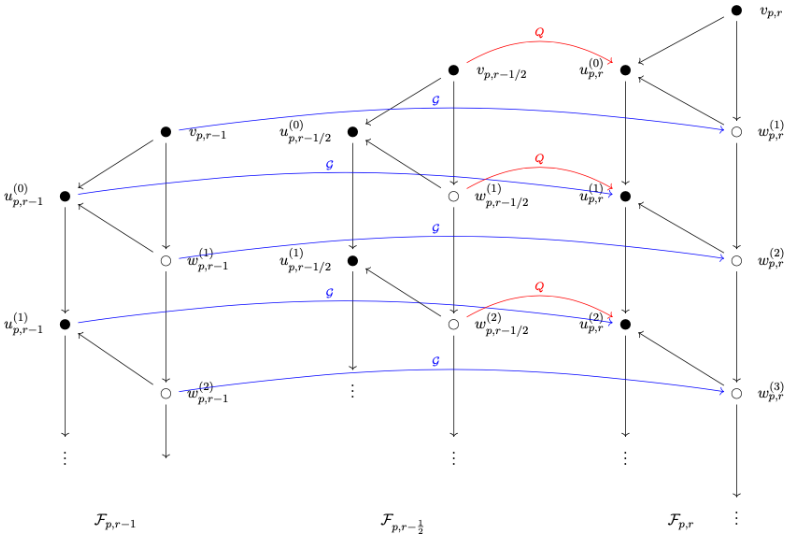
(ii) *$\langle w_{p,r} \rangle$ is the maximal submodule in $[p, r]$ i.e. $L[p, r] = V[p, r]/\langle w_{p,r} \rangle$.*

(ii) will follow from the character formula.

Structure of $V[p, r]$, for $p > 0$ odd



Characters

Theorem

Assume that $p \in \mathbb{Z} \setminus \{0\}$ *and* $r \in \mathbb{C}$. *Then we have:*

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\begin{aligned}
char_q L[p,r] &= q^{h_{p,r}}(1 - q^{\frac{|p|}{2}}) \prod_{k=1}^{\infty} \frac{(1+q^{k-1/2})^2}{(1-q^k)^2}, \quad \textit{if } p \textit{ is odd;} \\
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- "$\geq$" from realisation in case $p < 0$
- "$\leq$" from formulas for (sub)singular vectors in case $p > 0$

Singular vectors $p < 0$

In $\Pi(0)^{\frac{1}{2}}$ we have

$$\left(L(-2) - \frac{c_L - 27}{24}c(-2) - \frac{1}{\sqrt{2}}\Psi^-(-\tfrac{1}{2})G(-\tfrac{3}{2}) - \frac{c_L - 15}{12}\Psi^-(-\tfrac{3}{2})\Psi^-(-\tfrac{1}{2})\right)e^{-c} =$$

$$= -\frac{1}{2}L(-1)\left(d(-1) - \Psi^-(-\tfrac{1}{2})\Psi^+(-\tfrac{1}{2})\right)e^{-c}$$

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Since $(L(-1)v)(0) = 0$ for any v we get a nontrivial relation in $\mathcal{F}_{p,r}$, $p < 0$:

$$R[p, r]v_{p,r} = 0.$$

Formula $p < 0$

$$R[p,r] =$$
$$\sum_{i=1}^{-p} \left(L(-i) + \frac{c_L - 27}{24 c_{L,\alpha}} \alpha(-i) \right) S_{-p-i}\left(\frac{\alpha}{c_{L,\alpha}}\right) + S_{-p}\left(\frac{\alpha}{c_{L,\alpha}}\right)\left(L(0) + \frac{c_L - 3}{24 c_{L,\alpha}} \alpha(0) \right) +$$
$$+ \frac{1}{2 c_{L,\alpha}} \sum_{i=0}^{-p-1} \sum_{k=0}^{-p-i-1} \Psi\left(-i - \frac{1}{2}\right) G\left(-k - \frac{1}{2}\right) S_{-p-i-k-1}\left(\frac{\alpha}{c_{L,\alpha}}\right) +$$
$$- \frac{c_L - 15}{24 c_{L,\alpha}^2} \sum_{i=0}^{-p-1} \sum_{k=0}^{-p-i-1} i \Psi\left(-i - \frac{1}{2}\right) \Psi\left(-k - \frac{1}{2}\right) S_{-p-i-k-1}\left(\frac{\alpha}{c_{L,\alpha}}\right).$$

Singular vectors $p < 0$

Theorem

Let $p \in \mathbb{Z}_{<0}$, $r \in \mathbb{C}$. Then $u_{p,r} = R[p,r]v_{p,r}$, where is a non-trivial singular vector of weight $h[p,r] - p = h[p,r+1]$ in $V[p,r]$.

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If $p < 0$ is odd there exists a singular vector of weight $h[p,r] - \frac{p}{2} = h\left[p, r + \frac{1}{2}\right]$.
We couldn't obtain the explicit formula in this case.

$$G\left(\frac{p}{2}\right) vpr + \cdots$$

Structure $p < 0$

Theorem

Assume that $p \in \mathbb{Z} \setminus \{0\}$ is even. The maximal submodule of $V[p, r]$ is isomorphic to $V[p, r \pm 1]$.

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$p > 0$ we already showed.

For $p < 0$ we proved that $R[p, r]$ is not a zero divisor.

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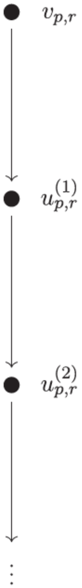
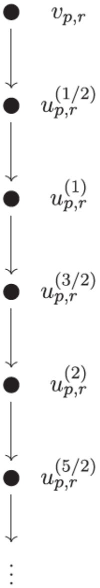
For $p < 0$ we proved that $R[p, r]$ is not a zero divisor.

Theorem

Assume that $p \in \mathbb{Z}_{<0}$ is odd. The maximal submodule of $V[p, r]$ is isomorphic to $V\left[p, r + \frac{1}{2}\right]$.



(a) $V[p,r]$, $p \in \mathbb{Z}_{<0}$ even. Singular vector $u_{p,r}^{(n)}$ generates $V[p,r+n]$ if $p<0$.



(b) $V[p,r]$, $p \in \mathbb{Z}_{<0}$ odd. Singular vector $u_{p,r}^{(n)}$ generates $V[p,r+n]$, $n \in \frac{1}{2}\mathbb{Z}_{>}0$.

Screenings on $\mathcal{F}_{p,r}$, $p < 0$ even

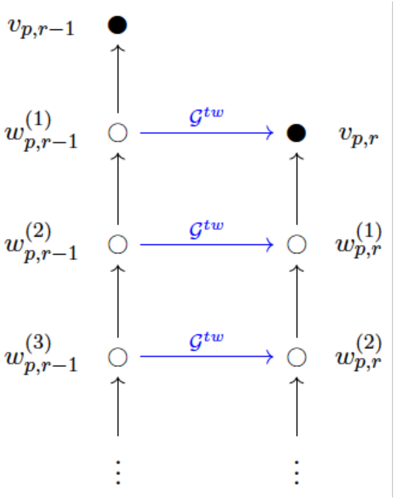


Figure: Subsingular vector $w^{(n-1)}$ generates $\mathrm{Ker}_{\mathcal{F}_{p,r}}(\mathcal{G}^{tw})^n$, $n \in \mathbb{Z}_{>0}$.

Screenings on $\mathcal{F}_{p,r}$, $p < 0$ odd

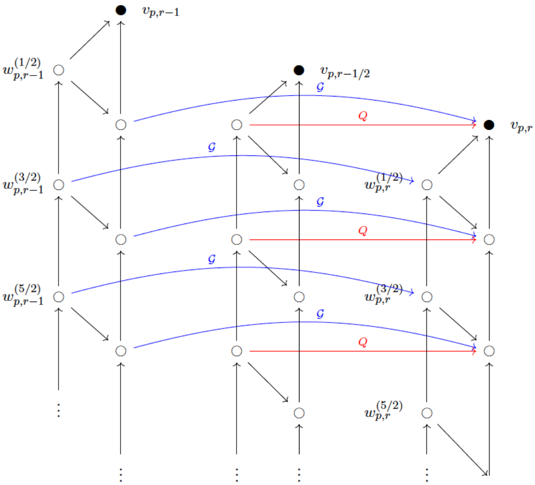


Figure: $\mathcal{G}w^{(n+1/2)}_{p,r-1} = w^{(n-1/2)}_{p,r}$ and $Qw^{(n+1/2)}_{p,r-1/2}$ are subsingular vectors.