# An Algorithm to Build the Auslander-Reiten Quiver of Some Equipped Posets.

Workshop on Representation Theory and Applications
São Paulo Brazil

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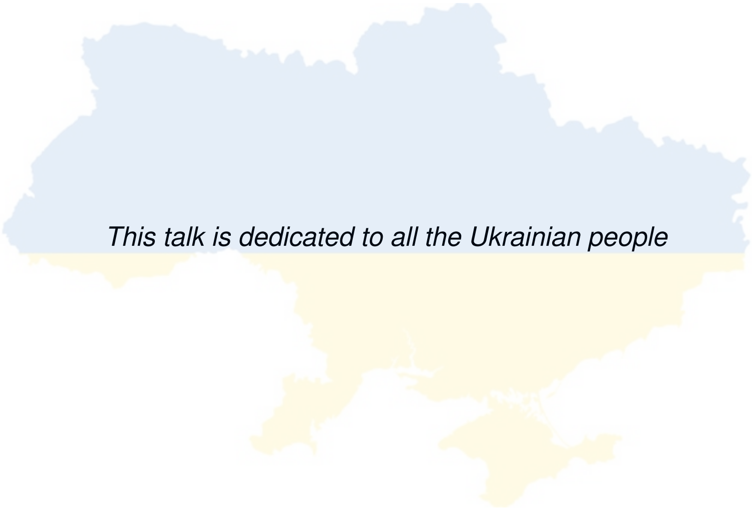

*This talk is dedicated to all the Ukrainian people*

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## Aims and Scope

In this talk, we describe an algorithm to build the Auslander-Reiten Quiver of Some Equipped Posets.

# Introduction

The representations theory of partially ordered sets or posets was introduced and developed in the early 1970s by Nazarova, Roiter and their students in Kiev.

To do that, Nazarova and Roiter introduced an algorithm known as the algorithm of differentiation with respect to a maximal point which allowed to Kleiner in 1972 to obtain a classification of posets of finite representation type.

# Introduction

The categorical properties of such an algorithm was given by Gabriel in 1973 [2, 3], founding in this way a new line of research in representation theory of posets which consists of proving that algorithms of differentiation induce a categorical equivalence between the category of representations of the original poset and the corresponding category of the derived poset [8, 9, 12].

Such functors constitute the main tool to classify posets of different types, for instance, the algorithm of differentiation with respect to a maximal point was also used by Nazarova to give a tame representation type criterion for ordinary posets.

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# Introduction

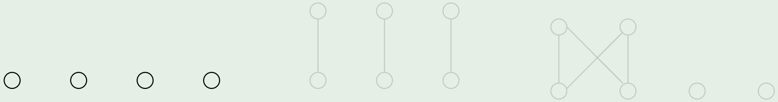
Soon afterwards between 1974 and 1977, Zavadskij defined the more general algorithm I (D-I) with respect to a suitable pair of points, this algorithm was used in 1981 by Nazarova and Zavadskij in order to give a criterion for the classification of posets of finite growth representation type [4, 5, 8].

# Introduction: Ordinary posets

## Definition

A partially ordered set (*Poset*) is an ordered pair $(\mathcal{P}, \leq)$ which consists of a not empty set $\mathcal{P}$ and a binary relation contained in $\mathcal{P} \times \mathcal{P}$, called order, such that $\leq$ is reflexive, antisymmetric and transitive.

## Example



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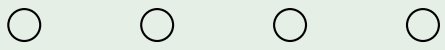

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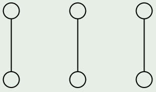

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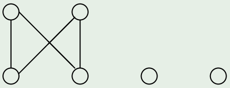
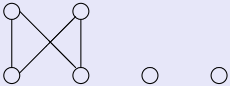


$G_1$

# Introduction: Ordinary posets

**Theorem (Nazarova-Zavadskij 1981)**

*A poset $\mathcal{P}$ of tame representation type is of finite growth representation type if and only if it does not contain the following subposet*



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# Introduction

The theory of representation of posets was developed in the 1980's and 1990's for posets with additional structures, for example, for posets with involution or for equipped posets by Bondarenko, Nazarova, Roiter, Zabarilo and Zavadskij among others.

## Equipped posets

# Equipped posets

In [14] A. V. Zabarilo and A.G. Zavadskij they presented the Equipped poset and their representations.

A poset $(\mathcal{P} \leq)$ is said to be **equipped** if the following three conditions hold:

1. The points of the set $\mathcal{P}$ can be either strong or weak;
2. The order relations between points can be either weak or strong;
3. If $x < y$ is a weak relation, then both points $x$ and $y$ are weak, and if $x < t < y$ in this case, then the relations $x < t$ and $t < y$ are also weak (and the point $t$ is automatically weak).

# Equipped posets

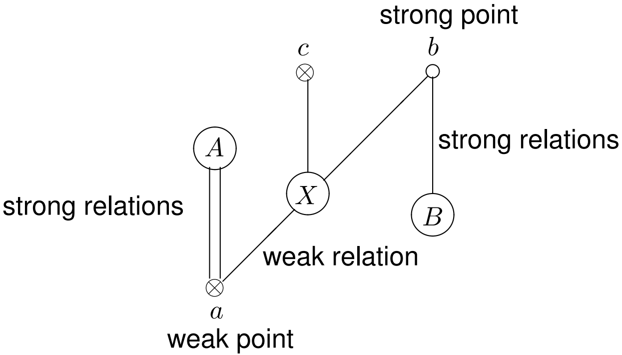
## Definition

A poset $(\mathcal{P}, \leq)$ is called *equipped* if all the order relations between its points $x \leq y$ are separated into strong (denoted $x \trianglelefteq y$) and weak (denoted $x \preceq y$) in such a way that

$$x \leq y \trianglelefteq z \quad \text{or} \quad x \trianglelefteq y \leq z \quad \text{implies} \quad x \trianglelefteq z.$$

i.e., a composition of a strong relation with any other relation is strong [6].

# Equipped posets



# Equipped posets

These equipped posets were introduced and classified by Zabarilo and Zavadskij, and they reflected the case of ”order relation multiplicities” not exceeding 2.

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# The Category of representations $(\text{rep}\,\mathcal{P})$

The *complexification* of a real vector space can be generalized to the case $(F, G)$ where $G = F(\mathbf{u})$ is a quadratic extension of $F$. In this case, we assume that **u** is a root of the minimal polynomial $m(t) = t^2 + \alpha t + \beta$, $\beta \neq 0$, $(\alpha, \beta \in F)$. In particular if $U_0$ is a $F$-space then the corresponding complexification is the $G$-vector space denoted $\widetilde{U_0}$. As in the case $(\mathbb{R}, \mathbb{C})$, we write $U_0 + \mathbf{u}U_0 = \widetilde{U_0}$.

To each $G$-subspace $W$ of $\widetilde{U_0}$ it is possible to associate the following $F$-subspaces of $U_0$,

$$W^+ = \operatorname{Re} W = \operatorname{Im} W \quad \text{and}$$

$$W^- = span\{\alpha \in U_0 \mid (\alpha, 0)^t \in W\} \subset W^+,$$

and for a $G$-space $Z$ we have the following property

$\widetilde{Z^+} = F(Z)$, is called the $F$-*hull* of $Z$ such that $Z \subset F(Z)$, The *complexification* of a real vector space can be generalized to the case $(F, G)$ where $G = F(\mathbf{u})$ is a quadratic extension of $F$. In this case, we assume that $\mathbf{u}$ is a root of the minimal polynomial $m(t) = t^2 + \alpha t + \beta$, $\beta \neq 0$, $(\alpha, \beta \in F)$. In particular if $U_0$ is a $F$-space then the corresponding complexification is the $G$-vector space denoted $\widetilde{U_0}$. As in the case $(\mathbb{R}, \mathbb{C})$, we write $U_0 + \mathbf{u}U_0 = \widetilde{U_0}$.

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## Definition

A representation $U \in \mathrm{rep}\,\mathcal{P}$ is a system of vectors spaces of the form:

$$U = (U_0\ ;\ U_x \mid x \in \mathcal{P}),$$

where $U_0$ is a finite dimensional $F$-space and for each $x \in \mathcal{P}$, $U_x$ is a $G$-subspace of $\widetilde{U_0}$, such that,

$$x \preceq y \Longrightarrow U_x \subset U_y$$
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In this case, a morphism $\varphi$ is defined in such a way that $\varphi : (U_0\ ;\ U_x \mid x \in \mathcal{P}) \longrightarrow (V_0\ ;\ V_x \mid x \in \mathcal{P})$ is an $F$-linear map $\varphi : U_0 \longrightarrow V_0$ such that:

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The sum of two representations $U, V \in \operatorname{rep} \mathcal{P}$ is given by the formula:

$$U \oplus V = (U_0 \oplus V_0; U_x \oplus V_x \mid x \in \mathcal{P})$$

A representation $U \neq 0$ is indecomposable if $U \simeq U_1 \oplus U_2$ implies that $U_1 = 0$ or $U_2 = 0$.

Thus, $\operatorname{rep} \mathcal{P}$ have the property of decomposition unique, i.e. $\operatorname{rep} \mathcal{P}$ is a Krull-Schmidt category. $\operatorname{Ind} \mathcal{P}$ denotes the set of all indecomposable representations of $\mathcal{P}$.

# Matrix problem

Each equipped poset $\mathcal{P}$ naturally defines a matrix problem of mixed type over the pair $(F, G)$. Consider a rectangular matrix $M$ separated into vertical stripes $M_x$, $x \in \mathcal{P}$, with $M_x$ being over $F$ (over $G$) if the point $x$ is strong (weak):

$$\begin{array}{c}x \longrightarrow y \\ \begin{array}{cccc}\otimes & \otimes & \bigcirc & \bigcirc\end{array}\end{array}$$

$$M = \begin{array}{|c|c|c|c|}\hline G & G & F & F \\ \hline\end{array},$$

such partitioned matrices $M$ are called *matrix representations of* $\mathcal{P}$ over $(F, G)$.

## The Main Problem

The main problem regarding equipped posets consists of giving a complete description of indecomposable representations and irreducible morphisms of the category of representations $\operatorname{rep} \mathcal{P}$ of a given equipped poset $\mathcal{P}$.

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# Some Indecomposable Representations

Here we describe some examples of indecomposable representations in the category $\mathrm{rep}\ \mathcal{P}$, where $\mathcal{P}$ is an equipped poset.

If $\mathcal{P}$ is an equipped poset and $A \subset \mathcal{P}$ then $P(A) = P(\min A) = (F = \mathbb{R}\ ;\ P_x \mid x \in \mathcal{P})$, $P_x = G = \mathbb{C}$ if $x \in A^\vee$ and $P_x = 0$ otherwise. In particular, $P(\varnothing) = (F\ ;\ 0, \ldots, 0)$.

# Some Indecomposable Representations

If $a, b \in \mathcal{P}^{\otimes}$, $c \in \mathcal{P}^{\circ}$ then $T(a)$, $T(a,b)$, $G_1(a,c)$, $G_2(a,c)$ denote indecomposable objects with matrix representation of the following form:

$$T(a) = \begin{array}{c} a \\ \begin{array}{|c|} \hline 1 \\ \mathbf{u} \\ \hline \end{array} \end{array}, \quad a \in \mathcal{P}^{\otimes},$$

$$\otimes$$

$$T(a,b) = \begin{array}{cc} a & b \\ \end{array} \begin{array}{|c|c|} \hline 1 & 0 \\ \mathbf{u} & 1 \\ \hline \end{array}, \text{ with } a \prec b.$$

$$\begin{array}{l} \otimes\, b \\ \,| \\ \otimes\, a \end{array}$$

$$G_1(a,c) = \begin{array}{cc} a & c \\ \end{array} \begin{array}{|c|c|} \hline 1 & 0 \\ \mathbf{u} & 1 \\ \hline \end{array}$$

$$G_2(a,c) = \begin{array}{cc} a & c \\ \end{array} \begin{array}{|c|cc|} \hline 1 & 0 & 1 \\ \mathbf{u} & 1 & 0 \\ \hline \end{array}$$

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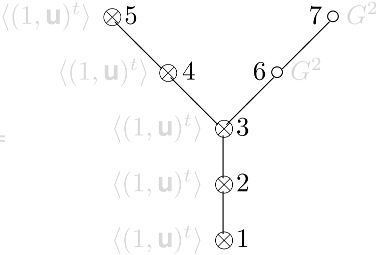
# Some Indecomposable Representations

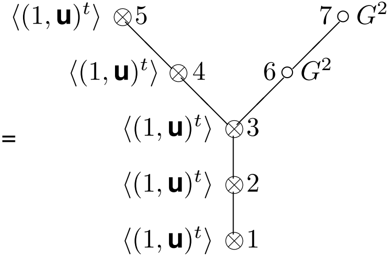
T(1) =




$(\mathbb{R},\mathbb{C}) \quad \mathbf{u} = \mathbf{i}$

P(4) =

T(1) =

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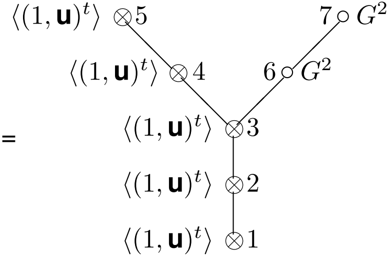


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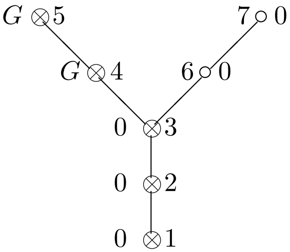


## Definition

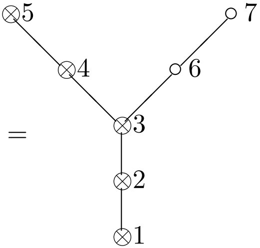
The *evolvent* of an equipped poset $\mathcal{P}$ is an ordinary poset:

$$\widehat{P} = \bigcup_{x \in \mathcal{P}} \{x', x''\}$$

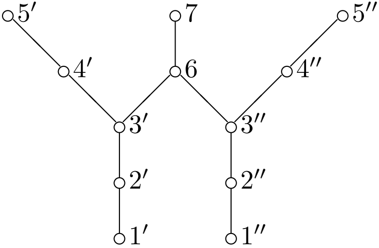
where $x' = x'' = x$ for a strong point $x \in \mathcal{P}$, and $x' \neq x''$ is a pair of new strong mutually incomparable points (replacing a weak point $x$), with the following order relations:

$(1)$ if $x \prec y$, then $x' < y'$ and $x'' < y''$;
$(2)$ if $x \lhd y$, then $x' < y'$; $x' < y''$; $x'' < y'$ and $x'' < y''$.

$\mathcal{P} =$




$\widehat{\mathcal{P}} =$



# Classification theorems

It was necessary to define a new class of algorithms to classify posets with these additional structures. In fact, Zavadskij introduced 17 algorithms. Algorithms, I-V (and some additional differentiations) were used by him and Bondarenko to classify posets with involution, whereas algorithms I, VII-XVII and completion were used to classify equipped posets with involution in 2003 [10].

In particular, algorithms I, VII, VIII, IX and completion were used to classify equipped posets of finite growth representation type without paying attention to the behavior of the morphisms of the corresponding categories.

The following theorems regard the classification of equipped posets [10, 11].

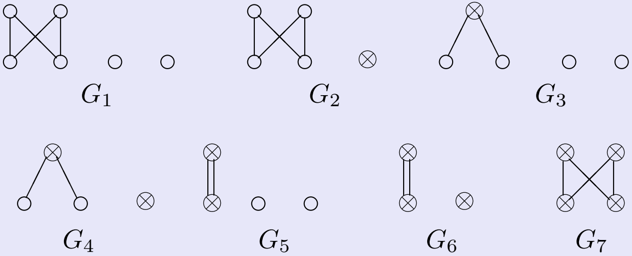
**Theorem (Zavadskij 2003)**

*An equipped poset $\mathcal{P}$ is tame (wild) if its evolvent $\widehat{\mathcal{P}}$ is tame (wild).*

## Theorem (Zavadskij 2005)

*For an equipped poset $\mathcal{P}$, the following statements are equivalent.*

- (a) *$\mathcal{P}$ is of finite growth (over an arbitrary field $k$).*
- (b) *The evolvent $\widehat{\mathcal{P}}$ is of finite growth.*
- (c) *The set $\mathcal{P}$ is not wild and contains none of the following equipped posets:*

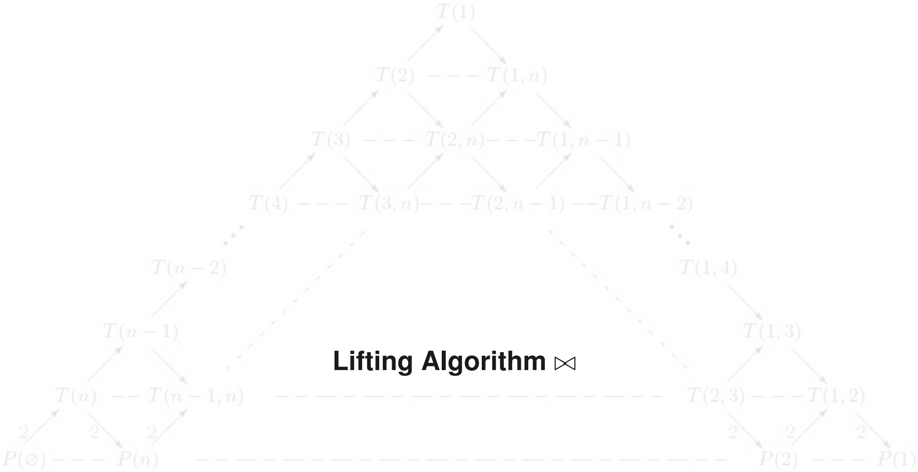



- (d) *$\mathcal{P}$ does not contain subsets of the form $N_1 \ldots, N_6$, $W_1 = (\tilde{1}, 1, 1, 1)$, $W_2 = (\tilde{1}, \tilde{1}, 1)$, $W_3 = (\tilde{1}, \tilde{1}, \tilde{1})$, $W_4 = (\tilde{1}, 1, 2)$, $W_5 = (\tilde{2}, 1, 1)$, $W_6 = (\tilde{1}, \tilde{2})$, $W_7 = (\tilde{2}, 3)$, $W_8 = (\tilde{3}, 2)$, $W_9 = (\tilde{4}, 1)$, and posets $G_1, \ldots, G_7$.*

# Algorithms of differentiation

In 1991 Zavadskij followed following Gabriel's ideas regarding the categorical properties of the algorithm of differentiation with respect to a maximal point [2] described the structure of the Auslander-Reiten quiver for ordinary posets of finite growth representation type [9].

To do that, he proved that the algorithm with respect to a suitable pair of points induces a categorical equivalence between some quotient categories.




# Lifting Algorithm $\bowtie$

We recall that according to Arnold $P(x) = (k; P(x)_y \mid x \in \mathcal{P})$[1]

$$P(x)_y = \begin{cases} k & \text{if } y \in x^{\nabla}, \\ 0 & \text{otherwise.} \end{cases} \qquad I(x)_y = \begin{cases} 0 & \text{if } y \in x_{\Delta}, \\ k & \text{otherwise.} \end{cases}$$

Following to Simson [7], arrows in $\Gamma(\mathcal{P})$ are valued arrows $[U] \xrightarrow{(d_{UV},\, d'_{UV})} [V]$ where $d'_{UV} = \dim \operatorname{Irr}(U,V)_{\overline{(U)}}$ and $d_{UV} = \dim_{\overline{(V)}} \operatorname{Irr}(U,V)$, and $\overline{U} = \operatorname{End} U/\operatorname{rad}\operatorname{End} U$. This means that there are irreducible morphisms $U \longrightarrow V^{d_{UV}}$ and $U^{d'_{UV}} \longrightarrow V$, we write $U \longrightarrow V$ if $d_{UV} = d'_{UV} = 1$.

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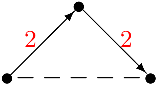
Henceforth, the following notations will be assumed in valued arrows of an Auslander-Reiten quiver.

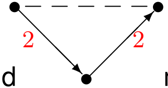
We recall that according to Arnold $P(x) = (k; P(x)_y \mid x \in \mathcal{P})$[1]

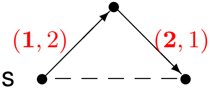
$$P(x)_y = \begin{cases} k & \text{if } y \in x^{\nabla}, \\ 0 & \text{otherwise.} \end{cases} \qquad I(x)_y = \begin{cases} 0 & \text{if } y \in x_{\triangle}, \\ k & \text{otherwise.} \end{cases}$$

Following to Simson [7], arrows in $\Gamma(\mathcal{P})$ are valued arrows $[U] \xrightarrow{(d_{UV},\, d'_{UV})} [V]$ where $d'_{UV} = \dim \operatorname{Irr}(U,V)_{\overline{(U)}}$ and $d_{UV} = \dim_{\overline{(V)}} \operatorname{Irr}(U,V)$, and $\overline{U} = \operatorname{End} U/\operatorname{rad} \operatorname{End} U$. This means that there are irreducible morphisms $U \longrightarrow V^{d_{UV}}$ and $U^{d'_{UV}} \longrightarrow V$, we write $U \longrightarrow V$ if $d_{UV} = d'_{UV} = 1$.

Henceforth, the following notations will be assumed in valued arrows of an Auslander-Reiten quiver.



and


means


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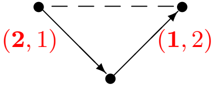


# Lifting Algorithm $\bowtie$

The following algorithm is a way to compress the information (in terms of the dimension vectors of the indecomposable representations) given by the Auslander-Reiten quiver of the evolvent of an equipped poset.

**Lifting Algorithm $\bowtie$ and the Auslander-Reiten quiver.**

Input. An equipped poset of type $\mathcal{P} \in \{F_{13}, \ldots, F_{18}\}$ or $\mathcal{A} \in \{k_6, k_8\}$.

Output. The Auslander-Reiten quiver $\Gamma(\mathcal{P})$ or $\mathbb{P}(\mathcal{A})$, $\mathbb{I}(\mathcal{A})$.

# Lifting Algorithm $\bowtie$

1. **choose** one of the following equipped posets $F_{13}, \ldots, F_{18}$, $k_6$, $k_8$, and denoted it by $\mathcal{P}$,
2. **compute** the evolvent $\widehat{\mathcal{P}}$ of $\mathcal{P}$,
3. **compute** the Auslander-Reiten quiver $\Gamma(\widehat{\mathcal{P}})$ of the ordinary poset $\widehat{\mathcal{P}}$, associating its corresponding dimension vectors,
4. **fix** a vertex $U = (U_0;\ U_x \mid x \in \widehat{\mathcal{P}})$ in $\Gamma(\widehat{\mathcal{P}})$ with vector dimension $d$ of the form $d_U = (d_0; d_x \mid x \in \widehat{\mathcal{P}})$,
5. (*lifting*; $\ltimes\Gamma(\widehat{\mathcal{P}})$)
   - a. ($\ltimes U$) for each $x \in \mathcal{P}^{\otimes}$ with associated evolvent points $x'$ and $x''$, **do** $d_x = d_{x'} + d_{x''}$
   - b. ($\ltimes f$) for each arrow $f : U \longrightarrow V \in \Gamma(\widehat{\mathcal{P}})$ **define** the arrow $\ltimes f : \ltimes U \to \ltimes V$ in $\Gamma(\mathcal{P})$, keeping neighbors and arrows orientation of $\Gamma(\widehat{\mathcal{P}})$.

# Lifting Algorithm $\bowtie$

6. **fix** two vertices $U$ and $V$ in $\ltimes\Gamma(\widehat{\mathcal{P}})$,
7. (*weak gluing*; $\rtimes\Gamma(\widehat{\mathcal{P}})$) if $d_U = d_V = d = (d_0; d_x \mid x \in \mathcal{P})$ then **do** $2d = (2d_0; 2d_x \mid x \in \mathcal{P})$ and identify $U$ and $V$ else keep $d_U$ and $d_V$ invariants,
8. **fix** a vertex $W$ in $\rtimes\Gamma(\widehat{\mathcal{P}})$ with vector dimension $s = (s_0; s_x \mid x \in \mathcal{P})$ given in step 7,
9. **define** $\mathbf{d} = (\mathbf{d_0}; \mathbf{d_x} \mid \mathbf{x} \in \mathcal{P})$ and do $\boldsymbol{d_0} = s_0$,

$$\boldsymbol{d_x} = \begin{cases} \frac{s_x}{2} & \text{if } x \in \mathcal{P}^{\otimes}, \\ s_x & \text{if } x \in \mathcal{P}^{\circ}. \end{cases}$$

10. assign vector $\mathbf{d}$ to a unique vertex $U$ of $\Gamma(\mathcal{P})$ keeping neighbors and arrows orientation of $\rtimes\Gamma(\widehat{\mathcal{P}})$,
11. **define** the dimension vector $l = (l_0 = \mathbf{d_0}; l_x = \frac{\mathbf{d_x}}{\dim \operatorname{rad} U_x} \mid x \in \mathcal{P})$.

# Some properties of the lifting algorithm $\bowtie$

1. $\ltimes\tau^{-l}P(x^{'}) = \ltimes\tau^{-l}P(x^{''})$ for any positive integer $l$.
2. Note that, $\bowtie(P(\widehat{\varnothing})) = P(\varnothing)$ and $\bowtie(I(\widehat{\mathcal{P}})) = I(\mathcal{P})$.
3. Arrows $\bowtie(f) : \bowtie(U) \to \bowtie(V)$ correspond to irreducible morphisms, were $f$ is a system of arrows in $\Gamma(\widehat{\mathcal{P}})$ as we previously described.
4. If $P(x)$ is an indecomposable projective representation of $\widehat{\mathcal{P}}$ then $\bowtie(P(x)) = P(x)$ is an indecomposable projective representation of $\mathcal{P}$.
5. If $P(x')$ and $P(x'')$ are indecomposable projective representations of $\widehat{\mathcal{P}}$, then $\bowtie(P(x') \oplus P(x'')) = T(x)$ is an indecomposable projective representation of $\mathcal{P}$.

# Auslander-Reiten quiver $\bowtie\Gamma(\widehat{F_{13}})$

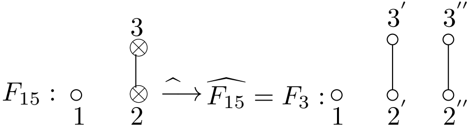
$$F_{13}: \underset{1}{\otimes} \xrightarrow{\ \widehat{\ }\ } \widehat{F_{13}} = F_7: \underset{1'}{\circ} \qquad \underset{1''}{\circ}$$

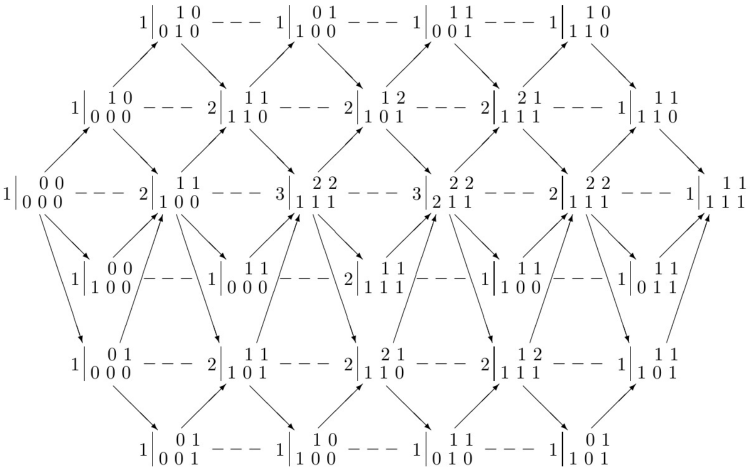
$$\begin{tikzcd}[column sep=small]
 & 1\mid 1\,0 \arrow[dr] & & & & 1\mid 1 \arrow[dr] & & & & 2\mid 1 \arrow[dr] & \\
\Gamma(F_7): 1\mid 0\,0 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 1\mid 1\,1 \arrow[r] & \ltimes\Gamma(F_7) & 1\mid 0 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 1\mid 2 \arrow[r] & \rtimes\Gamma(F_7) & 1\mid 0 \arrow[ur] \arrow[rr, dash, dashed] & & 1\mid 1 \\
 & 1\mid 0\,1 \arrow[ur] & & & & 1\mid 1 \arrow[ur] & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \underline{2}\mid 1 \arrow[dr, "2"] & \\
\bowtie\Gamma(F_7): & 1\mid 0 \arrow[ur, "2"] \arrow[rr, dash, dashed] & & \underline{1}\mid 1
\end{tikzcd}$$

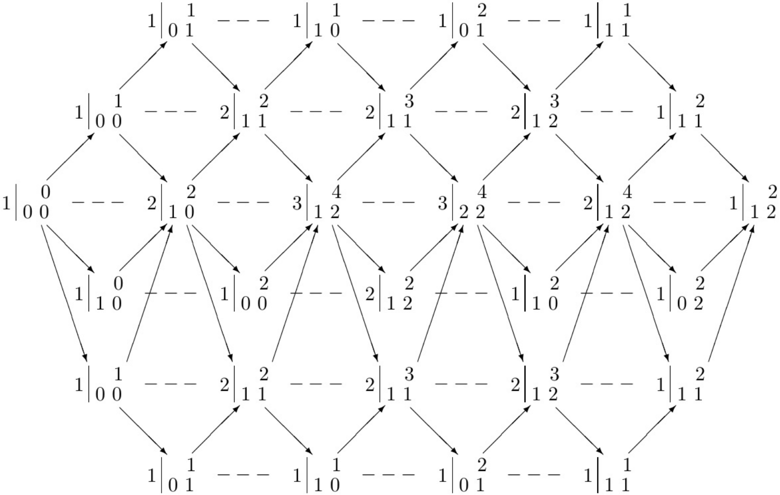
Figure: The Auslander-Reiten quiver $\Gamma(F_{13})$ of the equipped poset $F_{13}$ obtained from $\bowtie\Gamma(F_7)$ via the lifting algorithm $\bowtie$.

# Auslander-Reiten quiver $\bowtie\Gamma(\widehat{F_{15}})$

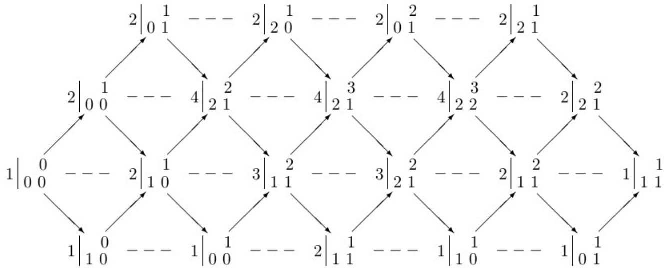


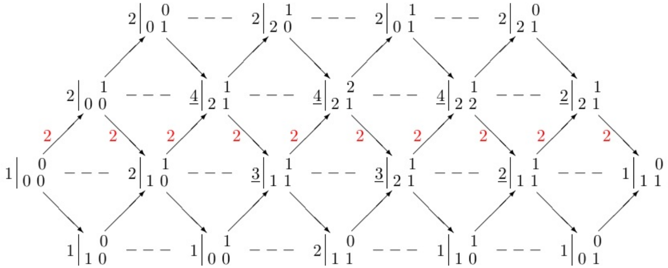



# Auslander-Reiten quiver $\bowtie\Gamma(\widehat{F_{15}})$

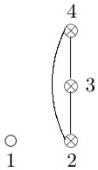


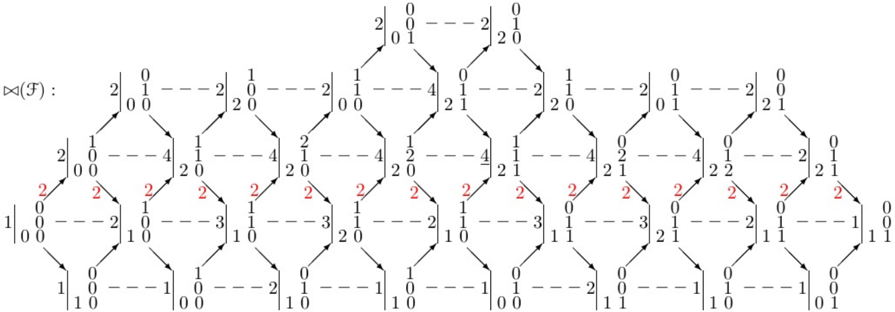
# Auslander-Reiten quiver $\bowtie\Gamma(\widehat{F_{15}})$



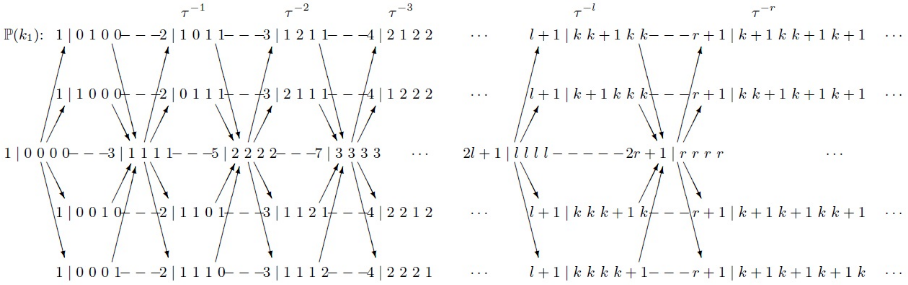


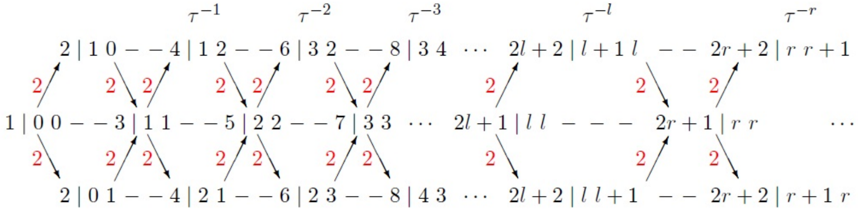
# Auslander-Reiten quiver $\bowtie\Gamma(\widehat{F_{18}})$

$F_{18} =$ 


$\bowtie(\mathcal{F})$ : 

# Preprojective component of the Auslander-Reiten quiver $\bowtie\Gamma(\widehat{k_6})$



**Theorem**

*Let $\mathcal{P}$ be a sincere equipped poset of finite representation type. Then the Auslander-Reiten quiver of $\mathcal{P}$ is $\bowtie(\widehat{\mathcal{P}})$.*

**Theorem**

*Let $\mathcal{P} \in \{k_i \mid i \in 6, 8\}$ be a sincere equipped poset of infinite representation type. If $\mathbb{P}(\widehat{\mathcal{P}})$, (resp. $\mathbb{I}(\widehat{\mathcal{P}})$) is the preprojective (resp. preinjective) component of $\widehat{\mathcal{P}}$, then the preprojective (resp. preinjective) component of $\mathcal{P}$ is $\bowtie(\mathbb{P}(\widehat{\mathcal{P}}))$ (resp. $\bowtie(\mathbb{I}(\widehat{\mathcal{P}}))$).*

### Theorem

*Let $\mathcal{P}$ be a sincere equipped poset of finite representation type. Then the Auslander-Reiten quiver of $\mathcal{P}$ is $\bowtie(\widehat{\mathcal{P}})$.*

### Theorem

*Let $\mathcal{P} \in \{k_i \mid i \in 6, 8\}$ be a sincere equipped poset of infinite representation type. If $\mathbb{P}(\widehat{\mathcal{P}})$, $($resp. $\mathbb{I}(\widehat{\mathcal{P}}))$ is the preprojective $($resp. preinjective$)$ component of $\widehat{\mathcal{P}}$, then the preprojective $($resp. preinjective$)$ component of $\mathcal{P}$ is $\bowtie(\mathbb{P}(\widehat{\mathcal{P}}))$ $($resp. $\bowtie(\mathbb{I}(\widehat{\mathcal{P}})))$.*

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# Thank You

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