CLASSICAL, QUANTUM AND CATEGORICAL (or representations, quantizations and categorifications)

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Lie algebras, quantum groups and their representations (classical and quantum)

Consider the Lie algebra

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0 \right\}$$

$$= span_{\mathbb{C}} \left\{ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

where $[H, E] = 2E$, $[H, F] = -2F$ y $[E, F] = H$.

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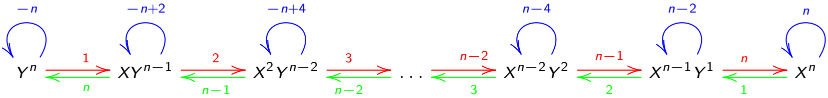
Its universal enveloping algebra is

$$U(\mathfrak{sl}_2) = span_{\mathbb{C}} \left\{ E, H, F \right\}$$

where $HE - EH = 2E$, $HF - FH = -2E$ y $EF - FE = H$.

The irreducible representations are of the form:



This representations is denoted $L(n)$ and we say that n is its weight.

A quantum group for $\mathfrak{sl}_2(\mathbb{C})$

Let

- v a variable.
- $\mathbb{C}(v)$ the field of rational functions with coefficients in $\mathbb{C}$.

$U_v(\mathfrak{sl}_2)$ is the associative algebra with unity over $\mathbb{C}(v)$ generated by elements E, F, K, K^{-1} subject to the relations:

$$KK^{-1} = K^{-1}K = 1$$

$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}}$$

$$KE = v^2 EK$$

$$KF = v^{-2} FK$$

Representations of $U_v(\mathfrak{sl}_2)$

When v is generic or takes the value of a complex number which is not a root of unity, the representation theory of $U_v(\mathfrak{sl}_2)$ is completely analogous to the one of $\mathfrak{sl}_2(\mathbb{C})$

The quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$

Consider $q \in \mathbb{C}$ such that $q^\ell = 1$. Let $\mathbf{U}_q(\mathfrak{sl}_2)$ be the (Lusztig version) quantum group that we get when we spacialize v to q.

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Here the representation theory becomes interesting. The category of finite dimensional modules over $\mathbf{U}_q(\mathfrak{sl}_2)$ is not semisimple

Examples of representations when $q^3 = 1$

$\Delta(0)$:

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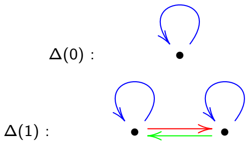

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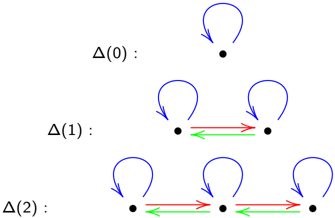

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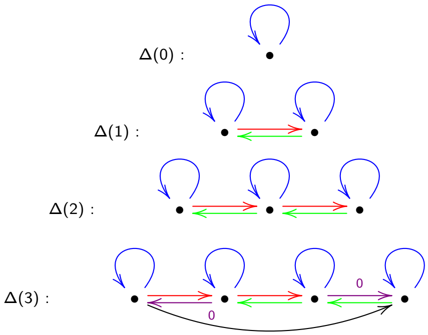

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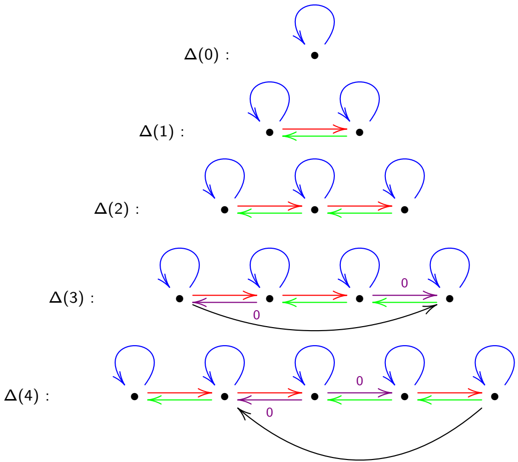

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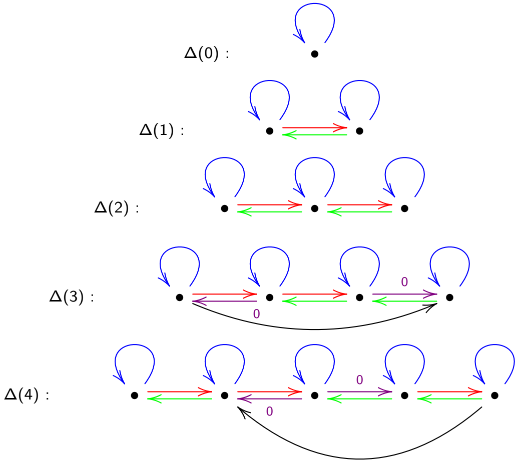

$\Delta(0), \Delta(1)$ and $\Delta(2)$ are irreducibles. $\Delta(3)$ and $\Delta(4)$ are not!

Application 1: Fusion (quantum and categorical)

Let us consider the category of finite dimensional representations for $\mathbf{U}_q(\mathfrak{sl}_2)$ when q is a root of unity.

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This is a non-semisimple category, but if we consider the modules $\Delta(i)$ for $i = 0, 1, \ldots, \ell - 2$ we get a semisimple category. This category is called a fusion category and its Grothendieck ring a fusion ring.

Example: fusion for $\mathbf{U}_q(\mathfrak{sl}_2)$, $q^5 = 1$

- Irreducible objects $\Delta(0)$, $\Delta(1)$, $\Delta(2)$, $\Delta(3)$
- Tensor products: $\Delta(0) \otimes \Delta(i) \cong \Delta(i)$ para $i = 0, 1, 2, 3$.
  $\Delta(1) \otimes \Delta(1) \cong \Delta(2) \otimes \Delta(2) \cong \Delta(0) \oplus \Delta(2)$,
  $\Delta(1) \otimes \Delta(2) \cong \Delta(1) \oplus \Delta(3)$, $\Delta(3) \otimes \Delta(2) \cong \Delta(1)$.

Construction of the fusion category: Tilting modules

The construction of the fusion category associated to a quantum group depends on some modules called tilting modules (modules such that it and its dual possess a filtration with quotients given by the modules $\Delta(i)$).

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- Every tilting module can be written as a direct sum of indecomposable tilting modules.
- The category of tilting modules is closed under tensor products.

Construction of the fusion category: Tilting modules

From the above, we can write any tilting module as:

$$T \cong \bigoplus_{0 \leq i \leq \ell-2} T(i)^{n_i} \oplus \bigoplus_{j \geq \ell-1} T(i)^{n_j}$$

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- The $T(j)$ are indecomposables and they form the non-semisimple part. This modules are called negligible tilting modules.

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$\mathcal{F}$ is semisimple and closed under direct sums, tensor products and duals.

Question: Can we avoid the use of tilting modules?

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Partial solution: Let $D^b(\mathbf{U}_q(\mathfrak{sl}2))$ the bounded derived category of finite dimensional representations of $\mathbf{U}_q(\mathfrak{sl}_2)$, let $\langle \mathcal{N} \rangle$ the triangulated subcategory of the derived category which contains the negligible tilting modules and is closed under direct summands and tensor products with arbitrary modules. Then:

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Theorem, A.

- $\langle \mathcal{N} \rangle$ is generated by modules of $\mathbf{U}_q(\mathfrak{sl}_2)$ belonging to the singular blocks (just depends on the representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$).
- $D^b(\mathbf{U}_q(\mathfrak{sl}2))/\langle \mathcal{N} \rangle$ categorifies the fusion ring.

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Let $\mathcal{C}$ a category in which the notion of negligible object has sense.

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Teorema, A.

Using this category we get known results for the fusion category of the small quantum group of $\mathfrak{sl}_2$.

The symmetric group and the Hecke algebra (classical and quantum)

Recall that the symmetric group S_n can be presented by generators $\{s_1, \ldots, s_{n-1}\}$ subject to the relations

- $s_i^2 = 1$
- $s_{i+1}s_is_{i+1} = s_is_{i+1}s_i$
- $s_is_j = s_js_i, |i-j| > 1$

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The Hecke algebra $\mathcal{H}_v(n)$, associated to the symmetric group S_n, is the associative algebra with unit over $\mathbb{Z}[v, v^{-1}]$ generated by the simbols $\{\delta_i | i = 1, \ldots, n-1\}$ and subject to the relations

- $\delta_i^2 = (v^{-1} - v)\delta_i + 1$
- $\delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1}$
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$\mathcal{H}_v(n)$ has another base called the Kazhdan-Lusztig basis, denoted $\{b_i | i = 1, \ldots, n\}$. This basis is of great importance in representation theory.

Soergel bimodules (categorical)

Soergel bimodules

Let $R = \mathbb{C}[x_1, \ldots, x_{n-1}]$ such that $\deg(x_i) = 2$. For each generator of the symmetric group s_i, $i \in \{1, 2, \ldots, n-1\}$ we define

$$B_i := R \otimes_{R^{s_i}} R(1)$$

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A Bott-Samelson bimodule associated to an expression $\underline{s} = (s_{i_1}, \ldots, s_{i_r})$ is defined by

$$BS(\underline{s}) = B_{i_1} B_{i_2} \cdots B_{i_r}$$

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A Soergel bimodule is a direct summand or a finite sum of graded shifts of Bott-Samelson bimodules. We denote the category of Soergel bimodules by $\mathbb{S}Bim$

The Soergel categorification theorem

What is the relation of the above? The answer is given by:

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Theorem (Soergel, Elias-Williamson)

$\mathcal{H}_v(n)$ is isomorphic to $K_0^{\oplus}(\mathbb{S}Bim)$ as $\mathbb{Z}[v, v^{-1}]$-algebras. The isomorphisms is given by $b_i \mapsto B_i$.

2-Kac-Moody algebra and 2-representations (categorical)

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The 2-Kac-Moody algebra, $\mathfrak{A}(\mathfrak{sl}_2)$ is the additive 2-category defined by generators and relations as follows:

- Objects: $n \in \mathbb{Z}$
- Generating 1-morphisms: $E_n : n \to n+2$ and $F_n : n \to n-2$ for each $n \in \mathbb{Z}$.
- Generating 2-morphisms: $x : E \to E$, $\tau : E^2 \to E^2$, $\varepsilon : EF \to 1$ and $\eta : 1 \to FE$

The 2-morphisms are subject to the relations:

- $\tau^2 = 0$; $\tau 1_E \circ 1_E \tau \circ \tau 1_E = 1_E \tau \circ \tau 1_E \circ 1_E \tau$
- $x 1_E \circ \tau - \tau \circ 1_E x = 1_{E^2}$; $\tau \circ x 1_E - 1_e x \circ \tau = 1_{E^2}$
- $\varepsilon 1_E \circ 1_E \eta = 1_E$; $1_F \varepsilon \circ \eta 1_F = 1_F$

Moreover, the following are isomorphisms

$$(n \geq 0) EF \to FE \oplus 1^{\oplus n}; \qquad (n < 0) EF \oplus 1^{\oplus -n} \to FE$$

If we consider graduations in the above category, in such a way that the morphisms x, τ, ε y η have degree $2, -2, 1-n$ and $n+1$ respectively, it is possible to prove that:

Teorema (Rouquier, Khovanov - Lauda)

$K_0(\mathfrak{A}(\mathfrak{sl}_2))$ is isomorphic to $U_v(\mathfrak{sl}_2)$ as $\mathbb{Z}[v, v^{-1}]$-algebras.

2-representations

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A 2-representation of $\mathfrak{A}(\mathfrak{sl}_2)$ in K-linear categories consists of:

- For each $n \in \mathbb{Z}$ a K-linear category $\mathcal{V}_n$.
- For each $n \in \mathbb{Z}$ a K-linear functor $F : \mathcal{V}_n \to \mathcal{V}_{n-2}$ which admits a right adjoint $E : \mathcal{V}_n \to \mathcal{V}_{n+2}$.
- Morphisms $x \in \mathrm{End}(F)$ and $\tau \in \mathrm{End}(F^2)$ which satisfy the same relations as in $\mathfrak{A}(\mathfrak{sl}_2)$.
- Isomorphisms
  $(n \geq 0) EF \to FE \oplus 1^{\oplus n}; \qquad (n < 0) EF \oplus 1^{\oplus -n} \to FE$

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$\mathcal{L}(n)$ categorify the irreducible representations of $\mathfrak{sl}_2$.

Application 2: Schur-Weyl duality (classical, quantum and categorical)

Schur-Weyl duality

Classical: Let V be the standard representation of GL_n. S_n acts on $V^{\otimes n}$. The double centralizing form of the Schur-Weyl duality claims that:

$$\mathbb{C}[S_n] \cong \mathrm{End}_{GL_n}(V^{\otimes n})$$

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Quantum: Let V_v the standard representation of $U_v(\mathfrak{sl}_n)$. $\mathcal{H}_v(n)$ acts on $V_v^{\otimes n}$. The quantum version of the double centralizing form of the Schur-Weyl duality claims that:

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Categorical: Let... in progress...

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Then, we conjecture:

$$\mathbb{S}Bim \cong \mathrm{End}_{\mathfrak{A}(\mathfrak{sl}_n)}(\mathcal{V}^{\otimes n})$$

this is a work in progress with N. Libedinsky.

Thank you!