Admissible representations of simple affine vertex algebras

Libor Křižka

Charles University in Prague

April 27, 2022
Saõ Paulo

Positive energy representations

- $\mathcal{V}$ – a $\mathbb{Z}$-graded vertex algebra
- $\mathcal{E}(\mathcal{V})$ – the category of graded $\mathcal{V}$-modules, $M \in \mathcal{E}(\mathcal{V})$ provided ...
    - ... is a $\mathcal{V}$-module
    - ... is a $\mathbb{C}$-graded vector space
    - ... $Y_M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$ has conformal dimension m for $a \in \mathcal{V}_m$, i.e. $\deg a^M_{(n)} = -n+m-1$
- $\mathcal{E}_+(\mathcal{V})$ – the category of positive energy $\mathcal{V}$-modules, $M \in \mathcal{E}_+(\mathcal{V})$ provided ...
    - ... M belongs to $\mathcal{E}(\mathcal{V})$
    - ... $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$ with $M_\lambda \neq 0$
- the top degree component

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda+n} \mapsto M_{\mathrm{top}} = M_\lambda$$

Zhu's correspondence

- $A(\mathcal{V})$ – the Zhu's algebra of $\mathcal{V}$
  - $A(\mathcal{V})$ is a unital associative algebra
  - $\pi_{\mathrm{Zhu}} \colon \mathcal{V} \to A(\mathcal{V})$ a canonical surjective mapping
  - $M \in \mathcal{E}_+(\mathcal{V}) \implies M_{\mathrm{top}} \in A(\mathcal{V})$, the action of $\pi_{\mathrm{Zhu}}(a) \in A(\mathcal{V})$ on M_{top} is given through $a^M_{(\deg a - 1)}$ for $a \in \mathcal{V}$
- Zhu's correspondence

$$M \mapsto M_{\mathrm{top}}$$

$$\left\{ \begin{matrix} \text{simple positive energy} \\ \mathcal{V}\text{-modules} \end{matrix} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{ \text{simple } A(\mathcal{V})\text{-modules} \right\}$$

Affine vertex algebras

- $\mathfrak{g}$ – a simple Lie algebra
- $\mathfrak{b}$ – a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$
- $\kappa_\mathfrak{g}$ – the Cartan–Killing form on $\mathfrak{g}$, $\kappa_\mathfrak{g} = 2h^\vee \kappa_0$
- κ – a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$, $\kappa = k\kappa_0$ for $k \in \mathbb{C}$
- $\widehat{\mathfrak{g}}_\kappa$ – the affine Kac–Moody algebra associated to $\mathfrak{g}$ of level κ
    - $\mathfrak{g}_\kappa = \mathfrak{g}((t)) \oplus \mathbb{C}c$
    - $[a_m, b_n] = [a,b]_{m+n} + \kappa(a,b)\delta_{m,-n}\, c$, $a_n = a \otimes t^n$ for $a \in \mathfrak{g}$, $n \in \mathbb{Z}$
    - c is the cental element of $\widehat{\mathfrak{g}}_\kappa$
- $\mathcal{V}^\kappa(\mathfrak{g})$ – the universal affine vertex algebra
    - $\mathcal{V}^\kappa(\mathfrak{g}) = U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}c)} \mathbb{C} \simeq U(\mathfrak{g} \otimes_\mathbb{C} t^{-1}\mathbb{C}[t^{-1}]) \otimes_\mathbb{C} \mathbb{C}$
    - $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$ for $a \in \mathfrak{g}$
    - $[a(z), b(w)] = [a,b](w)\delta(z-w) + \kappa(a,b)\partial_w \delta(z-w)$ for $a, b \in \mathfrak{g}$
- $\mathcal{L}_\kappa(\mathfrak{g})$ – the simple affine vertex algebra
- $\mathcal{V}^\kappa(\mathfrak{g})$ and $\mathcal{L}_\kappa(\mathfrak{g})$ are $\mathbb{N}_0$-graded vertex algebras

Zhu's correspondence

- $\mathcal{V}^\kappa(\mathfrak{g}) \implies A(\mathcal{V}^\kappa(\mathfrak{g})) \simeq U(\mathfrak{g})$
$$\left\{\begin{matrix}\text{simple positive energy} \\ \mathcal{V}^\kappa(\mathfrak{g})\text{-modules}\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\text{simple } U(\mathfrak{g})\text{-modules}\right\}$$
- $\mathcal{L}_\kappa(\mathfrak{g}) \implies A(\mathcal{L}^\kappa(\mathfrak{g})) \simeq U(\mathfrak{g})/I_\kappa(\mathfrak{g})$, $I_\kappa(\mathfrak{g})$ is a two-sided ideal of $U(\mathfrak{g})$
$$\left\{\begin{matrix}\text{simple positive energy} \\ \mathcal{L}_\kappa(\mathfrak{g})\text{-modules}\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\text{simple } U(\mathfrak{g})/I_\kappa(\mathfrak{g})\text{-modules}\right\}$$
- E is a $U(\mathfrak{g})/I_\kappa(\mathfrak{g})$-module $\iff$ E is a $\mathfrak{g}$-module, $I_\kappa(\mathfrak{g}) \subset \mathrm{Ann}_{U(\mathfrak{g})} E$

Associated variety

- $U(\mathfrak{g}) \implies \operatorname{gr} U(\mathfrak{g}) \simeq S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$
- I – a left ideal of $U(\mathfrak{g})$
- $\mathcal{V}(I)$ – the associated variety of I, $\mathcal{V}(I) \subset \mathfrak{g}^*$

$$\mathcal{V}(I) = \operatorname{Specm}(S(\mathfrak{g})/\operatorname{gr} I)$$

- I is a two-sided ideal $\implies \mathcal{V}(I)$ is invariant under the adjoint action of G (connected algebraic group with its Lie algebra $\mathfrak{g}$)
- $L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ – the simple highest weight module with highest weight $\lambda \in \mathfrak{h}^*$
- I – a primitive ideal of $U(\mathfrak{g})$
  - there exists $\lambda \in \mathfrak{h}^*$ such that $I = \operatorname{Ann}_{U(\mathfrak{g})} L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ [Duflo '77]
  - $\mathcal{V}(I) = \overline{\mathcal{O}^*}$ for a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ [Joseph '85]
- a simple $\mathfrak{g}$-module E belongs to a nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$ if $\mathcal{V}(I) = \overline{\mathcal{O}^*}$

Zhu's correspondence

- E is a simple $U(\mathfrak{g})/I_\kappa(\mathfrak{g})$-module $\iff$ E is a simple $\mathfrak{g}$-module, $I_\kappa(\mathfrak{g}) \subset \mathrm{Ann}_{U(\mathfrak{g})} E$
- E is a simple $\mathfrak{g}$-module $\implies \mathrm{Ann}_{U(\mathfrak{g})} E = \mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ for some $\lambda \in \mathfrak{h}^*$
- The goal is to determine all $\lambda \in \mathfrak{h}^*$ such that $I_\kappa(\mathfrak{g}) \subset \mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$
- difficult problem in general
- known for generic and admissible levels

Admissible levels

- $\widehat{\mathfrak{g}}_\kappa$ – the affine Kac–Moody algebra
- $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathbb{C}c$ – the Cartan subalgebra of $\widehat{\mathfrak{g}}_\kappa$
- $\widehat{\mathfrak{h}}^* \simeq \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0$, $\Lambda_0|_{\mathfrak{h}\otimes_{\mathbb{C}}\mathbb{C}1} = 0$, $\Lambda_0(c) = 1$
- $\widehat{\Delta}(\lambda)$ – the integral root system of $\lambda \in \widehat{\mathfrak{h}}^*$

$$\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{\mathrm{re}};\ \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\}, \quad \widehat{\rho} = \rho + h^\vee \Lambda_0$$

- a weight $\lambda \in \widehat{\mathfrak{h}}^*$ is *admissible* provided
  - i) λ is *regular dominant*, that is $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin -\mathbb{N}_0$ for all $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$
  - ii) the $\mathbb{Q}$-span of $\widehat{\Delta}(\lambda)$ contains $\widehat{\Delta}^{\mathrm{re}}$
- a level $\kappa = k\kappa_0$ is admissible if $\lambda = k\Lambda_0$ is an admissible weight
- [Kac–Wakimoto '89]

$$k + h^\vee = \frac{p}{q} \text{ with } p, q \in \mathbb{N},\ (p,q) = 1,\ p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

Admissible levels

Theorem (Arakawa '15)

Let $\kappa = k\kappa_0$ be an admissible level for $\mathfrak{g}$ with denominator $q \in \mathbb{N}$. Then there exists a nilpotent orbit $\mathcal{O}_q$ of $\mathfrak{g}$ such that $\mathcal{V}(I_k) = \overline{\mathcal{O}_q^*}$.

- E is a simple $U(\mathfrak{g})/I_\kappa(\mathfrak{g})$-module which belongs to $\mathcal{O} \implies I_\kappa(\mathfrak{g}) \subset \mathrm{Ann}_{U(\mathfrak{g})} E \implies \mathcal{O} \subset \overline{\mathcal{O}_q}$

Principal admissible weights

- a weights $\lambda \in \widehat{\mathfrak{h}}^*$ is principal admissible of an admissible level k if there is an element y of the extended affine Weyl group $\widetilde{W}$ of $\mathfrak{g}$ such that $\widehat{\Delta}(\lambda) = y(\widehat{\Delta}(k\Lambda_0))$

$$\overline{\mathrm{Pr}}_k = \{\lambda \in \mathfrak{h}^*;\, \lambda + k\Lambda_0 \in \mathrm{Pr}_k\}$$

Theorem (Arakawa '16)

Let $\kappa = k\kappa_0$ be an admissible level for $\mathfrak{g}$. Then the simple $\mathfrak{g}$-module $L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ with $\lambda \in \mathfrak{h}^*$ is a $U(\mathfrak{g})/I_\kappa(\mathfrak{g})$-module if and only if $\lambda \in \overline{\mathrm{Pr}}_k$.

- The goal is to describe the set $\overline{\mathrm{Pr}}_k$
- for $\mathfrak{g} = \mathfrak{sl}_2$ all highest weights were classified by [Adamović–Milas '95, Dong–Li–Mason '97]

Principal admissible weights

- $\mathcal{O}$ – a nilpotent orbit of $\mathfrak{g}$

$$\overline{\mathrm{Pr}}_k^{\mathcal{O}} = \{\lambda \in \overline{\mathrm{Pr}}_k;\ \mathcal{V}(\mathrm{Ann}_{U(\mathfrak{g})} L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)) = \overline{\mathcal{O}^*}\}$$

- decomposition

$$\overline{\mathrm{Pr}}_k = \bigsqcup_{\mathcal{O} \subset \overline{\mathcal{O}_q}} \overline{\mathrm{Pr}}_k^{\mathcal{O}},$$

Principal admissible weights for $\mathfrak{sl}_{n+1}$

- $\mathfrak{g} = \mathfrak{sl}_{n+1}$, $k \in \mathbb{Q}$

$$k + n + 1 = \frac{p}{q} \text{ with } p, q \in \mathbb{N},\ (p, q) = 1,\ p \geq n + 1.$$

- $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ – the set of simple roots
- $\mathcal{P}_{n+1}$ – the set of partitions of $n+1$, it parameterizes nilpotent orbits of $\mathfrak{g}$
- $\mathcal{O}_q = \mathcal{O}_{[q^r, s]}$, $n + 1 = qr + s$, $r, s \in \mathbb{N}_0$, $0 \leq s \leq q - 1$

$$\overline{\mathrm{Pr}}_k = \bigsqcup_{\mathcal{O} \subset \overline{\mathcal{O}_q}} \overline{\mathrm{Pr}}{}_k^{\mathcal{O}},$$

- $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r] \in \mathcal{P}_{n+1}$

$$\Sigma_\lambda = \{\alpha_1, \dots, \alpha_{\lambda_1 - 1}, \alpha_{\lambda_1 + 1}, \dots, \alpha_{\lambda_1 + \lambda_2 - 1}, \dots, \alpha_{\lambda_1 + \dots + \lambda_{r-1} + 1}, \dots, \alpha_{\lambda_1 + \dots + \lambda_r - 1}\}$$

- $\mathfrak{p}_\lambda$ – the standard parabolic subalgebra of $\mathfrak{g}$ determined by Σ_λ

Parabolic subalgebras

- $\mathfrak{p}$ – a standard (contains $\mathfrak{b}$) parabolic subalgebra of $\mathfrak{g}$
  - $\Sigma_\mathfrak{p} = \{\alpha \in \Pi;\, \mathfrak{g}_\alpha \subset \mathfrak{p}\}$
  - $\mathcal{O}_\mathfrak{p} = (G.\mathfrak{p}^\perp)^{\mathrm{reg}}$ – the Richardson orbit attached to $\mathfrak{p}$
  - $W_\mathfrak{p}$ – the subgroup of W generated by s_α for $\alpha \in \Sigma_\mathfrak{p}$
  - $W^\mathfrak{p}$ – the set of minimal coset representatives of $W_\mathfrak{p} \backslash W$
  - equivalence relation $\sim$

$$\mathfrak{p}_1 \sim \mathfrak{p}_2 \iff \text{there exists } w \in W \text{ such that } \Delta_{\Sigma_1} = w(\Delta_{\Sigma_2})$$

  - $[\mathfrak{p}]$ – the equivalence class attached to $\mathfrak{p}$
  - W_+ – the subgroup of W generated by the element $w_1 = s_1 s_2 \dots s_n$
    - $w_1^{n+1} = e$
    - $W_+ = \{e, w_1, \dots, w_n\}$, $w_j = w_1^j$
  - equivalence relation $\sim_+$

$$\mathfrak{p}_1 \sim_+ \mathfrak{p}_2 \iff \text{there exists } w \in W_+ \text{ such that } \Sigma_1 = w(\Sigma_2).$$

  - $[\mathfrak{p}]_+$ – the equivalence class attached to $\mathfrak{p}$
  - $W_+^\mathfrak{p} = \{w \in W_+;\, w(\Sigma_\mathfrak{p}) = \Sigma_\mathfrak{p}\}$

Theorem (Futorny, K., Morales)

Let $\mathcal{O}_\lambda$ be the nilpotent orbit of $\mathfrak{g}$ given by a partition $\lambda \in \mathcal{P}_{n+1}$. Then we have

$$\overline{\mathrm{Pr}}_k^{\mathcal{O}_\lambda} = \bigsqcup_{[\mathfrak{p}]_+ \in [\mathfrak{p}_{\lambda^t}]/\sim_+} \ \bigsqcup_{[w] \in W_+^{\mathfrak{p}} \backslash W^{\mathfrak{p}}} w^{-1} \cdot \Lambda_k(\mathfrak{p}),$$

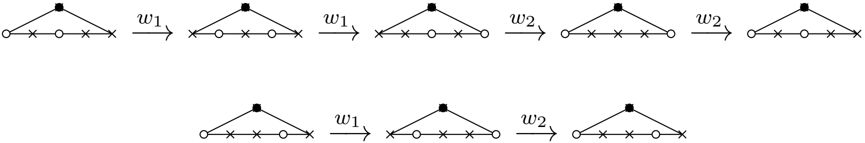
where $W_+^{\mathfrak{p}} \backslash W^{\mathfrak{p}}$ denotes the set of orbits of $W_+^{\mathfrak{p}}$ on $W^{\mathfrak{p}}$.

$$\mathrm{Pr}_{k,\mathbb{Z}} = \{\lambda \in \widehat{\mathfrak{h}}^*;\ \lambda(c) = k,\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{N}_0 \text{ for } \alpha \in \Pi,\ \langle \lambda, \theta^\vee \rangle \leq p - n - 1\}$$

$$\Lambda_k(\mathfrak{p}) = \{\lambda - (k + n + 1)\eta;\ \lambda \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ \eta \in P_+^\vee,\ (\eta, \theta) \leq q - 1,\ \Pi^\eta = \Sigma_{\mathfrak{p}}\}$$

Example 1

- $\mathfrak{g} = \mathfrak{sl}_6$, $\mathcal{O} = \mathcal{O}_{[4,2]}$
- $\mathfrak{p}_{[2,2,1,1]} =$ ∘—×—∘—×—×
- $[\mathfrak{p}_{[2,2,1,1]}] = \{$ ∘—×—∘—×—× , ∘—×—×—∘—× , ∘—×—×—×—∘ , ×—∘—×—∘—× , ×—∘—×—×—∘ , ×—×—∘—×—∘ $\}$
- Equivalence classes



- $\mathfrak{p}_1 =$ ∘—×—∘—×—× , $\mathfrak{p}_2 =$ ∘—×—×—∘—×

$[\mathfrak{p}_1]_+ = \{$ ∘—×—∘—×—× , ×—∘—×—∘—× , ×—×—∘—×—∘ , ∘—×—×—×—∘ $\}$, $W_+^{\mathfrak{p}_1} = \{e\}$

$[\mathfrak{p}_2]_+ = \{$ ∘—×—×—∘—× , ×—∘—×—×—∘ $\}$ $W_+^{\mathfrak{p}_2} = \{e, w_3\}$

- Admissible weights

$$\overline{\mathrm{Pr}}_k^{\mathcal{O}} = \bigsqcup_{w \in W^{\mathfrak{p}_1}} w^{-1} \cdot \Lambda_k(\mathfrak{p}_1) \sqcup \bigsqcup_{[w] \in W_+^{\mathfrak{p}_2} \backslash W^{\mathfrak{p}_2}} w^{-1} \cdot \Lambda_k(\mathfrak{p}_2)$$

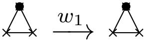
- Representative sets

$$\Lambda_k(\mathfrak{p}_1) = \Big\{ \lambda - \frac{p}{q}(a\omega_2 + b\omega_4 + c\omega_5);\ \lambda \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ a, b, c \in \mathbb{N},\ a + b + c \leq q - 1 \Big\}$$

$$\Lambda_k(\mathfrak{p}_2) = \Big\{ \lambda - \frac{p}{q}(a\omega_2 + b\omega_3 + c\omega_5);\ \lambda \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ a, b, c \in \mathbb{N},\ a + b + c \leq q - 1 \Big\}$$

Example 2

- $\mathfrak{g} = \mathfrak{sl}_3$, $\mathscr{O} = \mathscr{O}_{[3]} = \mathscr{O}_{\mathrm{reg}}$
- $\mathfrak{p}_{[1,1,1]} = \times\!\!-\!\!-\!\!\times = \mathfrak{b}$
- $[\mathfrak{p}_{[1,1,1]}] = \{ \times\!\!-\!\!-\!\!\times \}$
- Equivalence classes



,

- $$[\mathfrak{b}]_+ = \{ \times\!\!-\!\!-\!\!\times \}, \quad W_+^{\mathfrak{b}} = \{e, w_1, w_2\} = W_+$$

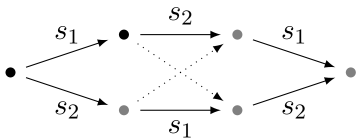
- Admissible weights

$$\overline{\mathrm{Pr}}_k^{\mathcal{O}} = \bigsqcup_{[w] \in W_+^{\mathfrak{b}} \backslash W^{\mathfrak{b}}} w^{-1} \cdot \Lambda_k(\mathfrak{b})$$

- Representative sets

$$\Lambda_k(\mathfrak{b}) = \left\{ \lambda - \frac{p}{q}(a\omega_1 + b\omega_2);\ \lambda \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ a, b \in \mathbb{N},\ a + b \leq q - 1 \right\}$$

- Hasse diagram for $W^{\mathfrak{b}}$



$$\overline{\mathrm{Pr}}_k^{\mathcal{O}} = \Lambda_k(\mathfrak{b}) \sqcup s_1 \cdot \Lambda_k(\mathfrak{b})$$

Example 3

- $\mathfrak{g} = \mathfrak{sl}_4$
- Hasse diagram of nilpotent orbits

$\mathcal{O}$	dim	$\mathfrak{p}$
$\mathcal{O}_{[4]}$	12	×—×—×
$\mathcal{O}_{[3,1]}$	10	×—×—○ ×—○—× ○—×—×
$\mathcal{O}_{[2^2]}$	8	○—×—○
$\mathcal{O}_{[2,1^2]}$	6	×—○—○ ○—○—×
$\mathcal{O}_{[1^4]}$	0	○—○—○

- Admissible weights

$$\overline{\mathrm{Pr}}_k = \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{zero}}} \sqcup \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{min}}} \sqcup \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{rect}}} \sqcup \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{subreg}}} \sqcup \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{reg}}}$$

$$\begin{aligned}
\overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{zero}}} &= \overline{\mathrm{Pr}}_{k,\mathbb{Z}}, \\
\overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{min}}} &= \bigsqcup_{w \in W_{\mathrm{min}}} w \cdot \Lambda_k(\mathfrak{p}_{\alpha_1}^{\mathrm{max}}), \quad W_{\mathrm{min}} = \{e, s_1, s_2 s_1, s_3 s_2 s_1\} \\
\overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{rect}}} &= \bigsqcup_{w \in W_{\mathrm{rect}}} w \cdot \Lambda_k(\mathfrak{p}_{\alpha_2}^{\mathrm{max}}), \quad W_{\mathrm{rect}} = \{e, s_2, s_1 s_2\}, \\
\overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{subreg}}} &= \bigsqcup_{w \in W_{\mathrm{subreg}}} w \cdot \Lambda_k(\mathfrak{p}_{\alpha_3}^{\mathrm{min}}), \\
W_{\mathrm{subreg}} &= \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_3 s_2, s_1 s_2 s_1, s_3 s_1 s_2, s_3 s_2 s_1, s_2 s_3 s_1 s_2, \\
&\quad s_3 s_1 s_2 s_1, s_2 s_3 s_1 s_2 s_1\}, \\
\overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{reg}}} &= \bigsqcup_{w \in W_{\mathrm{reg}}} w \cdot \Lambda_k(\mathfrak{b}), \quad W_{\mathrm{reg}} = \{e, s_1, s_2, s_3, s_1 s_2, s_1 s_3\}
\end{aligned}$$

- Representative sets

$$\overline{\mathrm{Pr}}_{k,\mathbb{Z}} = \{\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3;\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{N}_0,\\ \lambda_1 + \lambda_2 + \lambda_3 \leq p - 4\}$$

$$\Lambda_k(\mathfrak{p}^{\max}_{\alpha_1}) = \left\{\lambda - \frac{p}{q}\,a\omega_1;\ \lambda \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ a \in \mathbb{N},\ a \leq q - 1\right\}$$

$$\Lambda_k(\mathfrak{p}^{\max}_{\alpha_2}) = \left\{\lambda - \frac{p}{q}\,a\omega_2;\ \lambda \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ a \in \mathbb{N},\ a \leq q - 1\right\}$$

$$\Lambda_k(\mathfrak{p}^{\min}_{\alpha_3}) = \left\{\lambda - \frac{p}{q}(a\omega_1 + b\omega_2);\ \lambda \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ a, b \in \mathbb{N},\ a + b \leq q - 1\right\}$$

$$\Lambda_k(\mathfrak{b}) = \left\{\lambda - \frac{p}{q}(a\omega_1 + b\omega_2 + c\omega_3);\ \lambda \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ a, b, c \in \mathbb{N},\\ a + b + c \leq q - 1\right\}$$

Thank you for your attention!