Representations of Quantum Toroidal Superalgebras and $\mathfrak{s}$-Partitions

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Joint work with Evgeny Mukhin

April 25, 2022

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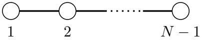

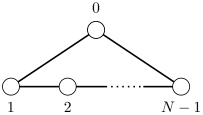

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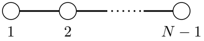

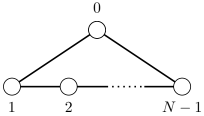

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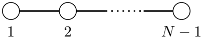

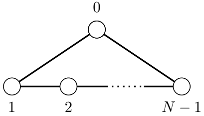

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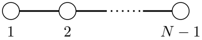

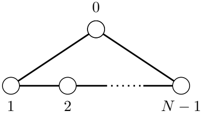

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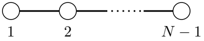

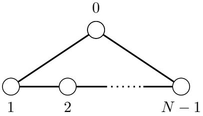

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$$|i| = |\alpha_i| = (1 - s_i s_{i+1})/2$$

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$$E_i(z) = \sum_{k\in\mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) = \sum_{k\in\mathbb{Z}} F_{i,k} z^{-k},$$

$$K_i^{\pm}(z) = K_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{r>0} H_{i,\pm r} z^{\mp r} \right).$$

$$\delta(z) = \sum_{r\in\mathbb{Z}} z^r, \qquad (z - w)\delta\left(\frac{z}{w}\right) = 0.$$

Recipe

$$U_q\mathfrak{sl}_{\mathbf{s}}$$

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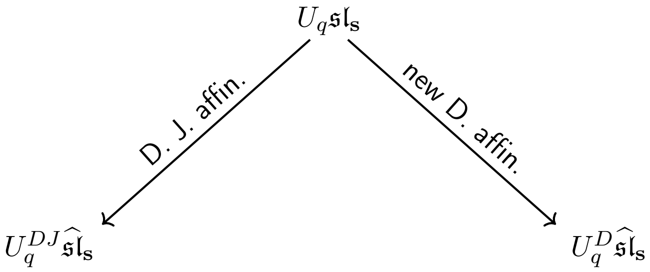
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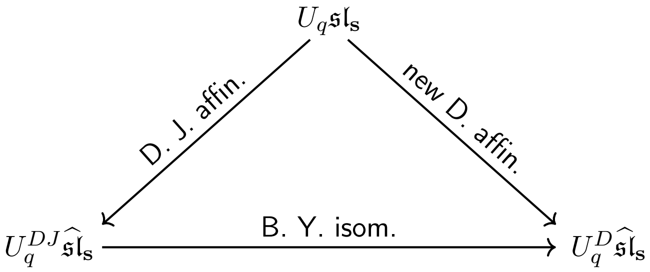
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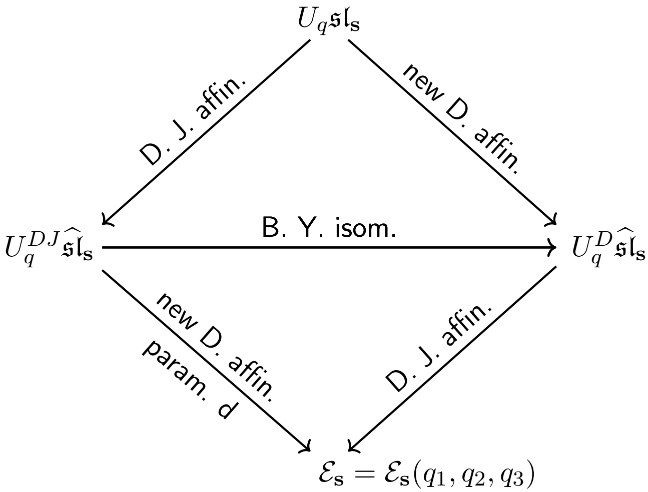

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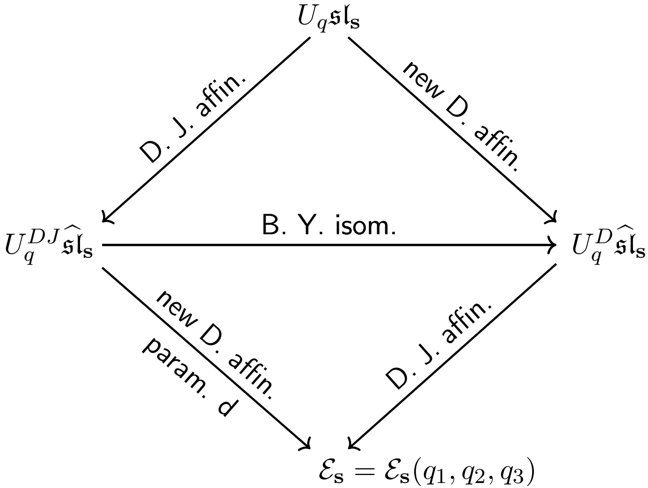

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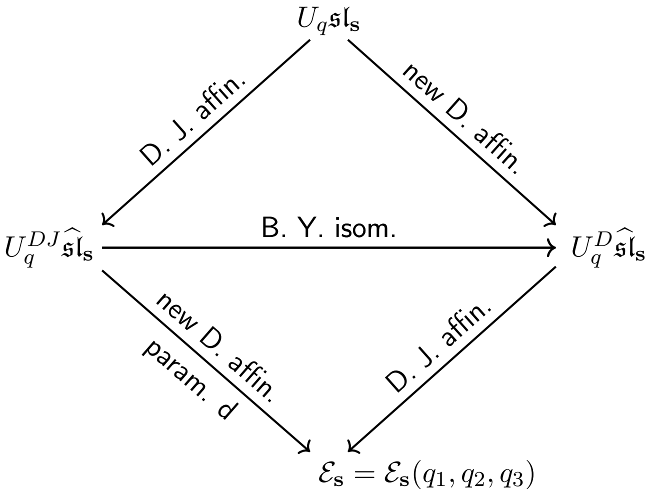

$$q_1 = d\,q^{-1}, \quad q_2 = q^2, \quad q_3 = d^{-1}q^{-1}.$$

We always assume that q_1, q_2 are generic. Note that $q_1 q_2 q_3 = 1$.

Vertical subalgebra

Vertical subalgebra

For $i \in \mathbb{Z}$, define $\bar{i} = \sum_{j=1}^{i} s_j$ if $i \geq 0$, and $\overline{i + m + n} = \bar{i} + m - n$.

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Proposition

There is an injective homomorphism of superalgebras $v_{\mathbf{s}} : U_q\widehat{\mathfrak{sl}}_{\mathbf{s}} \to \mathcal{E}_{\mathbf{s}}$ *given by*

$$v_{\mathbf{s}}(E_i(z)) = E_i(d^{-\bar{i}}z), \quad v_{\mathbf{s}}(F_i(z)) = F_i(d^{-\bar{i}}z),$$
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- $t = t_0 t_1 \cdots t_{N-1} \mapsto K = K_0 K_1 \cdots K_{N-1}$ are central.

Highest weight modules

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- A $U_q^{DJ}\widehat{\mathfrak{sl}}_{\mathbf{s}}$-module V is a highest weight module of highest weight $\Lambda = (\Lambda_i)_{i\in\hat{I}}$ if it is generated by a vector v such that

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- A $\mathcal{E}_{\mathbf{s}}$-module V is a highest (loop)-weight module of highest weight $\Lambda(z) = (\Lambda(z)_i)_{i\in\hat{I}}$ and level (k, c) if it is generated by a vector v such that

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- $U_q^{ver}\widehat{\mathfrak{sl}}_{\mathbf{s}}$ is a Hopf subalgebra.

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$N = 5$, $[u]_4$: $\cdots$

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$$\cdots \begin{array}{|c|c|c|c|c|c|}\hline 4 & \color{red}{0} & 1 & 2 & 3 & 4 \\ \hline \end{array} \xrightarrow{F_0(z)} \cdots \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & \color{red}{0} & 1 & 2 & 3 & 4 & 0 \\ \hline \end{array}$$

$$\begin{array}{cccccc} -1 & 0 & 1 & 2 & 3 & 4 \end{array} \qquad\qquad \begin{array}{ccccccc} -1 & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

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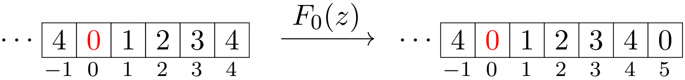

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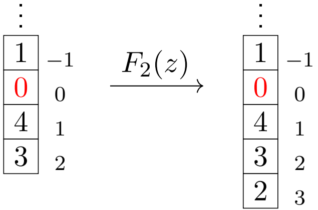

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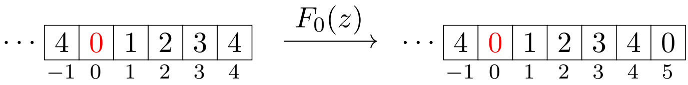

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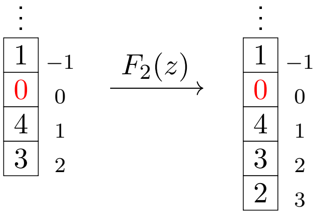

$$F_2(z)[u]^{-2} = s_3 \delta\left(q_3^{\overline{(-3)}} u/z\right)[u]^{-3}$$

Tensor Products

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$$\Delta(E_i(z)) = E_i(z) \otimes K_i^-(z) + 1 \otimes E_i(z),$$
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  - $\psi_{-s_{i+1}}(v/u)$ has a pole at $v = u$ and a zero at $v = q_2^{-s_{i+1}} u$.

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Proposition ([BM3])

Let $u, v \in \mathbb{C} \setminus \{0\}$.

1. *The coproduct defines an action of* $\mathcal{E}_{\mathbf{s}}$ *on* $V(u) \otimes V(v)$ *if and only if* $u \neq q_1^{(m-n)r} v$, $r \in \mathbb{Z}$.

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    - H.w.v.: $[u]_{i-1} \otimes [q_2 u]_{i-2} \otimes [q_2^2 u]_{i-3} \otimes \cdots$

- If $mn \neq 0$?
    - $[u]_i \otimes [q_2 u]_i \otimes [q_2^2 u]_i \otimes \cdots$ behaves badly in the limit.

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Proposition ([BM3])

Let $u, v \in \mathbb{C} \setminus \{0\}$.

1. *The coproduct defines an action of* $\mathcal{E}_{\mathbf{s}}$ *on* $V(u) \otimes W(v)$ *if and only if* $u \neq q_1^a q_3^b v$, $a, b \in \mathbb{Z}$, $a \equiv b \pmod{m-n}$.
2. *The coproduct defines an action of* $\mathcal{E}_{\mathbf{s}}$ *on* $W(u) \otimes V(v)$ *if and only if* $u \neq q_1^a q_3^b v$, $a, b \in \mathbb{Z}$, $a \equiv b \pmod{m-n}$.
3. *If the* $\mathcal{E}_{\mathbf{s}}$*-module* $V(u) \otimes W(v)$ *is well-defined, then it is an irreducible module of level zero.*
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- Renormalize and take the limit!

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  - $\ell(\boldsymbol{\lambda}) \geq \ell(\boldsymbol{\mu})$.
- If $\mathrm{s} \equiv 1$, s-partitions are Frobenius coordinates.

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- Given a pair of partitions $(\boldsymbol{\lambda}, \boldsymbol{\mu})$, both with at most k parts, define $|\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle^k = [u]_{\lambda_1 - 1} \otimes [u]^{-\mu_1} \otimes \cdots \otimes [q_2^{k-1} u]_{\lambda_k - 1} \otimes [q_2^{k-1} u]^{-\mu_k}$.

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Proposition ([BM3])

Let $v \in \mathring{\mathcal{F}}^{(k)}$. Then, $E_i^{(k)}(z)v$, $F_i^{(k)}(z)v$, and $K_i^{(k)}(z)v$, $i \in \hat{I}$, are well-defined vectors in $\mathcal{F}^{(k)}$. Moreover, the defining relations are satisfied when applied to v.

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$\phi^{(k)}: \mathcal{F}^{(k)} \hookrightarrow \mathcal{F}^{(k+1)}, \qquad |\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle^{(k)} \mapsto |\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle^{(k+1)}$.

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Lemma ([BM3])

For any $v \in \mathring{\mathcal{F}}^{(k)}$ *and* $i \in \hat{I}$, *we have*

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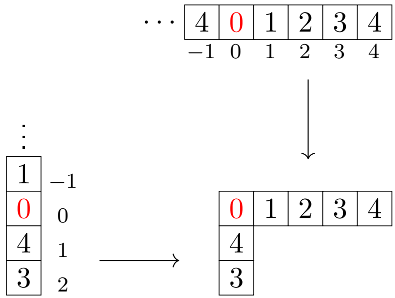

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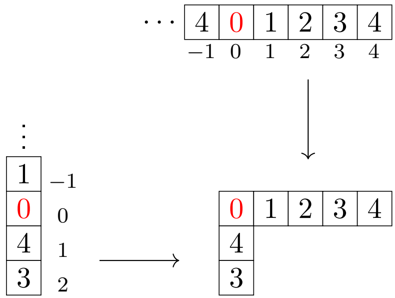
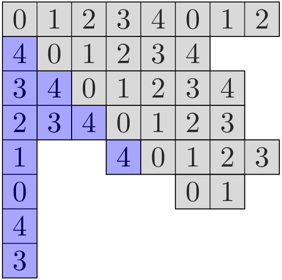


- Glue the hooks along the diagonal.

2D Young Diagrams

$|(8,5,5,4,4,2),(7,2,1,1)\rangle$:



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0	1	2	3	4	0	1	2
4	0	1	2	3	4		
3	4	0	1	2	3	4	
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1			4	0	1	2	3
0					0	1	
4							
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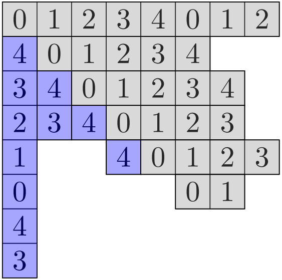

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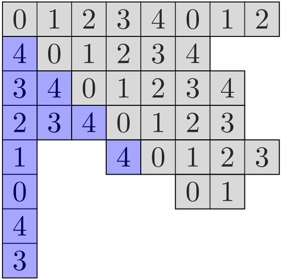

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Other level 1 modules

- In a similar way, a family of $\mathcal{E}_s$-modules which after restriction to $U_q\widehat{\mathfrak{gl}}_{m|n}$ are highest weight modules of highest weight

$$s_i(r\Lambda_{i-1} - (r+1)\Lambda_i),\ i \in \hat{I},\ r \in \mathbb{Z}.$$

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- For some modules in standard parity, the characters are known. We can show irreduciblity.

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$$\mathcal{M}(u) \subseteq \mathcal{F}_{\Lambda_0}(u) \otimes \mathcal{F}_{\Lambda_0}(uq_2^{-1}) \otimes \mathcal{F}_{\Lambda_0}(uq_2^{-2}) \otimes \cdots$$

- Let $f(k,z) = \dfrac{k - k^{-1}z}{1-z}$.

Theorem ([BM3])

The space $\mathcal{M}(u)$ is a highest weight $\mathcal{E}_{\mathbf{s}}$-module with highest weight $(f(k,u/z),1,\ldots,1)$. If k is generic ($k \neq q_1^a q_3^b$, $a,b \in \mathbb{Z}$), it is irreducible.

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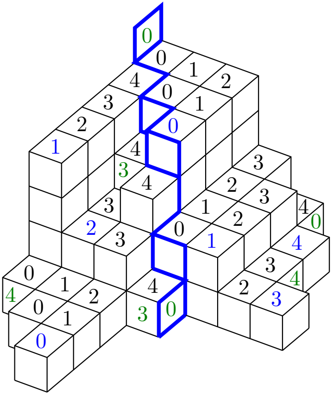
  - Horizontal boundaries: the q_2 shifts can have larger steps.

3D Young Diagrams



An example of a plane s-partition.

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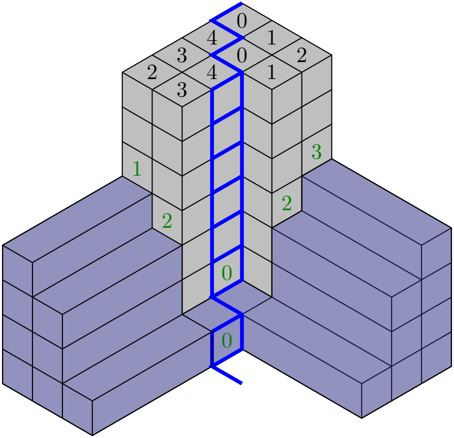

The vacuum of $\mathcal{M}_0^{(\boldsymbol{\gamma},\boldsymbol{\epsilon}),\boldsymbol{\alpha}}$, $\boldsymbol{\gamma} = (3,2)$, $\boldsymbol{\epsilon} = (3,2)$, $\boldsymbol{\alpha} = (3,2,2,1)$.

MacMahon ?

Theorem ([BM3])

Let $\sigma(i,j) = (-1)^{\sum_{k=i}^{j} |k|}$. *The character of the module* $\mathcal{M}(u)$ *is given by*

$$\chi(\mathcal{M}(u)) = \prod_{r=1}^{\infty} \prod_{k=0}^{r-1} (1 - \sigma(k+1-r,k) q^r)^{-\sigma(k+1-r,k)}.$$

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- If $\mathrm{s} \equiv 1$:
  - Plane s-partitions are the usual plane partitions.
  - The generating function is the MacMahon function (1916)

$$\chi(\mathcal{M}(u)) = \sum_{r=0}^{\infty} \mathrm{pl}(r)q^r = \prod_{r=1}^{\infty} \frac{1}{(1-q^r)^r}$$

$pl(r)$ = number of plane partitions with r boxes.

Percy MacMahon

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Thank you!