Representations of the Lie superalgebra $W(\infty)$

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The Lie algebra $\mathfrak{gl}(\infty)$

$I := \{1, 2, 3, \dots\}$: index set

V, V_* : countable-dim. v.s. with bases $\{\xi_i\}_{i\in I}$, $\{\partial_i\}_{i\in I}$

$\langle\cdot,\cdot\rangle : V \otimes V_* \to \mathbb{C}$ defined by $\langle \xi_i, \partial_j\rangle = \delta_{i,j}$ for all $i, j \in I$.

Definition

$\mathfrak{gl}(\infty) := V \otimes V_*$ with bracket (extended linearly):

$$[\xi_i \otimes \partial_j, \xi_k \otimes \partial_\ell] = \langle \xi_k, \partial_j\rangle \xi_i \otimes \partial_\ell - \langle \xi_i, \partial_\ell\rangle \xi_k \otimes \partial_j.$$

$\mathfrak{sl}(\infty) := \mathrm{Ker}\langle\cdot,\cdot\rangle$

$\mathfrak{gl}(\infty) \cong$ infinite matrices $\left(a_{i,j}\right)_{i,j\in I}$ with finitely many $a_{i,j} \neq 0$

$$\xi_i \otimes \partial_j^* \leftrightarrow E_{i,j}$$

$\mathfrak{gl}(\infty) \cong \varinjlim \mathfrak{gl}(n)$ where $\mathfrak{gl}(n) \hookrightarrow \mathfrak{gl}(n+1)$ satisfies $E_{i,j} \to E_{i,j}$

$\mathfrak{gl}(\infty)$-modules

V, V_* : defining representations of $\mathfrak{gl}(\infty)$, $\mathfrak{sl}(\infty)$

We have Schur–Weyl duality:

Theorem (Penkov, Styrkas)

We have an isomorphism of $\mathfrak{gl}(\infty) \times (S_p \times S_q)$-modules:

$$V^{\otimes p} \otimes V_*^{\otimes q} \cong \bigoplus_{|\boldsymbol{\lambda}|=p, |\boldsymbol{\mu}|=q} (\mathbb{S}_{\boldsymbol{\lambda}}(V) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_*)) \otimes (Y_{\boldsymbol{\lambda}} \otimes Y_{\boldsymbol{\mu}}),$$

where

- $|\boldsymbol{\lambda}|$ *is the size of the Young diagram* $\boldsymbol{\lambda}$*,*
- $\mathbb{S}_{\boldsymbol{\lambda}}$ *is the Schur functor corresponding to* $\boldsymbol{\lambda}$*,*
- $Y_{\boldsymbol{\lambda}}$ *and* $Y_{\boldsymbol{\mu}}$ *are irreducible* S_p*- and* S_q*-modules.*

$\widetilde{V}_{\lambda,\mu} := \mathbb{S}_{\boldsymbol{\lambda}}(V) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_*)$ *is an indecomposable module.*

Simple $\mathfrak{gl}(\infty)$-modules

The *socle filtration* of M is defined by $soc^0 M := \operatorname{soc} M$ and

$$\operatorname{soc}^i M := p_i^{-1}(\operatorname{soc}(M/(\operatorname{soc}^{i-1} M))),$$

where $p_i : M \to M/(\operatorname{soc}^{i-1} M)$ is the natural projection.

Theorem (Penkov, Styrkas)

The socle filtration of $\widetilde{V}^{\lambda,\mu} = (\mathbb{S}_{\boldsymbol{\lambda}}(V) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_*))$ *has layers*

$$\operatorname{soc}^k(\widetilde{V}_{\lambda,\mu})/\overline{\operatorname{soc}}^{k-1}(\widetilde{V}_{\lambda,\mu}) \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',|\gamma|=k} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\gamma} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\gamma} V_{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$$

where $N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\gamma}$ *are the Littlewood-Richardson coefficients.*

The socle of $\widetilde{V}_{\lambda,\mu}$ *is simple and denoted by* $V_{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

Simple as highest weight modules

Let's fix

- $\mathfrak{h}_{\mathfrak{gl}} \subset \mathfrak{gl}(\infty)$ Cartan subalgebra of diagonal matrices

$$\mathfrak{gl}(\infty) = \mathfrak{h}_{\mathfrak{gl}} \oplus \bigoplus_{i \neq j} \mathfrak{gl}(\infty)_{\varepsilon_i - \varepsilon_j}$$

- $\prec : 1 \prec 3 \prec 5 \prec \cdots \prec 6 \prec 4 \prec 2$ linear ordering on I

$$\Delta(\prec) := \{\varepsilon_i - \varepsilon_j \mid i \prec j\}$$

- $\mathfrak{b}(\prec)^0 = \mathfrak{h}_{\mathfrak{gl}} \oplus \bigoplus_{\alpha \in \Delta(\prec)} \mathfrak{gl}(\infty)_\alpha$: Borel subalgebra of $\mathfrak{gl}(\infty)$

V, V_* are highest weight $\mathfrak{gl}(\infty)$-modules w.r.t. $\mathfrak{b}(\prec)^0$ with highest weight ε_1, $-\varepsilon_2$ (respectively).

$V_{\lambda,\mu}$ is a highest weight $\mathfrak{gl}(\infty)$-module w.r.t. $\mathfrak{b}(\prec)^0$ with h.w.

$$\sum_{i \in \mathbb{Z}_{>0}} \lambda_i \varepsilon_{2i-1} - \sum_{j \in \mathbb{Z}_{>0}} \mu_j \varepsilon_{2j}$$

The category $\mathbb{T}_{\mathfrak{gl}(\infty)}$

M is *integrable* if for all $x \in \mathfrak{gl}(\infty)$, $m \in M$,

$$\dim \operatorname{span}_{\mathbb{C}}\{x^i m \mid i \in I\} < \infty.$$

M satisfies the *large annihilator condition (l.a.c.)* if for each $m \in M$, $\exists\ n >> 0$,

$$\langle \xi_i \partial_j = E_{i,j} \mid j, i \geq n \rangle \subset \operatorname{Ann}(m).$$

Definition

The category $\mathbb{T}_{\mathfrak{gl}(\infty)}$ is the full subcategory of $\mathfrak{gl}(\infty)$-mod consisting of modules M which satisfy:

1. M has finite length;
2. M is integrable;
3. M satisfies the l.a.c.

$\mathbb{T}_{\mathfrak{sl}(\infty)}$ is defined similarly.

Some properties of $\mathbb{T}_{\mathfrak{sl}(\infty)}$

[Dan-Cohen, Penkov, Serganova]

- $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is an abelian symmetric monoidal category;
- $\mathbb{T}_{\mathfrak{sl}(\infty)}$ consists of tensor modules: subquotients of

$$\bigoplus_{i}^{\textit{finite}} V^{\otimes p_i} \otimes V_*^{\otimes q_i};$$

- $V^{\otimes p} \otimes V_*^{\otimes q}$ is injective in $\mathbb{T}_{\mathfrak{sl}(\infty)}$;
- each indecomposable injective object of $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is isomorphic to some $\widetilde{V}_{\lambda,\mu} := \mathbb{S}_{\boldsymbol{\lambda}}(V) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_*)$;
- $M \in \mathbb{T}_{\mathfrak{sl}(\infty)}$ is an $\mathfrak{h}_{\mathfrak{sl}}$-weight module with integral weights

Equivalence of $\mathbb{T}_{\mathfrak{gl}(\infty)}$ and $\mathbb{T}_{\mathfrak{sl}(\infty)}$

The restriction functor

$$\mathrm{R}_{\mathfrak{sl}} : \mathbb{T}_{\mathfrak{gl}(\infty)} \to \mathbb{T}_{\mathfrak{sl}(\infty)}$$

defined by restricting to $\mathfrak{sl}(\infty)$ is well-defined.

If $M \in \mathbb{T}_{\mathfrak{gl}(\infty)}$, then M is an $\mathfrak{h}_{\mathfrak{gl}}$-weight module.

$\mu, \lambda \in \mathrm{Supp}\, M$ are equal if and only if $\mu|_{\mathfrak{h}_{\mathfrak{sl}}} = \lambda|_{\mathfrak{h}_{\mathfrak{sl}}}$.

Any object in $\mathbb{T}_{\mathfrak{sl}(\infty)}$ can be extended in a unique way to an object of $\mathbb{T}_{\mathfrak{gl}(\infty)}$

The categories $\mathbb{T}_{\mathfrak{gl}(\infty)}$ and $\mathbb{T}_{\mathfrak{sl}(\infty)}$ are equivalent.

The categories $\widetilde{\mathbb{T}}_{\mathfrak{gl}(\infty)}$ and $\widetilde{\mathbb{T}}_{\mathfrak{sl}(\infty)}$

$\widetilde{\mathbb{T}}_{\mathfrak{gl}(\infty)}$: Grothendieck envelope of $\mathbb{T}_{\mathfrak{gl}(\infty)}$ = full subcat of $\mathfrak{gl}(\infty)$-mod, objects are arbitrary sums of objects in $\mathbb{T}_{\mathfrak{gl}(\infty)}$.

$\widetilde{\mathbb{T}}_{\mathfrak{sl}(\infty)}$: Grothendieck envelope of $\mathbb{T}_{\mathfrak{sl}(\infty)}$.

$\widetilde{\mathbb{T}}_{\mathfrak{gl}(\infty)}$ and $\widetilde{\mathbb{T}}_{\mathfrak{sl}(\infty)}$ are equivalent.

We have a description of $\widetilde{\mathbb{T}}_{\mathfrak{gl}(\infty)}$

Proposition (C. Hoyt)

The category $\widetilde{\mathbb{T}}_{\mathfrak{gl}(\infty)}$ is the full subcategory of $\mathfrak{gl}(\infty)$-mod consisting of modules M which satisfy:

1. *M is an $\mathfrak{h}_{\mathfrak{gl}}$-weight module;*
2. *M is integrable;*
3. *M satisfies the l.a.c.*

The Lie superalgebra $W(n)$

Definition ($\Lambda(n)$)

The Grassmann algebra $\Lambda(n)$ is the free commutative (unital) superalgebra with n odd generators $\xi_1, \ldots, \xi_n$.

Definition ($W(n)$)

Let $W(n)$ be the Lie superalgebra of super derivations of $\Lambda(n)$.

$W(n)$ appears in Kac's list of simple f.d. Lie superalgebras.

Let ∂_i be the derivation defined by $\partial_i(\xi_j) = \delta_{ij}$. Then

$$W(n) = \operatorname{span}\{\xi_{i_1} \cdots \xi_{i_k} \partial_j \mid i, j \in \mathbb{Z}_n, \;\; k \geq 0, \; i_1 < \cdots < i_k\}$$

Setting $\deg \xi_i = 1$, $\deg \partial_j = -1$ yields a $\mathbb{Z}$-grading

$$W(n) = \bigoplus_{k=-1}^{n-1} W(n)^k, \quad \text{where } W(n)^0 \cong \mathfrak{gl}(n).$$

The Lie superalgebra $W(\infty)$

Definition

Consider $W(n) \hookrightarrow W(n+1)$ where $p(\xi)\partial_j \to p(\xi)\partial_j$, and let

$$\mathfrak{g} := W(\infty) = \varinjlim W(n).$$

- $\mathfrak{g}$ is a locally finite Lie superalgebra.
- $\mathfrak{g}$ is locally simple $\Rightarrow$ simple.
- $\mathfrak{g} =_{v.s.} \Lambda(V) \otimes V_*$, where $\Lambda(V) = \Lambda[\xi_i \mid i \in I]$.

$\{\xi_{i_1} \cdots \xi_{i_k}\partial_j \mid i, j \in I,\ k \geq 0,\ i_1 < \cdots < i_k\}$ is a basis for $W(\infty)$.

Setting $\deg \xi_i = 1$, $\deg \partial_i = -1$ yields a $\mathbb{Z}$-grading of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{k \geq -1} \mathfrak{g}^k.$$

We identify $\mathfrak{g}^0 = \mathfrak{gl}(\infty)$ via $\xi_i \partial_j \leftrightarrow E_{i,j}$. Then

$$\mathfrak{h} = \operatorname{span}\{\xi_i \partial_i \mid i \in I\}$$

is a Cartan subalgebra for $\mathfrak{g}$ and $\mathfrak{h} = \mathfrak{h}_{\mathfrak{gl}}$.

As $\mathfrak{g}^0$-modules, we have $\mathfrak{g}^k \cong \Lambda^{k+1}(V) \otimes V_*$.

l.a.c. for $W(\infty)$ modules

For each $n \in \mathbb{Z}_{>0}$, let $\mathfrak{t}_n := \langle \partial_j,\ \xi_{i_1} \cdots \xi_{i_k} \partial_j \mid j, i_t \geq n \rangle$.

Definition

Let $\mathfrak{s} \subset \mathfrak{g}$ be a subalgebra. We say that a $\mathfrak{g}$-module M satisfies the *large annihilator condition (l.a.c) for* $\mathfrak{s}$ if: for each $m \in M$,

$$(\mathfrak{t}_n \cap \mathfrak{s}) \subset \mathrm{Ann}_{\mathfrak{g}}(m) \qquad n >> 0.$$

For a $\mathfrak{g}$-module M, let $M^{\mathfrak{t}_n} = \{m \in M \mid \mathfrak{t}_n \cdot m = 0\}$.

Remark

M satisfies the l.a.c. for $\mathfrak{g}$ if and only if $M = \bigcup_{i>0} M^{\mathfrak{t}_n}$.

The categories $\mathbb{T}_W$, $\mathbb{T}_W^{\geq}$, $\mathbb{T}_W^{\leq}$

Set

$$\mathfrak{g}^{\geq} := \bigoplus_{k\geq 0} \mathfrak{g}^k, \qquad \mathfrak{g}^{\leq} := \mathfrak{g}^{-1} \oplus \mathfrak{g}^0.$$

Definition

Let $\widetilde{\mathbb{T}}_W$ (respectively, $\widetilde{\mathbb{T}}_W^{\geq}$, $\widetilde{\mathbb{T}}_W^{\leq}$) be the full subcategory of $\mathfrak{g}$-mod consisting of modules M which satisfy:

1. M has a $\mathbb{Z}$-grading $M = \bigoplus_{k\in\mathbb{Z}} M^k$;
2. M is an $\mathfrak{h}$-weight module;
3. M is integrable over $\mathfrak{g}^0$;
4. M satisfies the l.a.c. for $\mathfrak{g}$. (respectively, for $\mathfrak{g}^{\geq}$, $\mathfrak{g}^{\leq}$)

Let $\mathbb{T}_W$ be the full subcategory of $\widetilde{\mathbb{T}}_W$ consisting of finite-length $\mathfrak{g}$-modules. Similarly, define $\mathbb{T}_W^{\geq}$, $\mathbb{T}_W^{\leq}$.

Properties of $\mathbb{T}_W$ and $\mathbb{T}_W^{\leq}$

- $\mathbb{T}_W$ and $\widetilde{\mathbb{T}}_W$ are abelian symmetric monoidal categories.
- $\mathbb{T}_W$ is a subcategory of $\mathbb{T}_W^{\geq}$ and $\mathbb{T}_W^{\leq}$.
- $\mathbb{T}_W$ is precisely the full subcategory of $\mathfrak{g}$-mod whose objects are in both $\mathbb{T}_W^{\geq}$ and $\mathbb{T}_W^{\leq}$.

Lemma

If $M \in \widetilde{\mathbb{T}}_W, \widetilde{\mathbb{T}}_W^{\geq}, \widetilde{\mathbb{T}}_W^{\leq}$, *then* $M|_{\mathfrak{g}^0} \in \widetilde{\mathbb{T}}_{\mathfrak{gl}(\infty)}$.

Induced modules

Definition

Let X be a module over $\mathfrak{g}^0 = \mathfrak{gl}(\infty)$, and extend trivially to $\mathfrak{g}^{\geq}$ (resp., $\mathfrak{g}^{\leq}$) action. Define the induced $\mathfrak{g}$-modules

$$K^+(X) := \mathrm{Ind}_{\mathfrak{g}^{\geq}}^{\mathfrak{g}}(X) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{g}^{\geq})} X \quad \text{small}$$
$$K^-(X) := \mathrm{Ind}_{\mathfrak{g}^{\leq}}^{\mathfrak{g}}(X) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{g}^{\leq})} X \quad \text{big}$$

If X is simple, $L^{\pm}(X)$ denotes the unique simple quotient.

Let $\mathfrak{g}_n = W(n)$. Then $\mathfrak{g}_n^0 = \mathfrak{gl}(n)$.

For a $\mathfrak{g}_n^0$-module X_n, the $\mathfrak{g}_n$-modules $K_n^{\pm}(X_n)$ and $L_n^{\pm}(X_n)$ are defined analogously.

Properties of $K^{\pm}(-)$

$K^{\pm}(-)$ is an exact functor.

Proposition

For each n, let $f_n : X_n \hookrightarrow X_{n+1}$ be an embedding of $\mathfrak{g}_n^0$-modules, and consider the natural embedding $f_n : K^{\pm}(X_n) \hookrightarrow K^{\pm}(X_{n+1})$ of $\mathfrak{g}_n$-modules, where $f_n(uv) = uf_n(v)$ for every $u \in \mathbf{U}(\mathfrak{g}_n)$ and $v \in X_n$.

Then we have an isomorphism of $\mathfrak{g}$-modules

$$K^{\pm}(\varinjlim_n X_n) \cong \varinjlim_n K_n^{\pm}(X_n),$$

where the limits are taken over the family $\{f_n\}$.

More properties of $K^{\pm}(-)$

For $X \in \mathbb{T}_{\mathfrak{gl}(\infty)}$ with weight decomposition $X = \bigoplus_{\mu \in \mathfrak{h}^*} X_\mu$, we define a $\mathbb{Z}$-grading $X = \bigoplus_{k \in \mathbb{Z}} X^k$ by

$$X^k := \bigoplus_{\mu \in \mathfrak{h}^*,\ \sum \mu_i = k} X_\mu .$$

If X simple, then $X = X^k$ for some $k \in \mathbb{Z}$, and we write $|X| := k$.

Proposition (C. Hoyt)

Let $X \in \mathbb{T}_{\mathfrak{gl}(\infty)}$ such that $|X| = k$. The module $K^{\pm}(X)$ admits a a $\mathbb{Z}$-grading

$$K^+(X) = \bigoplus_{j \le 0} T_+^j, \qquad K^-(X) = \bigoplus_{j \ge 0} T_-^j$$

such that each $T_\pm^j$ considered as a $\mathfrak{g}^0$-module is in $\mathbb{T}_{\mathfrak{gl}(\infty)}$ and each simple subquotient Y of $T_\pm^j$ satisfies $|Y| = k + j$.

Results for $K^{\pm}(-)$

Theorem (C. Hoyt)

If $X \in \mathbb{T}_{\mathfrak{gl}(\infty)}$, then

- $K^{+}(X)$ *lies in* $\widetilde{\mathbb{T}}_{W}^{\geq}$,
- $K^{-}(X)$ *lies in* $\widetilde{\mathbb{T}}_{W}^{\leq}$.

The induced $\mathfrak{g}$-module $K^{+}(X)$ is not in $\mathbb{T}_{W}^{\leq}$ or $\mathbb{T}_{W}$, since it does not satisfy the l.a.c. for $\mathfrak{g}^{\leq}$. Similarly for $K^{-}(X)$.

When are these modules simple?

Proposition

If $X \in \mathbb{T}_{\mathfrak{gl}(\infty)}$, then $K^{-}(X)$ is not simple.

Borel subalgebras

Let $\mathfrak{b}^0$ be a Borel subalgebra of $\mathfrak{g}^0 = \mathfrak{gl}(\infty)$, and

$$\mathfrak{g}^{>} := \bigoplus_{k>0} \mathfrak{g}^k, \qquad \mathfrak{g}^{<} := \mathfrak{g}^{-1}.$$

Define Borel subalgebras for $\mathfrak{g}$ by

$$\mathfrak{b}^{\max} := \mathfrak{b}^0 \oplus \mathfrak{g}^{>}, \qquad \mathfrak{b}^{\min} := \mathfrak{b}^0 \oplus \mathfrak{g}^{<}.$$

$V_{\mathfrak{b}^0}(\gamma)$: simple $\mathfrak{b}^0$-highest weight $\mathfrak{g}^0$-module with h.w. $\gamma \in \mathfrak{h}^*$.

$L_{\mathfrak{b}}(\gamma)$: simple $\mathfrak{b}$-highest weight $\mathfrak{g}$-module with h.w. γ (here, $\mathfrak{b} = \mathfrak{b}^{\max}, \mathfrak{b}^{\min}$).

$\mathfrak{b}(<)^0, \mathfrak{b}(\prec)^0$: Borel subalgebras of $\mathfrak{g}^0$ corresponding to the orders $<$ and $\prec$ on I. Set

$$\mathfrak{b}(\prec)^0_n := \mathfrak{b}(\prec)^0 \cap \mathfrak{gl}(n), \text{ and } \mathfrak{b}(<)^0_n := \mathfrak{b}(<)^0 \cap \mathfrak{gl}(n).$$

Modules for $W(n)$

Bernstein and Leites described irreducible f.d. $W(n)$-modules and realized these representations as tensor fields modules.

Serganova studied the category of $\mathbb{Z}$-graded $W(n)$-modules and described the structure of induced modules. She proved:

- $K_n^+(V_{\mathfrak{b}(<)_n^0}(\nu))$ is simple if and only if the weight ν is typical.
- $K_n^+(V_{\mathfrak{b}(<)_n^0}(\nu))$ has length 2 when ν is atypical.
- atypical weights are of the form

$$a\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n, \quad \text{for some } a \in \mathbb{C}$$

Simplicity condtions for $K^+(V_{\lambda,\mu})$

Recall the simple module $V_{\lambda,\mu} \in \mathbb{T}_{\mathfrak{gl}(\infty)}$ corresponding to λ, μ

Theorem (C. Hoyt)

$K^+(V_{\lambda,\mu})$ is simple if and only if $(\lambda, \mu) \neq (\emptyset, (\mu_1))$, that is, if and only if $V_{\lambda,\mu} \ncong S^k(V_*)$ for some $k \in \mathbb{Z}_{\geq 0}$.

Moreover, we have a $\mathfrak{g}$-module homomorphism $K^+(S^{k+1}(V_*)) \to K^+(S^k(V_*))$ by $1 \otimes \partial_2^{k+1} \mapsto \partial_2 \otimes \partial_2^k$.

$\Rightarrow$ simple modules in $\mathbb{T}_W$ cannot be constructed via small induced modules

The functor Ψ

For $\mathfrak{k} \subset \mathfrak{g}$, let $M^{\mathfrak{k}} = \{m \in M \mid \mathfrak{k} \cdot m = 0\}$.

We define the functor $\Psi : \mathbb{T}_W^{\leq} \to \widetilde{\mathbb{T}}_{\mathfrak{gl}(\infty)}$ by $\Psi(M) := M^{\mathfrak{g}^<}$.

Proposition (C. Hoyt)

If $M \in \mathbb{T}_W^{\leq}$, then $\Psi(M) \neq 0$. If $M \in \mathbb{T}_W^{\leq}$ is simple, then

1. *$\Psi(M)$ is a simple $\mathfrak{g}^0$-module,*
2. *$\Psi(M)$ is in $\mathbb{T}_{\mathfrak{gl}(\infty)}$, and*
3. *$M \cong L^-(\Psi(M))$.*

- $\Lambda(V)_+ := \Lambda(V)/\mathbb{C} \cong L^-(V)$, since $(\Lambda(V)_+)^{\mathfrak{g}^<} = V$.
- $\mathfrak{g} \cong L^-(V_*)$, since $\mathfrak{g}^{\mathfrak{g}^<} = \mathfrak{g}^{-1} \cong_{\mathfrak{g}^0} V_*$.

A direct computation shows that $\Lambda(V)_+$ and $\mathfrak{g}$ are in $\mathbb{T}_W$.

Simple modules in $\mathbb{T}_W^{\leq}$

Set $L^-_{\lambda,\mu} := L^-(V_{\lambda,\mu})$.

Note: $L^-_{\lambda,\mu}$ is a highest weight $\mathfrak{g}$-module w.r.t. $\mathfrak{b}(\prec)^{\min}$ with the same h.w. as $V_{\lambda,\mu}$.

Theorem (C. Hoyt)

- *$L^-_{\lambda,\mu}$ is in $\mathbb{T}_W^{\leq}$ for any partitions λ and μ*
- *If $M \in \mathbb{T}_W^{\leq}$ is simple, then $M \cong L^-_{\lambda,\mu}$ for some λ and μ.*

Corollary

Every simple module of $\mathbb{T}_W^{\leq}$ (and hence of $\mathbb{T}_W$) is a highest weight module w.r.t. $\mathfrak{b}(\prec)^{\min}$.

Simple modules in $\mathbb{T}_W$

Proposition (C. Hoyt)

For any pair of partitions λ and μ, there exist $m, n \in \mathbb{Z}_{>0}$ such that the $\mathfrak{g}$-module $L^-_{\lambda,\mu}$ is a subquotient of $L^-(V)^{\otimes m} \otimes L^-(V_*)^{\otimes n}$. In particular, $L^-_{\lambda,\mu}$ lies in $\mathbb{T}_W$.

$L^-(V) \cong \Lambda(V)_+$ and $L^-(V_*) \cong \mathfrak{g}$ play a similar role for the category $\mathbb{T}_W$ as that of V, V_* for the category $\mathbb{T}_{\mathfrak{gl}(\infty)}$.

Corollary

The simple objects of $\mathbb{T}_W$ and $\mathbb{T}_W^{\leq}$ coincide.

This does not hold if we replace $\mathbb{T}_W^{\leq}$ with $\mathbb{T}_W^{\geq}$: $K^+(V_{\lambda,\mu})$ is simple when $(\lambda, \mu) \neq (\emptyset, (\mu_1))$, but $K^+(V_{\lambda,\mu})$ is never in $\mathbb{T}_{\mathfrak{g}}$.

Tensor fields modules

X : $\mathfrak{g}^0$-module

$\mathfrak{g}$-module of tensor fields w.r.t X is

$$\mathcal{T}(X) := \Lambda(\infty) \otimes X$$

where, for $f \in \Lambda(\infty)$ and $v \in X$, the action is

$$\underline{\xi}^{\underline{e}}\partial_j \cdot (fv) = \underline{\xi}^{\underline{e}}\partial_j(f)v + (-1)^{p(\underline{\xi}^{\underline{e}}\partial_j)p(f)} \sum_{i \in I} \partial_i(\underline{\xi}^{\underline{e}}) f E_{i,j} v,$$

Note: $X \in \widetilde{\mathbb{T}}_{\mathfrak{gl}(\infty)} \Rightarrow \mathcal{T}(X) \in \widetilde{\mathbb{T}}_W$

Coinduced modules

Consider X as a $\mathfrak{g}^{\geq}$-module

Define
$$\mathrm{Coind}_{\mathfrak{g}^{\geq}}^{\mathfrak{g}}(X) := \mathrm{Hom}_{\mathfrak{g}^{\geq}}(\mathbf{U}(\mathfrak{g}), X).$$

For $t = \sum f_i v_i \in \mathcal{T}(X)$, we set

$$t(0) := \sum f_i(0) v_i \in X.$$

We have a natural embedding of $\mathfrak{g}$-modules:

$$\varphi : \mathcal{T}(X) \hookrightarrow \mathrm{Coind}_{\mathfrak{g}^{\geq}}^{\mathfrak{g}}(X),$$

where for $t \in \mathcal{T}(X)$ and $u \in \mathbf{U}(\mathfrak{g})$,

$$\varphi(t)(u) := (-1)^{p(t)p(u)}(u \cdot t)(0).$$

Simple modules as tensor fields

In the finite-dimensional case, we have a nice situation:

$$\mathcal{T}(X_n) \cong \mathrm{Coind}_{\mathfrak{g}_n^{\geq}}^{\mathfrak{g}_n}(X_n) \cong K_n^+(X_n^*)^*$$

Proposition (C. Hoyt)

If $(\lambda, \mu) \neq ((\lambda_1), \emptyset)$, then $\mathcal{T}(V_{\lambda,\mu})$ is a locally simple $\mathfrak{g}$-module, and hence simple.

Corollary

If $(\lambda, \mu) \neq ((\lambda_1), \emptyset)$, then $\mathcal{T}(V_{\lambda,\mu})$ is in $\mathbb{T}_W$.

Theorem (C. Hoyt)

For any (λ, μ) we have that $L^-_{\lambda,\mu}$ is isomorphic to a submodule of $\mathcal{T}(V_{\lambda,\mu})$. Moreover, if $(\lambda, \mu) \neq ((\lambda_1), \emptyset)$, then $L^-_{\lambda,\mu} \cong \mathcal{T}(V_{\lambda,\mu})$.

Some categories

$\mathfrak{g}\,\mathsf{mod}$ (resp. $\mathfrak{g}^0\,\mathsf{mod}$) : category of all $\mathfrak{g}$-modules (resp. $\mathfrak{g}^0\,\mathsf{mod}$)

$\mathsf{Int}_{\mathfrak{g}^0}$: full subcategory of $\mathfrak{g}^0\,\mathsf{mod}$ consisting of integrable $\mathfrak{g}^0$-modules

$\mathsf{Int}_{\mathfrak{g},\mathfrak{g}^0}$: full subcategory of $\mathfrak{g}\,\mathsf{mod}$ consisting of $\mathfrak{g}^0$-integrable modules

$\mathsf{Int}^{\mathsf{wt}}_{\mathfrak{g},\mathfrak{g}^0}$: full subcategory of $\mathsf{Int}_{\mathfrak{g},\mathfrak{g}^0}$ consisting of $\mathfrak{h}$-weight modules

Some functors

Define the functors:

$\Gamma_{\mathfrak{g},\mathfrak{g}^0} : \mathfrak{g}\,\mathrm{mod} \to \mathrm{Int}_{\mathfrak{g},\mathfrak{g}^0}$, where

$$\Gamma_{\mathfrak{g},\mathfrak{g}^0}(M) = \{m \in M \mid \dim \mathrm{span}\{g^i m \mid i \geq 0\} < \infty,\ \forall g \in \mathfrak{g}^0\}$$

$\Gamma_{\mathfrak{h}} : \mathrm{Int}_{\mathfrak{g},\mathfrak{g}^0} \to \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{g}^0}$, where

$$\Gamma_{\mathfrak{h}}(M) := \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$$

$\Theta : \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{g}^0} \to \mathbb{T}_W$, where

$$\Theta(M) = \bigcup_{n > 0} M^{\mathfrak{t}_n}$$

Let $\Gamma = \Phi \circ \Gamma_{\mathfrak{h}} \circ \Gamma_{\mathfrak{g},\mathfrak{g}^0} : \mathfrak{g}\,\mathrm{mod} \to \mathbb{T}_W$

Injective $W(\infty)$-modules

Proposition (C. Hoyt)

If $I \in \mathfrak{g}\,\mathrm{mod}$ is injective, then $\Gamma(I)$ is injective in $\mathbb{T}_W$. Moreover, the category $\mathbb{T}_W$ has enough injectives.

Proposition (C. Hoyt)

If $I \in \mathrm{Int}_{\mathfrak{g}^0}$ is injective, then $\Gamma(\mathrm{Coind}_{\mathfrak{g}^0}^{\mathfrak{g}}(I))$ is injective in $\mathbb{T}_W$

For $V_{\lambda,\mu} \xrightarrow[\text{Serganova}]{\text{Penkov}} ((V_{\lambda,\mu})_*)^*$ is the injective hull of $V_{\lambda,\mu}$ in $\mathrm{Int}_{\mathfrak{g}^0}$.

Combining this with our classification, we obtain

Theorem (C. Hoyt)

Each simple module $L^-_{\lambda,\mu}$ of $\mathbb{T}_W$ is isomorphic to a submodule of the injective module $\Gamma\left(\mathrm{Coind}_{\mathfrak{g}^0}^{\mathfrak{g}}(((V_{\lambda,\mu})_*)^*)\right) \in \mathbb{T}_W$.

Some questions

- Description of $M|_{\mathfrak{g}^0}$ for $M \in \mathbb{T}_W$.
- Block decomposition of $\mathbb{T}_W$.
- Is $\mathbb{T}_W$ is Koszul? This is the case for $\mathfrak{gl}(\infty|\infty)$, and $\mathfrak{osp}(\infty|\infty)$ (Serganova).
- Is $\mathbb{T}_W$ equivalent to $\mathbb{T}_{\mathfrak{gl}(\infty)}$?

Thank you