Simple subquotients of relation modules

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Workshop on Representation Theory and Applications
April 28th

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Conventions

- We fix $n \geq 2$.
- $\mathfrak{gl}(n)$ will denote the Lie algebra of $n \times n$ matrices over $\mathbb{C}$.
- For $a, b \in \mathbb{C}$ we will write $a \geq b$ if $a - b \in \mathbb{Z}_{\geq 0}$.

Simple Finite dimensional modules for $\mathfrak{gl}(n)$

Definition

$$
\begin{array}{ccccccc}
l_{n1} & l_{n2} & & \cdots & & l_{n,n-1} & l_{nn} \\
 & l_{n-1,1} & & \cdots & & l_{n-1n-1} & \\
 & & \cdots & \cdots & \cdots & & \\
 & & l_{21} & & l_{22} & & \\
 & & & l_{11} & & &
\end{array}
$$

is called a Gelfand-Tsetlin tableau. A Gelfand-Tsetlin tableau is called standard if the entries satisfied

$$l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0} \quad \textit{and} \quad l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{>0}.$$

Theorem (Gelfand-Tsetlin-1950)

If $L(\lambda)$ is a finite dimensional irreducible representation of $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. The vector space with basis consisting of all standard tableaux $T(L)$'s with top row $l_{nj} = \lambda_j + j - 1$ has a $\mathfrak{gl}(n)$-module structure with action of the generators of $\mathfrak{gl}(n)$ given by the Gelfand-Tsetlin formulas. Moreover, this module is isomorphic to $L(\lambda)$.

$$E_{k,k+1}(T(L)) = -\sum_{i=1}^{k} \left(\frac{\prod_{j=1}^{k+1}(l_{ki} - l_{k+1,j})}{\prod_{j\neq i}^{k}(l_{ki} - l_{kj})} \right) T(L + \delta^{ki}),$$

$$E_{k+1,k}(T(L)) = \sum_{i=1}^{k} \left(\frac{\prod_{j=1}^{k-1}(l_{ki} - l_{k-1,j})}{\prod_{j\neq i}^{k}(l_{ki} - l_{kj})} \right) T(L - \delta^{ki}),$$

$$E_{kk}(T(L)) = \left(k - 1 + \sum_{i=1}^{k} l_{ki} - \sum_{i=1}^{k-1} l_{k-1,i} \right) T(L),$$

Where $T(L \pm \delta^{ki})$ is the tableau obtained by $T(L)$ adding ± 1 to the (k, i)'s position of $T(L)$ (if a new tableau is not standard then the result of the action is zero). The formulas above are called Gelfand-Tsetlin formulas for $\mathfrak{gl}(n)$.

Can we modify the concept of standard relations and obtain a well define set of tableaux where the Gelfand-Tsetlin formulas define a module structure?

I. Gelfand, M. Graev, "Finite-dimensional simple representations of the unitary and complete linear group and special functions associated with them." Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya 29.6 (1965): 1329-1356.

I. Gelfand, M. Graev, “Finite-dimensional simple representations of the unitary and complete linear group and special functions associated with them.” Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya 29.6 (1965): 1329-1356.

F. Lemire, J. Patera, “Formal analytic continuation of Gel’fand’s finite dimensional repre- sentations of $\mathfrak{gl}(n, \mathbb{C})$, Journal of Mathematical Physics. 20 (1979), 820–829.

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- V. Mazorchuk, Tableaux realization of generalized Verma modules, Canad. J. Math. 50 (1998) 816–828.

- V. Mazorchuk, Quantum deformation and tableau realization of simple dense $\mathfrak{gl}(n, \mathbb{C})$- modules, Journal of Algebra and Its Applications, (2003), vol.1 01.

Set $\mathfrak{V} := \{(i,j) \mid 1 \leq j \leq i \leq n\}$.

$$\mathcal{R}^{+} := \{((i,j);(i-1,t)) \mid 1 \leq j \leq i,\ 2 \leq i \leq n,\ 1 \leq t \leq i-1\}$$

$$\mathcal{R}^{-} := \{((i,j);(i+1,s)) \mid 1 \leq j \leq i \leq n-1,\ 1 \leq s \leq i+1\}$$

$$\mathcal{R}^{0} := \{((n,i);(n,j)) \mid 1 \leq i \neq j \leq n\}$$

and let $\mathcal{R} := \mathcal{R}^{-} \cup \mathcal{R}^{0} \cup \mathcal{R}^{+} \subset \mathfrak{V} \times \mathfrak{V}$. From now any $\mathcal{C} \subseteq \mathcal{R}$ will be called a set of relations.

Associated with any $\mathcal{C} \subseteq \mathcal{R}$ we can construct a directed graph $G(\mathcal{C})$ with set of vertices $\mathfrak{V}$ and an arrow going from (i,j) to (r,s) if and only if $((i,j);(r,s)) \in \mathcal{C}$.

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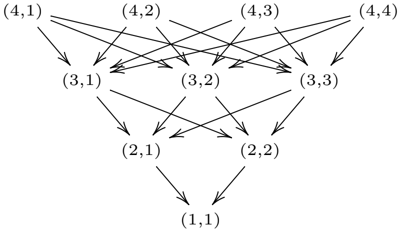
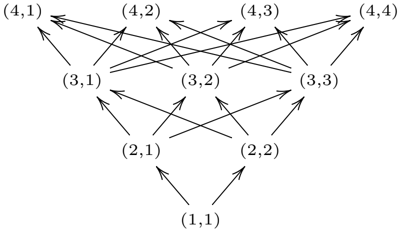
For convenience we will picture the vertex set as disposed in a triangular arrangement with n rows and k-th row given by $\{(k, 1), \ldots, (k, k)\}$.

Set $\mathcal{R}^+ = \{((i,j);(i-1,t)) \mid 1 \leq j \leq i,\ 2 \leq i \leq n,\ 1 \leq t \leq i-1\}$



Set $\mathcal{R}^{-} = \{((i,j);(i+1,s)) \mid 1 \leq j \leq i \leq n-1,\ 1 \leq s \leq i+1\}$



Definition

We will say that $T(L)$ satisfies $\mathcal{C}$ if:

- $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}$ for any $((i,j);(r,s)) \in \mathcal{C}^+ \cup \mathcal{C}^0$.
- $l_{ij} - l_{rs} \in \mathbb{Z}_{>0}$ for any $((i,j);(r,s)) \in \mathcal{C}^-$.

- By $\mathcal{B}_{\mathcal{C}}(T(L))$ we denote the set of all tableaux of the form $T(L+z)$, $z \in \{z \in \mathbb{Z}^{\frac{n(n+1)}{2}} \mid z_{ni} = 0, i = 1, \ldots, n\}$ satisfying $\mathcal{C}$.

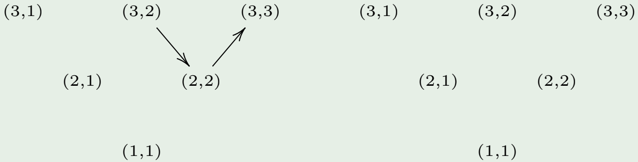
- By $\mathcal{B}_{\mathcal{C}}(T(L))$ we denote the set of all tableaux of the form $T(L+z)$, $z \in \{z \in \mathbb{Z}^{\frac{n(n+1)}{2}} \mid z_{ni} = 0, i = 1, \ldots, n\}$ satisfying $\mathcal{C}$.

- By $V_{\mathcal{C}}(T(L))$ we denote the complex vector space spanned by $\mathcal{B}_{\mathcal{C}}(T(L))$.

Example

π	2	1
2	2	
0		

satisfies, where $G(\mathcal{C})$ is given by any of the following graphs



Definition

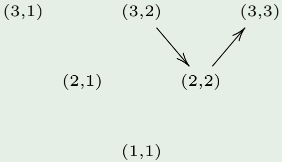
We will say that $T(L)$ is a $\mathcal{C}$-realization if:

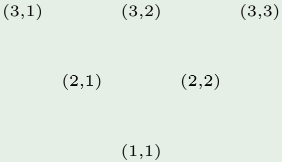
- $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0}$ for any $((i,j);(r,s)) \in \mathcal{C}^+ \cup \mathcal{C}^0$.
- $l_{ij} - l_{rs} \in \mathbb{Z}_{>0}$ for any $((i,j);(r,s)) \in \mathcal{C}^-$.
- For any $1 \leq k \leq n-1$ we have, $l_{ki} - l_{kj} \in \mathbb{Z}$ if and only if (k,i) and (k,j) in the same connected component of $G(\mathcal{C})$.

Example

$$\begin{array}{ccccc} \pi & & 2 & & 1 \\ & \sqrt{2} & & 2 & \\ & & 0 & & \end{array}$$

Is a $\mathcal{C}$-realization, where $G(\mathcal{C})$ is given by any of the following graphs





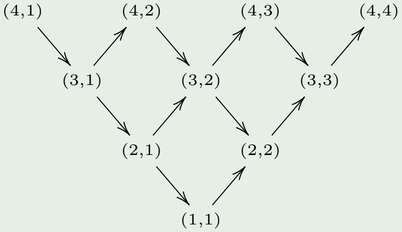
Definition $\mathcal{C} \subseteq \mathcal{R}$ is call admissible if:

- There exist a $\mathcal{C}$-realization $T(L)$.

- For any $\mathcal{C}$-realization $T(L)$, the vector space $V_{\mathcal{C}}(T(L))$ has a structure of a $\mathfrak{gl}_n$-module, endowed with the action of the generators of $\mathfrak{gl}_n$ given by the Gelfand-Tsetlin formulas.

Example

$$\mathcal{S}^+ := \{(i+1,j);(i,j)) \mid 1 \leq j \leq i \leq n-1\}$$
$$\mathcal{S}^- := \{((i,j);(i+1,j+1)) \mid 1 \leq j \leq i \leq n-1\}.$$



How to construct admissible sets of relations?

Definition

Let $\mathcal{C}$ be any set of relations and $(i,j) \in \mathfrak{V}$ be a maximal or a minimal pair with respect to $G(\mathcal{C})$. Denote by $\mathcal{C}_{ij}$ the set of relations obtained from $\mathcal{C}$ by removing all relations that involve (i,j).

Definition

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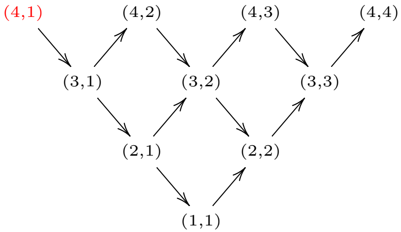
We say that $\widetilde{\mathcal{C}} \subsetneq \mathcal{C}$ is obtained from $\mathcal{C}$ by the RR-method if it is obtained by a sequence removing of relations of the form $\mathcal{C}' \to \mathcal{C}'_{ij}$ for different indexes.

Theorem (Futorny, R., Zhang)

Let $\mathcal{C}_1$ be any admissible set of relations. If $\mathcal{C}_2$ is obtained from $\mathcal{C}_1$ by the RR-method then $\mathcal{C}_2$ is admissible.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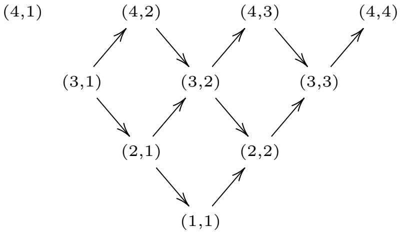
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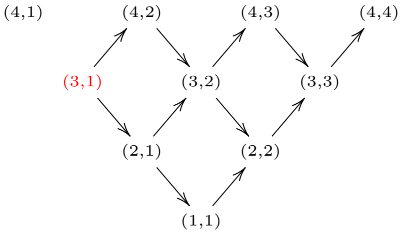
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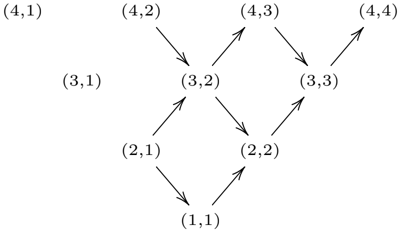
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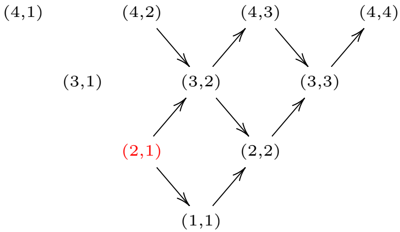
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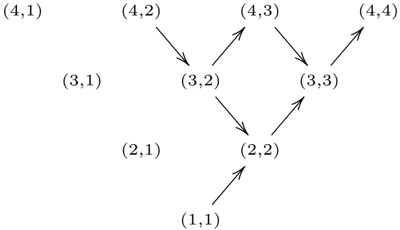
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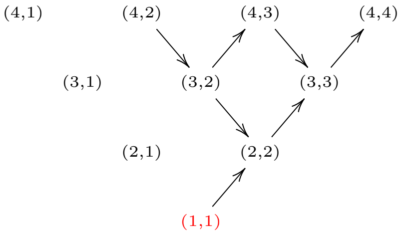
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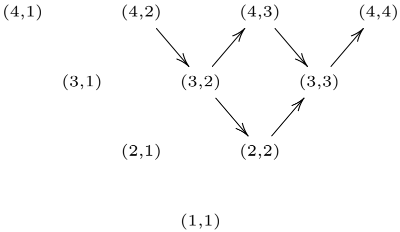
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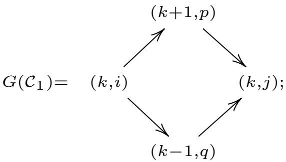
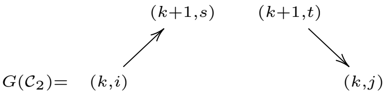
(1,1)

Description of admissible sets of relations

$\Diamond$-condition.

For every adjoining pair (k,i) and (k,j), $1 \leq k \leq n-1$, there exist p,q such that $\mathcal{C}_1 \subseteq \mathcal{C}$ or, there exist $s<t$ such that $\mathcal{C}_2 \subseteq \mathcal{C}$, where the graphs associated to $\mathcal{C}_1$ and $\mathcal{C}_2$ are as follows





Theorem (Futorny, R., Zhang)

A reduced set of relations $\mathcal{C}$ without cycles and crosses is admissible if and only if $G(\mathcal{C})$ is a union of disconnected sets satisfying $\Diamond$-condition.

Generic modules

(5,1) (5,2) (5,3) (5,4) (5,5)

(4,1) (4,2) (4,3) (4,4)

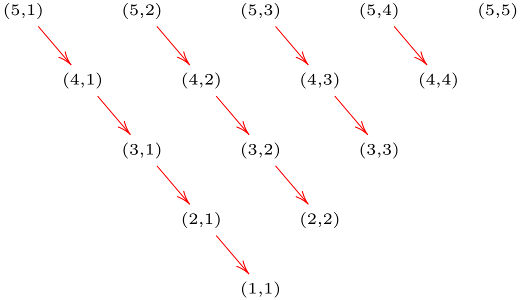
(3,1) (3,2) (3,3)

(2,1) (2,2)

(1,1)

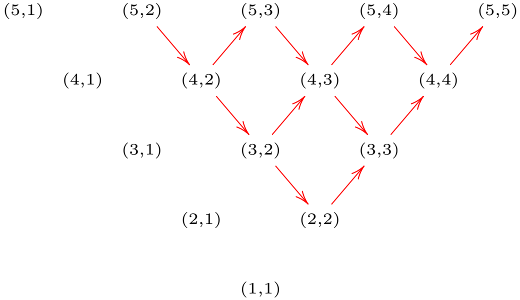
Y. Drozd, S. Ovsienko, V. Futorny, Harish-Chandra subalgebras and Gelfand-Zetlin modules, Math. Phys. Sci. 424 (1994) 72-89.

Generic Verma modules



V. Mazorchuk, Tableaux realization of generalized Verma modules, Canad. J. Math. 50 (1998) 816–828.

Cuspidal modules



V. Mazorchuk, Quantum deformation and tableaux realization of simple dense $\mathfrak{gl}(n, \mathbb{C})$-modules, J. Algebra Appl. 1 (01) (2003).

Structure of relation modules

Let for $m \leqslant n$, $\mathfrak{gl}_m$ be the Lie subalgebra of $\mathfrak{gl}(n)$ spanned by $\{E_{ij} \,|\, i, j = 1, \ldots, m\}$.

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n$$

which induces a chain of the corresponding Universal enveloping algebras

$$U_1 \subset U_2 \subset \ldots \subset U_n.$$

Let us denote by Z_m the center of U_m.

Definition

The standard Gelfand-Tsetlin subalgebra Γ of U is the subalgebra generated by $\bigcup_{i=1}^{n} Z_i$.

Definition

A Gelfand-Tsetlin module is a U-module M such that

$$M = \bigoplus_{\chi \in \Gamma^*} M(\chi),$$

with $M(\chi)$ the set of all vectors of generalized Γ-eigenvalue χ.

$$M(\chi) = \{v \in M : \forall g \in \Gamma \ , \exists k \in \mathbb{N} \text{ such that } (g - \chi(g))^k v = 0\}.$$

Theorem (Futorny, R., Zhang)

For any admissible $\mathcal{C}$ the module $V_{\mathcal{C}}(T(L))$ is a Gelfand-Tsetlin module with diagonalizable action of the generators of the Gelfand-Tsetlin subalgebra Γ.

Simple subquotients!

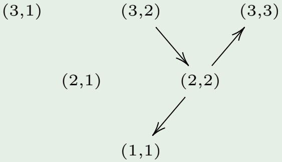
Associated with any Gelfand-Tsetlin tableau $T(L)$ we have a directed graph $G(T(L))$ with set of vertices $\mathfrak{V}$ and an arrow going from (i,j) to (r,s) if

- $i = r + 1$, and $l_{i,j} - l_{r,s} \in \mathbb{Z}_{\geq 0}$, or
- $i = r - 1$, and $l_{i,j} - l_{r,s} \in \mathbb{Z}_{>0}$.

Example

π	2	1
$\sqrt{2}$	2	
0		

has as associated graph



Let us fix an admissible set of relations $\mathcal{C}$, a $\mathcal{C}$-realization $T(L)$ and the relation module $V := V_{\mathcal{C}}(T(L))$.

For any $T(U) \in \mathcal{B}_{\mathcal{C}}(T(L))$ we define

$$\Omega^{+}(T(U)) := \{(r, s, t) \mid u_{rs} - u_{r-1,t} \in \mathbb{Z}_{\geq 0}\}$$

Theorem

For any basis element $T(R) \in \mathcal{B}_{\mathcal{C}}(T(L))$*, a basis for the* $U(\mathfrak{gl}(n))$*-module generated by* $T(R)$ *is given by:*

$$\{T(S) \in \mathcal{B}_{\mathcal{C}}(T(L)) \mid \Omega^{+}(T(R)) \subseteq \Omega^{+}(T(S))\}$$

Theorem

For any basis element $T(R) \in \mathcal{B}_{\mathcal{C}}(T(L))$, a basis for the simple subquotient of $V_{\mathcal{C}}(T(L))$ containing $T(R)$ is given by:

$$\{T(S) \in \mathcal{B}_{\mathcal{C}}(T(L)) \mid \Omega^{+}(T(R)) = \Omega^{+}(T(S))\}$$

References

V. Futorny, D. Grantcharov, and L. E. Ramirez, Simple Generic Gelfand-Tsetlin modules of $\mathfrak{gl}(n)$. SIGMA, **18**, (2015).

V. Futorny, L. E. Ramirez, and J. Zhang, Combinatorial construction of Gelfand-Tsetlin modules for $\mathfrak{gl}_n$, Adv. Math. **343** (2019), 681–711.

Thanks for your attention!