Trees Are Real And Totally Ordered Graphs Are Prime

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Open Question: Classify the simple prime modules in the category of finite-dimensional representations for a quantum affine algebra.

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Recall $U_q(\tilde{\mathfrak{g}}) = U_q(\tilde{\mathfrak{n}}^-)U_q(\tilde{\mathfrak{h}})U_q(\tilde{\mathfrak{n}}^+)$ and $U_q(\tilde{\mathfrak{h}})$ is abelian.

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$$\boldsymbol{\omega}_{a,r} := (1 - aq^{1-r}u)\cdots(1 - aq^{r-3}u)(1 - aq^{r-1}u) = \prod_{l=1}^{r}(1 - aq^{r+1-2l}u).$$

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Every polynomial has a unique factorization as a product of such elements so that, for any two factors, say $\boldsymbol{\omega}_{a,r}$ and $\boldsymbol{\omega}_{b,s}$, we have

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There exists a finite set $\mathscr{R}_{i,j}^{r,s} \subseteq \mathbb{Z}_{>0}$ such that

$$V_q(\boldsymbol{\omega}_{i,a,r}) \otimes V_q(\boldsymbol{\omega}_{j,b,s}) \text{ is reducible} \quad \Leftrightarrow \quad b = a q^m \text{ with } |m| \in \mathscr{R}_{i,j}^{r,s}$$

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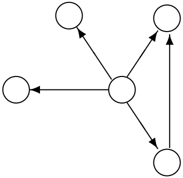

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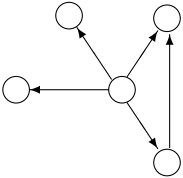

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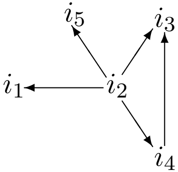

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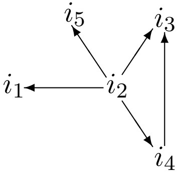

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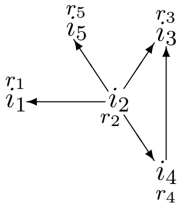

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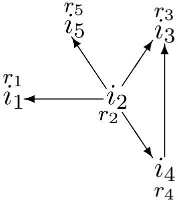

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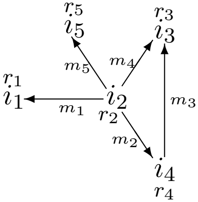

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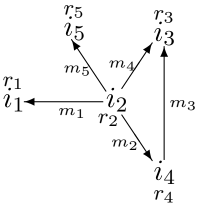

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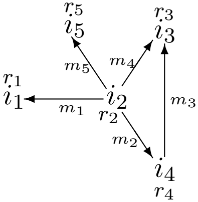

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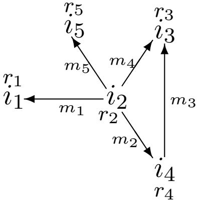

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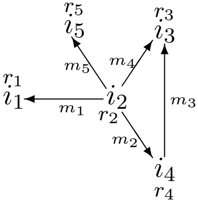

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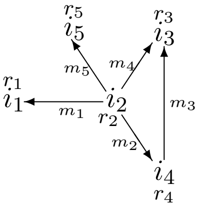

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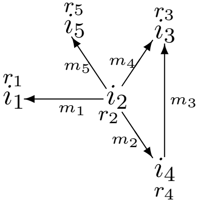

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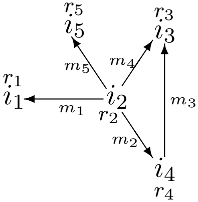

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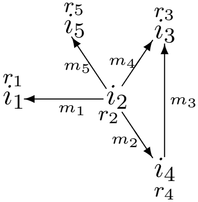

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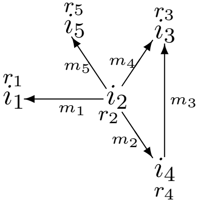

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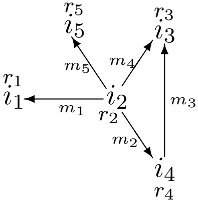

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in that case, $\epsilon(\boldsymbol{\omega}_{i,a,r}, \boldsymbol{\omega}_{j,b,s}) = m$.

Examples

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If $V_q(\boldsymbol{\pi})$ is prime, $G(\boldsymbol{\pi})$ is connected.

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$$m_j - m_{j'} + 1 \notin \mathscr{R}^{r_j-1,r_{j'}}_{i_j,i_{j'},I_j} \quad \text{if} \quad r_j > 1.$$

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A simple module whose tensor square is simple is said to be real

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Then, $V_q(\boldsymbol{\pi}') \otimes V_q(\boldsymbol{\pi}'')$ is not highest-ℓ-weight.

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Type $A_n, n \geq 2$, the following is prime, but has non-prime connected subgraphs

$$\begin{tikzcd}
\overset{2}{2} \arrow[dr, "4"] & \overset{2}{1} \arrow[l, "3"'] \arrow[r, "4"] & \overset{1}{2} \arrow[dl, "3"'] \\
& \overset{1}{1} &
\end{tikzcd}$$