On the double of the Jordan plane

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I. Preliminaries.
Let \( k = \overline{k} \) be an algebraically closed field, \( \text{char } k \neq 2 \).

The **Jordan plane** is the (graded, quadratic) algebra
\[
J = k\langle x, y | yx - xy + \frac{1}{2}x^2 \rangle.
\]

- (M. Artin & W. F. Schelter) The classification of the AS-regular algebras of rank 2 consists of the quantum planes and the Jordan plane.

Notice that \( \text{GK-dim } J = 2 \).
(D. Gurevich) Let \( V \) be a vector space with basis \( \{x, y\} \) and let \( c \in GL(V \otimes V) \) be given by

\[
\begin{align*}
c(x \otimes x) &= x \otimes x, & \quad c(y \otimes x) &= x \otimes y, \\
c(x \otimes y) &= (y + x) \otimes x, & \quad c(y \otimes y) &= (y + x) \otimes y.
\end{align*}
\]

Then \( c \) (called the braiding) satisfies the braid equation

\[
(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).
\]

From this it is easy to see that \( J \) is a braided Hopf algebra.
Actually $V \in \hat{k}_\Gamma \mathcal{YD}$, where $\Gamma = \langle g \rangle \simeq \mathbb{Z}$ acts on $V$ by $g \cdot x = x$, $g \cdot y = y + x$. Thus

$$\widetilde{H} := J \# \hat{k}_\Gamma = \hat{k}\langle x, y, g^{\pm 1} | \ g^{\pm 1}g^{\mp 1} - 1, \ gx - xg, \ gy - yg - xg, \ yx - xy + \frac{1}{2} x^2 \rangle$$

is a Hopf algebra with

$$\Delta(g^{\pm 1}) = g^{\pm 1} \otimes g^{\pm 1},$$
$$\Delta(x) = x \otimes 1 + g \otimes x,$$
$$\Delta(y) = y \otimes 1 + g \otimes y.$$

The Hopf algebra $\widetilde{H}$ is the bosonization of the Jordan plane.
Assume that $\text{char } \mathbb{k} = 0$. Then $J \cong \mathcal{B}(V)$ is a Nichols algebra (Cibils, Lauve & Witherspoon); that is, all primitives are in degree 1. Indeed, given $\epsilon \in \mathbb{k}^\times$ and $\ell \in \mathbb{N}_{\geq 2}$, the block $\mathcal{V}(\epsilon, \ell)$ is a vector space with a basis $(x_i)_{i \in \mathbb{I}_\ell}$ and a braiding

$$c(x_i \otimes x_j) = \begin{cases} \epsilon x_1 \otimes x_i, & j = 1, \\ (\epsilon x_j + x_{j-1}) \otimes x_i, & j \geq 2, \end{cases} \quad i \in \mathbb{I}_\ell.$$  

**Theorem.** (A.–Angiono–Heckenberger)  
$\text{GK-dim } \mathcal{B}(\mathcal{V}(\epsilon, \ell)) < \infty \iff \ell = 2$ and $\epsilon \in \{\pm 1\}$.  

- $\mathcal{B}(\mathcal{V}(1, 2)) \cong J$ is the **Jordan plane**.

- $\mathcal{B}(\mathcal{V}(-1, 2)) = \mathbb{k}\langle x_1, x_2 | x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21} \rangle$ is the **super Jordan plane**. Here $x_{21} = x_2x_1 + x_1x_2$. 
This result is important for the question of classifying Hopf algebras with finite Gelfand-Kirillov dimension. Indeed, we have

**Theorem.** (A.–Angiono–Heckenberger) Assume that \( \text{char } k = 0 \). The classification of the braided vector spaces \( V \) which are direct sums of blocks and points and that have \( \text{GK-dim } B(V) < \infty \) is known (up to a Conjecture on diagonal type).

**Remark.** (A.–Angiono–Heckenberger), (A.–Bagio–Della Flora–Flôres) The analogous braided vector spaces have often *finite-dimensional* Nichols algebras.
II. The restricted Jordan plane (odd characteristic).

Assume now that $p = \text{char } k > 2$. Recall that $V$ has a basis $\{x, y\}$ and braiding given by

\[
\begin{align*}
    c(x \otimes x) &= x \otimes x, & c(y \otimes x) &= x \otimes y, \\
    c(x \otimes y) &= (y + x) \otimes x, & c(y \otimes y) &= (y + x) \otimes y.
\end{align*}
\]

**Theorem.** (Cibils, Lauve & Witherspoon) The Nichols algebra $\mathcal{B}(V)$ is the quotient of $T(V)$ by the ideal generated by

\[
    x^p, \quad y^p, \quad yx - xy + \frac{1}{2} x^2.
\]

That is, $\mathcal{B}(V) \simeq J/\langle x^p, y^p \rangle$; this is the restricted Jordan plane (in characteristic $p$).
Actually $V \in \mathcal{YD}_{kC_p}$, where $C_p = \langle g \rangle$ has order $p$. Thus

$$H := \mathcal{B}(V) \# kC_p = k\langle x, y, g | g^p - 1, gx - xg, gy - yg - xg, x^p, y^p, yx - xy + \frac{1}{2}x^2 \rangle$$

is a Hopf algebra (of dimension $p^3$) with

$$\Delta(g) = g \otimes g \quad (g \in G(H)),$$

$$\Delta(x) = x \otimes 1 + g \otimes x, \quad (x \in \mathcal{P}_{g,1}(H))$$

$$\Delta(y) = y \otimes 1 + g \otimes y \quad (x \in \mathcal{P}_{g,1}(H)).$$

$H$ is the bosonization of the restricted Jordan plane.

Clearly there is a surjective map of Hopf algebras $\tilde{H} \rightarrow H$. 
Let $L$ be a finite-dimensional Hopf algebra. V. G. Drinfeld introduced the double of $L$, a quasitriangular Hopf algebra denoted by $D(L)$. Actually $\mathbb{L}_{L} \mathcal{YD} \simeq D(L)\mathcal{M}$. The underlying coalgebra is $L \otimes L^{* \text{op}}$ and the algebra is a kind of bi-semidirect product.

Here quasitriangular means that any $L$-module comes with a solution of the braid equation.
Proposition. (A.–Peña Pollastri) The algebra $D(H)$ is generated by

$$
\begin{array}{c}
g, x, y, \\
H
\end{array} \quad \quad \quad \quad \begin{array}{c}
u, \nu, \zeta \\
H^*\end{array}
$$

with defining relations

$$g^p = 1, \quad \zeta^p = 0, \quad x^p = 0, \quad y^p = 0, \quad u^p = 0, \quad v^p = 0. \quad (1)$$

$$\zeta g = g\zeta, \quad (2)$$

$$gx = xg, \quad gy = yg + xg, \quad \zeta y = y\zeta + y, \quad \zeta x = x\zeta + x, \quad (3)$$

$$ug = gu, \quad vg = gv + gu, \quad v\zeta = \zeta v + v, \quad u\zeta = \zeta u + u,$$

$$yx = xy - \frac{1}{2}x^2, \quad vu = uv - \frac{1}{2}u^2, \quad (4)$$

$$ux = xu, \quad vx = xv + (1 - g) + xu,$$

$$uy = yu + (1 - g), \quad vy = yv - g\zeta + yu.$$
The comultiplication is defined by

\[ g \in G(D(H)), \]
\[ u, \zeta \in \mathcal{P}(D(H)), \]
\[ x, y \in \mathcal{P}_{g,1}(D(H)), \]
\[ \Delta(v) = v \otimes 1 + 1 \otimes v + \zeta \otimes u. \]
Recall that a short exact sequence of Hopf algebras is a collection

\[ A \xrightarrow{\iota} C \xrightarrow{\pi} B \]

where

(i) \( \iota \) is injective. 

(ii) \( \pi \) is surjective. 

(iii) \( \ker \pi = C\iota(A)^{+} \).

(iv) \( \iota(A) = C^{\text{co} \pi} \).

One also says that \( C \) is an extension of \( B \) by \( A \).

When \( A \) is commutative and \( B \) is cocommutative, one says that the extension is \textit{abelian}. 

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Let \( \{h, e, f\} \) be the Cartan generators of \( \mathfrak{sl}_2(\mathbb{k}) \), whose restricted enveloping algebra is denoted \( u(\mathfrak{sl}_2(\mathbb{k})) \).

**Proposition.** (A.–Peña Pollastri) The subalgebra \( R \) of \( D(H) \) generated by \( g, x \) and \( u \) is a normal local commutative Hopf subalgebra of \( D(H) \) of dimension \( p^3 \) with defining relations

\[
g^p = 1, \quad x^p = 0, \quad u^p = 0.
\]

(6)

It gives rise to the (abelian) exact sequence of Hopf algebras

\[
R \overset{\iota}{\hookrightarrow} D(H) \overset{\pi}{\twoheadrightarrow} u(\mathfrak{sl}_2(\mathbb{k}))
\]

where \( \pi(\zeta) = h, \pi(y) = \frac{1}{2}e \) and \( \pi(v) = f \).
Since $R$ is normal and local, we conclude

**Theorem.** (A.–Peña Pollastri) Irrep $D(H) \simeq \text{Irrep } u(\mathfrak{sl}_2(k))$. Thus there are exactly $p$ isomorphism classes of simple $D(H)$-modules which have dimensions $1, 2, \ldots, p$.

These simple modules can be constructed as quotients of Verma modules.
III. The double of the Jordan plane, \( \text{char } k \neq 2 \)

**Definition.** (A.–Peña Pollastri) The double of the Jordan plane is the algebra \( \mathcal{D} \) generated by 

\[
g^\pm 1, x, y, \quad \{z \} \quad \tilde{H}, \quad u, v, \zeta \quad \tilde{K}
\]

with defining relations (2), (3), (4) and

\[
g^\pm 1 g^\mp 1 = 1 \quad (7)
\]

The comultiplication is defined by (5).
Consider the algebraic groups

- $G_a = \text{additive group } (\mathbb{k}, +)$,

- $G_m = \text{multiplicative group } (\mathbb{k}^\times, \cdot)$,

- $H_3 = \text{Heisenberg group of dimension 3}$,

- $G = (G_a \times G_a) \rtimes G_m$,

- $B = ((G_a \times G_a) \rtimes G_m) \rtimes H_3$,

with suitable semidirect products.

The algebra of regular functions on $G$ is denoted by $\mathcal{O}(G)$, etc.
• The subalgebra of $\mathcal{D}$ generated by $x$, $u$ and $g$ is a normal Hopf subalgebra isomorphic to $\mathcal{O}(G)$ (as Hopf algebras).

• Let $\pi: \mathcal{D} \longrightarrow U(\mathfrak{sl}_2(\mathbb{k}))$ be the Hopf algebra map given by

\[
x \mapsto 0, \quad u \mapsto 0, \quad g \mapsto 1, \quad y \mapsto \frac{1}{2}e, \quad v \mapsto f, \quad \zeta \mapsto h.
\]

**Proposition.** (A.–Peña Pollastri) There is an exact sequence

\[
\mathcal{O}(G) \xrightarrow{i} \mathcal{D} \xrightarrow{\pi} U(\mathfrak{sl}_2(\mathbb{k})).
\]
Assume that char \( k = p > 2 \).

**Proposition.** (A.–Peña Pollastri) There is a commutative diagram

\[
\begin{array}{c}
\mathcal{O}(G) \leftarrow \mathcal{O}(B) \rightarrow \mathcal{O}(G_3) \\
\mathcal{O}(G) \downarrow \downarrow \downarrow \\
\mathcal{O}(G) \downarrow \downarrow \downarrow \\
D \downarrow \downarrow \downarrow \\
\mathcal{R} \downarrow \downarrow \downarrow \\
D(H) \rightarrow \mathcal{U}(\mathfrak{sl}_2(\mathbb{k}))
\end{array}
\]

where all columns and rows are exact sequences.
Properties of $\mathcal{D}$, $\operatorname{char} k = 0$

- The algebra $\mathcal{D}$ admits an exhaustive ascending filtration $(\mathcal{D}_n)_{n \in \mathbb{N}_0}$ such that $\operatorname{gr} \mathcal{D} \simeq k[T^\pm] \otimes k[X_1, \ldots, X_5]$.

- $\mathcal{D}$ is a noetherian domain.

- $\mathcal{D}$ is an Ore extension, hence strongly noetherian, AS-regular and Cohen-Macaulay.

- $\mathcal{D}$ is a PI-algebra.

- $\mathcal{D}$ satisfies the Gelfand-Kirillov property i. e. its skew field of fractions $\operatorname{Frac} \mathcal{D}$ is $k$-isomorphic to a Weyl skew field $\mathcal{D}_{n,s}(k)$. 
**Theorem.** (A.–Dumas–Peña Pollastri) \(\text{Irrep } D \simeq \text{Irrep } U(\mathfrak{sl}_2(\mathbb{k}))\).

For the proof we use the following result. Let \(A\) be an algebra. Let \(F \subseteq A\) be a family of elements satisfying
- the elements of \(F\) commute with each other;
- any \(x \in F\) acts nilpotently on any \(M \in A\mathcal{M}\), \(\text{dim } M < \infty\);
- \(F\) is normal: \(AF = FA\).

Let \(L \in \text{Irrep } A\). Then the representation \(\rho : A \to \text{End } L\) factorizes through \(A/AF\). Thus the projection \(A \to A/AF\) induces a bijection

\[
\text{Irrep } A \simeq \text{Irrep } A/AF.
\]

Then we check that \(F = \{x, u, g - 1\}\) fulfills the preceding hypothesis.
Recall that the algebra $\mathcal{D}$ generated by $g^{\pm 1}, x, y, u, v, \zeta$. Let

$$q = ux + 2(1 + g), \quad s = xv + uy + \left(-\frac{1}{2}ux + g - 1\right)\zeta - 2(1 + g).$$

Then we can show that the following elements belong to $\mathcal{Z}(\mathcal{D})$:

$$z = q^2g^{-1}, \quad \theta = s^2g^{-1}, \quad \omega = qg^{-1}s.$$

**Theorem.** (A.–Dumas–Peña Pollastri) The center of $\mathcal{D}$ is the commutative subalgebra generated by $z$, $\omega$ and $\theta$, which is isomorphic to the quotient $\mathbb{k}[X, Y, Z]/(XZ - Y^2)$. 
V. The super Jordan plane (& its restricted version in odd char).

Assume that char $\mathbb{k} \neq 2$. Let $V$ have a basis $\{x, y\}$ and braiding

\[
\begin{align*}
    \phi(x \otimes x) &= -x \otimes x, & \phi(y \otimes x) &= -x \otimes y, \\
    \phi(x \otimes y) &= (-y + x) \otimes x, & \phi(y \otimes y) &= (-y + x) \otimes y.
\end{align*}
\]

Let $x_{21} = x_2 x_1 + x_1 x_2$. The super Jordan plane is the algebra

\[
sJ = \mathbb{k} \langle x_1, x_2 | x_1^2, x_2 x_{21} - x_{21} x_2 - x_1 x_{21} \rangle
\]

**Theorem.** (A.–Angiono–Heckenberger)

- If char $\mathbb{k} = 0$, then the Nichols algebra $\mathcal{B}(V) \simeq sJ$.
- If char $\mathbb{k} = p > 2$, then $\mathcal{B}(V) \simeq sJ/\langle x_2^{2p}, x_{21}^p \rangle$.

This is the restricted super Jordan plane in characteristic $p$; it has dimension $4p^2$.  

Assume that $\text{char } \mathbb{k} = p > 2$. 

Now $V \in \mathbb{k}C_{2p}^2 \gamma \mathbb{D}$. Let $H$ be the bosonization of $\mathcal{B}(V)$, i. e.

$$H = \mathcal{B}(V) \# \mathbb{k}C_{2p} = \mathbb{k}\langle x, y, \gamma | \gamma^{2p} - 1, \gamma x + x \gamma, \gamma y + y \gamma - x \gamma, \\
x_1^2, x_2^{2p} = 0, x_2 x_{21} - x_2 x_{21} - x_{21} x_1, x_2^{2p}, x_{21}^p \rangle$$

is a Hopf algebra (of dimension $4p^3$) with

$$\gamma \in G(H), \quad x, y \in \mathcal{P}_{\gamma, 1}(H).$$
Facts.

- The Drinfeld double $D(H)$ can be computed explicitly.

- $D(H)$ has a normal Hopf subalgebra $Z_0 \cong \mathbb{k}C_2$; the quotient is $D := D(H)/D(H)Z_0^+$. 

- There is a Hopf superalgebra $\mathcal{D}$ such that $D \cong \mathcal{D} \# \mathbb{k}C_2$. 
Let $R$ be the super commutative Hopf superalgebra

$$R := \mathbb{k}[X_1, X_2, T]/(X_1^p, X_2^p, T^p - 1) \otimes \Lambda(Y_1, Y_2)$$

with $|X_1| = |X_2| = |T| = 0$, $|Y_1| = |Y_2| = 1$, and comultiplication

$$\Delta(X_1) = X_1 \otimes 1 + T^2 \otimes X_1 + Y_1 T \otimes Y_1, \quad \Delta(T) = T \otimes T,$$

$$\Delta(X_2) = X_2 \otimes 1 + 1 \otimes X_2 + Y_2 \otimes Y_2,$$

$$\Delta(Y_2) = Y_2 \otimes 1 + 1 \otimes Y_2, \quad \Delta(Y_1) = Y_1 \otimes 1 + T \otimes Y_1.$$ 

**Theorem.** (A.–Peña Pollastri) There exist Hopf superalgebra maps $\iota$ and $\pi$ such that

$$R \xleftarrow{\iota} D \xrightarrow{\pi} u(osp(1|2))$$

(8)

is an exact sequence of Hopf superalgebras.
Since $\mathcal{R}$ is normal and local, we conclude

**Theorem.** (A.–Peña Pollastri) $\text{Irrep } \mathcal{D} \simeq \text{Irrep } \mathfrak{u}(\mathfrak{osp}(1|2))$. Hence there are $p$ isomorphism classes of simple $\mathcal{D}$-modules which have dimensions $1, 3, 5, \ldots, 2p - 1$. 
Proposition. (A.–Peña Pollastri) There is a commutative diagram of Hopf superalgebras

\[
\begin{array}{ccc}
\mathcal{O}(G) & \rightarrow & \mathcal{O}(G) \\
\downarrow & & \downarrow \\
\mathcal{O}(G) & \rightarrow & \mathcal{F}
\end{array}
\]

where all columns and rows are exact sequences.
V. The restricted Jordan plane (even characteristic).
Assume now that \( p = \text{char } \mathbb{k} = 2 \). Recall that \( V \) has a basis \( \{x, y\} \) and braiding given by

\[
\begin{align*}
   c(x \otimes x) &= x \otimes x, & c(y \otimes x) &= x \otimes y, \\
   c(x \otimes y) &= (y + x) \otimes x, & c(y \otimes y) &= (y + x) \otimes y.
\end{align*}
\]

**Theorem.** (Cibils, Lauve & Witherspoon) The Nichols algebra \( \mathcal{B}(V) \) is the quotient of \( T(V) \) by the ideal generated by

\[
\begin{align*}
   x_1^2 &= 0, & x_2^4 &= 0, & x_2^2x_1 &= x_1x_2^2 + x_1x_2x_1, & x_1x_2x_1x_2 &= x_2x_1x_2x_1.
\end{align*}
\]

This is the restricted Jordan plane in characteristic 2; it has dimension 16. It is closer to the restricted Jordan plane in odd characteristic.
Now $V \in \kC_2 \mathcal{YD}$. Let $H$ be the bosonization of $\mathcal{B}(V)$, i.e.

$$H = \mathcal{B}(V) \# \kC_2 = \mathbb{k}\langle x, y, g \mid g^2 - 1, gx - xg, gy - yg - xg, x_1^2, x_2^4, x_2^2x_1 - x_1x_2^2 + x_1x_2x_1, x_1x_2x_1x_2 = x_2x_1x_2x_1 \rangle$$

is a Hopf algebra (of dimension 32) with

$$g \in G(H), \quad x, y \in \mathcal{P}_{g,1}(H).$$
Let $\mathfrak{s}$ be the derived Lie algebra of the 4-dimensional Witt Lie algebra. Then $\mathfrak{s}$ has a basis $\{a, b, c\}$ and bracket
\[
\begin{align*}
[a, b] &= c, & [a, c] &= a, & [b, c] &= b. 
\end{align*}
\] (9)

$\mathfrak{s}$ is the unique (up to isomorphism) simple Lie algebra of dim. 3.

The Lie algebra $\mathfrak{s}$ is not restricted.

The minimal 2-envelope of $\mathfrak{s}$ is the Lie algebra $\mathfrak{m}$ with basis $\{b', b, c, a, a'\}$, bracket (9) and
\[
\begin{align*}
[a', b] &= a, & [a', b'] &= c, & [a, b'] &= b, & [a', a] &= [a', c] &= [b', b] &= [b', c] &= 0;
\end{align*}
\]
and 2-operation $(\ )^{[2]} : \mathfrak{m} \to \mathfrak{m}$ given by
\[
\begin{align*}
(a')^{[2]} &= (b')^{[2]} = 0, & c^{[2]} &= c, & a^{[2]} &= a', & b^{[2]} &= b'.
\end{align*}
\]
The restricted enveloping algebra $u(m)$ is isomorphic to $\mathbb{k}\langle a, b, c \rangle/I$ where $I$ is generated by the relations

$$ab + ba = c, \ ac + ca = a, \ bc + cb = b, \ a^4 = b^4 = 0, \ c^2 + c = 0.$$ 

That is,

$$u(m) \simeq U(s)/\langle a^4, b^4, c^2 + c \rangle.$$
**Proposition.** (A.–Bagio–Della Flora–Flöres) The algebra $D(H)$ is generated by $x, y, g, u, v, \gamma$ with defining relations explicitly described.

**Proposition.** (A.–Bagio–Della Flora–Flöres) There is a normal local commutative Hopf subalgebra $T$ of $D(H)$ such that $D(H)$ fits in an abelian extension

$$T \overset{\iota}{\hookrightarrow} D(H) \overset{\pi}{\twoheadrightarrow} u(m),$$

Since $T$ is normal and local, we conclude

**Theorem.** (A.–Bagio–Della Flora–Flöres) Irrep $D(H) \simeq$ Irrep $u(m)$.

Even more the functor $\pi^* : u(m)\mathcal{M} \to D(H)\mathcal{M}$ is tensor and sends indecomposables to indecomposables.
References.


