Admissible tame representations of vertex algebras

Oscar Armando Hernández Morales.

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Let $\mathfrak{g}$ be a simple Lie algebra and $\kappa$ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ and $\widehat{\mathfrak{g}}_\kappa$ the affine Kac–Moody algebra of level $\kappa$ associated to $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$, we define its integral root system $\widehat{\Delta}(\lambda)$ by

$$\widehat{\Delta}(\lambda) = \{ \alpha \in \widehat{\Delta}_{\text{re}}; \langle \lambda + \hat{\rho}, \alpha^\vee \rangle \in \mathbb{Z} \},$$

where $\hat{\rho} = \rho + h^\vee \Lambda_0$. Further, let $\widehat{\Delta}(\lambda)_+ = \widehat{\Delta}(\lambda) \cap \widehat{\Delta}_{\text{re}}^+$ be the set of positive roots of $\widehat{\Delta}(\lambda)$ and $\widehat{\Pi}(\lambda) \subset \widehat{\Delta}(\lambda)_+$ be the set of simple roots. Then we say that a weight $\lambda \in \mathfrak{h}^*$ is admissible (Kac-Wakimoto, 1989) provided

i) $\lambda$ is regular dominant, that is $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle \notin -\mathbb{N}_0$ for all $\alpha \in \widehat{\Delta}_{\text{re}}^+$;

ii) the $\mathbb{Q}$-span of $\widehat{\Delta}(\lambda)$ contains $\widehat{\Delta}_{\text{re}}$.
For a \( g \)-module \( E \), let us consider the induced \( \hat{g}_\kappa \)-module

\[
\mathcal{M}_{\kappa, g}(E) = U(\hat{g}_\kappa) U(g[[t]] \oplus \mathbb{C} c) E,
\]

where \( E \) is considered as the \( g[[t]] \oplus \mathbb{C} c \)-module on which \( g \otimes \mathbb{C} t \mathbb{C}[[t]] \) acts trivially and \( c \) acts as the identity. Since \( \mathcal{M}_{\kappa, g}(E) \) has a unique maximal \( \hat{g}_\kappa \)-submodule having zero intersection with \( E \), we denote by \( \mathbb{L}_{\kappa, g}(E) \) the corresponding quotient.

We say that a \( g \)-module \( E \) is admissible of level \( k \) if \( \mathbb{L}_{\kappa, g}(E) \) is an \( \mathcal{L}_\kappa(g) \)-module, or equivalently if \( E \) is an \( Z_k \)-module, where \( Z_k := \text{Zhu}(\mathcal{L}_\kappa(g)) \) denoted the Zhu’s algebra of simple affine vertex algebra \( \mathcal{L}_\kappa(g) \). As we have \( Z_k \cong U(g)/I_k \), where \( I_k \) is a two-sided ideal of \( U(g) \), we obtain that a \( g \)-module \( E \) is admissible of level \( k \) if and only if the ideal \( I_k \) is contained in the annihilator \( \text{Ann}_{U(g)} E \).
Admissible simple highest weight \( \mathfrak{g} \)-modules of level \( k \) were classified in (Arakawa, 2016) as follows. Let us denote by \( \Pr_k \) the set of admissible weights \( \lambda \in \hat{\mathfrak{h}}^* \) of level \( k \). Besides, let us introduce the subset

\[
\overline{\Pr}_k = \{ \lambda \in \mathfrak{h}^*; \lambda + k\Lambda_0 \in \Pr_k \}
\]

of \( \mathfrak{h}^* \), which is the canonical projection of \( \Pr_k \) to \( \mathfrak{h}^* \). Then we have the following statement.

**Theorem (Arakawa, 2016)**

Let \( k \in \mathbb{Q} \) be an admissible number for \( \mathfrak{g} \). Then the simple highest weight \( \mathfrak{g} \)-module \( L^g_b(\lambda) \) with highest weight \( \lambda \in \mathfrak{h}^* \) is admissible of level \( k \) if and only if \( \lambda \in \overline{\Pr}_k \).
Classification of simple weight modules

Theorem (Fernando, 1990)

Every simple weight $\mathfrak{g}$-module with finite-dimensional weight spaces is isomorphic to induced $\mathfrak{g}$-module, for some parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with Levi factor $\mathfrak{l}$ of AC-type, and some simple dense $\mathfrak{l}$-module $\mathcal{N}$.

Theorem (Kawasetsu and Ridout, 2021)

A simple weight $\mathfrak{g}$-module $\mathcal{M}$, with finite-dimensional weight spaces, is a $Z_k$-module if and only if either of the following statements hold:

1. $\mathcal{M}$ is a highest-weight $Z_k$-module, with respect to some Borel subalgebra of $\mathfrak{g}$.

2. There is a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with non-abelian Levi factor $\mathfrak{l}$ of AC-type, and a corresponding irreducible semisimple parabolic family $\mathcal{P}$ of $\mathfrak{g}$-modules such that $\mathcal{M}$ is isomorphic to a submodule of $\mathcal{P}$ and some submodule of $\mathcal{P}$ is an $\mathfrak{l}$-bounded highest-weight $Z_k$-module.
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Standard Flag

Let us consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ with a triangular decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h} \oplus \mathfrak{n}$ and with the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that the corresponding Cartan matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is given by $a_{ii} = 2$, $a_{ij} = -1$ if $|i - j| = 1$ and $a_{ij} = 0$ if $|i - j| \geq 2$. Further, let us denote by $\mathfrak{g}_k$ for $k = 1, 2, \ldots, n$ the Lie subalgebra of $\mathfrak{g}$ generated by the root subspaces $\mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_k}$ and $\mathfrak{g}_{-\alpha_1}, \ldots, \mathfrak{g}_{-\alpha_k}$. Then we obtain a finite sequence

$$\mathcal{F} : 0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

of Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}_k \simeq \mathfrak{sl}_{k+1}$ for $k = 1, 2, \ldots, n$. We have also the induced triangular decomposition

$$\mathfrak{g}_k = \mathfrak{p}_k \oplus \mathfrak{h}_k \oplus \mathfrak{n}_k$$

of the Lie algebra $\mathfrak{g}_k$ for $k = 1, 2, \ldots, n$, where $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{g}_k$, $\mathfrak{n}_k = \mathfrak{n} \cap \mathfrak{g}_k$ and $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{g}_k$. Besides, we have a sequence $U(\mathfrak{g}_1) \subset U(\mathfrak{g}_2) \subset \cdots \subset U(\mathfrak{g}_n)$ of $\mathbb{C}$-subalgebras of the universal enveloping algebra $U(\mathfrak{g})$. 
Let us denote by $\mathfrak{z}_{g_k}$ the center of $U(g_k)$ for $k = 1, 2, \ldots, n$. Then the Gelfand–Tsetlin subalgebra $\Gamma$ of $U(g)$ to respect $\mathcal{F}$ is generated by $\mathfrak{z}_{g_k}$ for $k = 1, 2, \ldots, n$ and by the Cartan subalgebra $\mathfrak{h}$. It is known that $\Gamma$ is a maximal commutative $\mathbb{C}$-subalgebra of $U(g)$ (Drozd-Futorny-Ovsienko, 1994).

**Definition**

A finitely generated $U(g)$-module $M$ is called a $\Gamma$-Gelfand–Tsetlin module if $M$ splits into a direct sum of $\Gamma$-modules:

$$M = \bigoplus_{m \in \text{Specm} \Gamma} M(m),$$

where

$$M(m) = \{ v \in M \mid m^k v = 0 \text{ for some } k \geq 0 \}.$$
Tame $\Gamma$-Gelfand–Tsetlin module

**Definition**
We say that a $\Gamma$-Gelfand–Tsetlin $\mathfrak{g}$-module $M$ is *tame* if $\Gamma$ has a simple spectrum on $M$, i.e. all $\Gamma$-multiplicities are equal to 1.

**Remark**
If $M$ is tame $\Gamma$-Gelfand–Tsetlin $\mathfrak{g}$-module then $\Gamma$-weights of $M$ parameterize a basis of $M$.

Finite-dimensional $\mathfrak{g}$-modules are examples of tame $\Gamma$-Gelfand–Tsetlin $\mathfrak{g}$-modules. For infinite-dimensional $\mathfrak{g}$-modules the situation is very more complicated, for example the Verma $\Gamma$-Gelfand–Tsetlin $\mathfrak{g}$-module $M(-\rho)$ is not tame (Futorny-Grantcharov-Ramirez-Zadunaisky, 2020). On the other hand, *relation* modules are tame $\Gamma$-Gelfand–Tsetlin $\mathfrak{g}$-modules (Futorny-Ramirez-Zhang, 2019).
Strongly tame $\Gamma$-Gelfand–Tsetlin module

Relation $\Gamma$-Gelfand–Tsetlin $g$-modules can be characterized as those $g$-modules having an eigenbasis for $\Gamma$ formed by a set of tableaux $T(\nu)$ for $\nu \in \mathbb{C}^{(n+1)(n+2)/2}$, where $T(\nu)$ is the tableau

$$
\begin{array}{cccc}
\nu_{n+1,1} & \nu_{n+1,2} & \ldots & \nu_{n+1,n-1} & \nu_{n+1,n+1} \\
\nu_{n,1} & \ldots & & \nu_{n,n} \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

where $\nu_{i,j} \in \mathbb{C}$ for all $1 \leq j \leq i \leq n$. 

The classical Gelfand-Tsetlin action

and admitting the action of the Chevalley generators in the form

\[ e_k(T(v)) = - \sum_{i=1}^{k} \left( \prod_{j=1}^{k+1} \frac{(v_{k,i} - v_{k+1,j})}{\prod_{j \neq i}^{k} (v_{k,i} - v_{k,j})} \right) T(v + \delta^{k,i}), \]

\[ f_k(T(v)) = \sum_{i=1}^{k} \left( \prod_{j=1}^{k-1} \frac{(v_{k,i} - v_{k-1,j})}{\prod_{j \neq i}^{k} (v_{k,i} - v_{k,j})} \right) T(v - \delta^{k,i}), \tag{1} \]

\[ h_k(T(v)) = \left( 2 \sum_{i=1}^{k} v_{k,i} - \sum_{i=1}^{k-1} v_{k-1,i} - \sum_{i=1}^{k+1} v_{k+1,i} - 1 \right) T(v), \]

where \( \delta^{k,i} \) is the vector having 1 at the position \((k, i)\) and 0 elsewhere, for \( k = 1, 2, \ldots, n \). If a denominator equals zero, then the summand is assumed to be zero. For such a relation \( \mathfrak{g} \)-module, we will use the name strongly tame \( \Gamma \)-Gelfand–Tsetlin \( \mathfrak{g} \)-module.
The following assertion classifies all simple highest weight strongly tame \( \Gamma \)-Gelfand–Tsetlin \( g \)-module, where \( b \) is the standard Borel subalgebra of \( g \).

**Theorem (Futorny, H. M. and Ramirez, 2021)**

The simple highest weight module \( L^g_b(\lambda) \) is a strongly tame \( \Gamma \)-Gelfand–Tsetlin \( g \)-module if and only if one of the following conditions holds:

1. \( \langle \lambda + \rho, \alpha \vee \rangle \notin \mathbb{Z}_{\leq 0}, \) for all \( \alpha \in \Delta_+ \setminus \{\alpha_{k,n} | k = 1, \ldots, n\} \).
2. There exist unique \( i, j \) with \( 1 \leq i \leq j < n \) such that:
   - \( \langle \lambda + \rho, \alpha^\vee_k \rangle \in \mathbb{Z}_{>0} \) for each \( k > j \),
   - \( \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0} \) for all \( \alpha \in \Delta_+ \setminus \{\alpha_{i,k} | k \geq j\} \),
   - \( \langle \lambda + \rho, \alpha^\vee_{i,n} \rangle \in \mathbb{Z}_{\leq 0} \).

where \( \alpha_{r,s} := \alpha_r + \ldots + \alpha_s \) for \( 1 \leq r \leq s \leq n \).
Corollary (Futorny, H. M. and Křižka, 2021)

If $\lambda \in \mathfrak{h}^*$ dominant integral or dominant regular, then $L_b^g(\lambda)$ is a strongly tame $\Gamma$-Gelfand–Tsetlin $g$-module.

Let $k$ be admissible, i.e., $k + n + 1 = \frac{p}{q}$ with $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$, $p > n$ and $\overline{Pr}_k$ the set of admissible weight of $g$ (Arakawa, 2015). Given that $\lambda \in \overline{Pr}_k$ implies that $\lambda$ is dominant regular, we have that

Corollary (Futorny, H. M. and Křižka, 2021)

For $\lambda \in \overline{Pr}_k$, the simple $g$-module $L_b^g(\lambda)$ is a strongly tame $\Gamma$-Gelfand–Tsetlin $g$-module.
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All simple finite-dimensional admissible $g$-module of level $k$ is a strongly tame Gelfand-Tsetlin module in the zero nilpotent orbit.

For $n = 1$. We have:

- All simple $g$-module is bounded.
- All simple $g$-module is strongly tame Gelfand-Tsetlin bounded module.
- All simple infinite-dimensional admissible $g$-module of level $k$ is in the minimal nilpotent orbit. In this case, minimal orbit is the principal orbit.

For now suppose that $n > 1$ and $M$ is a simple infinite-dimensional bounded $g$-modules.
Proposition (Futorny, H. M. and Křížka, 2022)

Let $M$ be a simple infinite-dimensional bounded $\mathfrak{g}$-module. $M$ is admissible if and only $M \cong D^x_F L(\lambda)$ for some admissible highest weight bounded $\mathfrak{g}$-module $L(\lambda)$.

Theorem (Futorny, H. M. and Křížka, 2022)

If $M$ is simple dense admissible bounded $\mathfrak{g}$-module then $M \cong D^x_F L(\lambda)$ such that $L(\lambda)$ and $D_F L(\lambda)$ are $\Gamma_{st}$-Gelfand–Tsetlin strongly tame $\mathfrak{g}$-module.

The question now is when $M$ is a $\Gamma_{st}$-Gelfand–Tsetlin strongly tame $\mathfrak{g}$-module?
Theorem (Futorny, H. M. and Křižka, 2022)

Let $L(\lambda)$ be an admissible simple highest weight bounded $\mathfrak{g}$-module, such that $\langle \lambda, \alpha_1^\vee \rangle \not\in \mathbb{Z}$, $F = \{ f_{\alpha_{1j}} \mid j = 1, 2, \ldots, n \}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for some set of complex numbers $\{ x_i \mid i = 1, 2, \ldots, n \}$, then

- $D^\mathbf{x}_F L(\lambda)$ is strongly tame $\Gamma_{\text{st}}$-Gelfand–Tsetlin $\mathfrak{g}$-module if and only if
  \[ \sum_{j=i}^{n} x_j + \langle \lambda, \alpha_1^\vee \rangle \not\in \mathbb{Z}, \text{ for all } i = 2, \ldots, n; \]

- The $\mathfrak{g}$-module $D^\mathbf{x}_F L(\lambda)$ is simple strongly tame $\Gamma_{\text{st}}$-Gelfand–Tsetlin $\mathfrak{g}$-module if and only if $x_i \not\in \mathbb{Z}$ for all $i = 1, 2, \ldots, n$ and
  \[ \sum_{j=i}^{n} x_j + \langle \lambda, \alpha_1^\vee \rangle \not\in \mathbb{Z}, \text{ for all } i = 1, 2, \ldots, n. \]
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