Admissible tame representations of vertex algebras

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Admissible weight

Let $\mathfrak{g}$ be a simple Lie algebra and κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ and $\widehat{\mathfrak{g}}_\kappa$ the affine Kac–Moody algebra of level κ associated to $\mathfrak{g}$. For $\lambda \in \widehat{\mathfrak{h}}^*$, we define its integral root system $\widehat{\Delta}(\lambda)$ by

$$\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{\mathrm{re}};\ \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\},$$

where $\widehat{\rho} = \rho + h^\vee \Lambda_0$. Further, let $\widehat{\Delta}(\lambda)_+ = \widehat{\Delta}(\lambda) \cap \widehat{\Delta}^{\mathrm{re}}_+$ be the set of positive roots of $\widehat{\Delta}(\lambda)$ and $\widehat{\Pi}(\lambda) \subset \widehat{\Delta}(\lambda)_+$ be the set of simple roots. Then we say that a weight $\lambda \in \widehat{\mathfrak{h}}^*$ is *admissible* (Kac-Wakimoto, 1989) provided

i) λ is *regular dominant*, that is $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin -\mathbb{N}_0$ for all $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$;

ii) the $\mathbb{Q}$-span of $\widehat{\Delta}(\lambda)$ contains $\widehat{\Delta}^{\mathrm{re}}$.

Admissible module

For a $\mathfrak{g}$-module E, let us consider the induced $\widehat{\mathfrak{g}}_\kappa$-module

$$\mathbb{M}_{\kappa,\mathfrak{g}}(E) = U(\widehat{\mathfrak{g}}_\kappa)_{U(\mathfrak{g}[[t]]\oplus\mathbb{C}c)}E,$$

where E is considered as the $\mathfrak{g}[[t]] \oplus \mathbb{C}c$-module on which $\mathfrak{g} \otimes_{\mathbb{C}} t\mathbb{C}[[t]]$ acts trivially and c acts as the identity. Since $\mathbb{M}_{\kappa,\mathfrak{g}}(E)$ has a unique maximal $\widehat{\mathfrak{g}}_\kappa$-submodule having zero intersection with E, we denote by $\mathbb{L}_{\kappa,\mathfrak{g}}(E)$ the corresponding quotient.

We say that a $\mathfrak{g}$-module E is *admissible of level* k if $\mathbb{L}_{\kappa,\mathfrak{g}}(E)$ is an $\mathcal{L}_\kappa(\mathfrak{g})$-module, or equivalently if E is an Z_k-module, where $Z_k := \mathrm{Zhu}(\mathcal{L}_\kappa(\mathfrak{g}))$ denoted the *Zhu's algebra* of *simple affine vertex algebra* $\mathcal{L}_\kappa(\mathfrak{g})$. As we have $Z_k \simeq U(\mathfrak{g})/I_k$, where I_k is a two-sided ideal of $U(\mathfrak{g})$, we obtain that a $\mathfrak{g}$-module E is admissible of level k if and only if the ideal I_k is contained in the annihilator $\mathrm{Ann}_{U(\mathfrak{g})}E$.

Arakawa's classification

Admissible simple highest weight $\mathfrak{g}$-modules of level k were classified in (Arakawa, 2016) as follows. Let us denote by Pr_k the set of admissible weights $\lambda \in \widehat{\mathfrak{h}}^*$ of level k. Besides, let us introduce the subset

$$\overline{\mathrm{Pr}}_k = \{\lambda \in \mathfrak{h}^*; \, \lambda + k\Lambda_0 \in \mathrm{Pr}_k\}$$

of $\mathfrak{h}^*$, which is the canonical projection of Pr_k to $\mathfrak{h}^*$. Then we have the following statement.

Theorem (Arakawa, 2016)

Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$. Then the simple highest weight $\mathfrak{g}$-module $L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$ is admissible of level k if and only if $\lambda \in \overline{\mathrm{Pr}}_k$.

Classification of simple weight modules

Theorem (Fernando, 1990)

Every simple weight $\mathfrak{g}$-module with finite-dimensional weight spaces is isomorphic to induced $\mathfrak{g}$-module, for some parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with Levi factor $\mathfrak{l}$ of AC-type, and some simple dense $\mathfrak{l}$-module $\mathcal{N}$.

Theorem (Kawasetsu and Ridout, 2021)

A simple weight $\mathfrak{g}$-module $\mathcal{M}$, with finite-dimensional weight spaces, is a Z_k-module if and only if either of the following statements hold:

- *$\mathcal{M}$ is a highest-weight Z_k-module, with respect to some Borel subalgebra of $\mathfrak{g}$.*
- *There is a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with non-abelian Levi factor $\mathfrak{l}$ of AC-type, and a corresponding irreducible semisimple parabolic family $\mathcal{P}$ of $\mathfrak{g}$-modules such that $\mathcal{M}$ is isomorphic to a submodule of $\mathcal{P}$ and some submodule of $\mathcal{P}$ is an $\mathfrak{l}$-bounded highest-weight Z_k-module.*

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Standard Flag

Let us consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ with a triangular decomposition $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ and with the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that the corresponding Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is given by $a_{ii} = 2$, $a_{ij} = -1$ if $|i - j| = 1$ and $a_{ij} = 0$ if $|i - j| \geq 2$. Further, let us denote by $\mathfrak{g}_k$ for $k = 1, 2, \ldots, n$ the Lie subalgebra of $\mathfrak{g}$ generated by the root subspaces $\mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_k}$ and $\mathfrak{g}_{-\alpha_1}, \ldots, \mathfrak{g}_{-\alpha_k}$. Then we obtain a finite sequence

$$\mathcal{F} : 0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

of Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}_k \simeq \mathfrak{sl}_{k+1}$ for $k = 1, 2, \ldots, n$. We have also the induced triangular decomposition

$$\mathfrak{g}_k = \bar{\mathfrak{n}}_k \oplus \mathfrak{h}_k \oplus \mathfrak{n}_k$$

of the Lie algebra $\mathfrak{g}_k$ for $k = 1, 2, \ldots, n$, where $\bar{\mathfrak{n}}_k = \bar{\mathfrak{n}} \cap \mathfrak{g}_k$, $\mathfrak{n}_k = \mathfrak{n} \cap \mathfrak{g}_k$ and $\mathfrak{h}_k = \mathfrak{h} \cap \mathfrak{g}_k$. Besides, we have a sequence $U(\mathfrak{g}_1) \subset U(\mathfrak{g}_2) \subset \cdots \subset U(\mathfrak{g}_n)$ of $\mathbb{C}$-subalgebras of the universal enveloping algebra $U(\mathfrak{g})$.

Γ-Gelfand–Tsetlin module

Let us denote by $\mathfrak{z}_{\mathfrak{g}_k}$ the center of $U(\mathfrak{g}_k)$ for $k = 1, 2, \ldots, n$. Then the Gelfand–Tsetlin subalgebra Γ of $U(\mathfrak{g})$ to respect $\mathcal{F}$ is generated by $\mathfrak{z}_{\mathfrak{g}_k}$ for $k = 1, 2, \ldots, n$ and by the Cartan subalgebra $\mathfrak{h}$. It is known that Γ is a maximal commutative $\mathbb{C}$-subalgebra of $U(\mathfrak{g})$ (Drozd-Futorny-Ovsienko, 1994).

Definition

A finitely generated $U(\mathfrak{g})$-module M is called a Γ-*Gelfand–Tsetlin module* if M splits into a direct sum of Γ-modules:

$$M = \bigoplus_{\mathrm{m} \in \operatorname{Specm} \Gamma} M(\mathrm{m}),$$

where

$$M(\mathrm{m}) = \{v \in M \mid \mathrm{m}^k v = 0 \text{ for some } k \geq 0\}.$$

Tame Γ-Gelfand–Tsetlin module

Definition

We say that a Γ-Gelfand–Tsetlin $\mathfrak{g}$-module M is *tame* if Γ has a simple spectrum on M, i.e. all Γ-multiplicities are equal to 1.

Remark

If M is tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-module then Γ-weights of M parameterize a basis of M.

Finite-dimensional $\mathfrak{g}$-modules are examples of tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-modules. For infinite-dimensional $\mathfrak{g}$-modules the situation is very more complicated, for example the Verma Γ-Gelfand–Tsetlin $\mathfrak{g}$-module $M(-\rho)$ is not tame (Futorny-Grantcharov-Ramirez-Zadunaisky, 2020). On the other hand, *relation* modules are tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-modules (Futorny-Ramirez-Zhang, 2019).

Strongly tame Γ-Gelfand–Tsetlin module

Relation Γ-Gelfand–Tsetlin $\mathfrak{g}$-modules can be characterized as those $\mathfrak{g}$-modules having an eigenbasis for Γ formed by a set of tableaux $T(v)$ for $v \in \mathbb{C}^{\frac{(n+1)(n+2)}{2}}$, where $T(v)$ is the tableau

$$
\begin{array}{ccccccc}
\boxed{v_{n+1,1}} & \boxed{v_{n+1,2}} & & \cdots & & \boxed{v_{n+1,n-1}} & \boxed{v_{n+1,n+1}} \\
& \boxed{v_{n,1}} & & \cdots & & \boxed{v_{n,n}} & \\
& & \cdots & \cdots & \cdots & & \\
& & \boxed{v_{3,1}} & \boxed{v_{3,2}} & \boxed{v_{3,3}} & & \\
& & & \boxed{v_{2,1}} \; \boxed{v_{2,2}} & & & \\
& & & \boxed{v_{1,1}} & & &
\end{array}
$$

where $v_{i,j} \in \mathbb{C}$ for all $1 \leq j \leq i \leq n$.

The classical Gelfand-Tsetlin action

and admiting the action of the Chevalley generators in the form

$$\begin{aligned} e_k(T(v)) &= -\sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k+1}(v_{k,i}-v_{k+1,j})}{\prod_{j\neq i}^{k}(v_{k,i}-v_{k,j})}\right) T(v+\delta^{k,i}), \\ f_k(T(v)) &= \sum_{i=1}^{k}\left(\frac{\prod_{j=1}^{k-1}(v_{k,i}-v_{k-1,j})}{\prod_{j\neq i}^{k}(v_{k,i}-v_{k,j})}\right) T(v-\delta^{k,i}), \\ h_k(T(v)) &= \left(2\sum_{i=1}^{k} v_{k,i} - \sum_{i=1}^{k-1} v_{k-1,i} - \sum_{i=1}^{k+1} v_{k+1,i} - 1\right) T(v), \end{aligned} \tag{1}$$

where $\delta^{k,i}$ is the vector having 1 at the position (k,i) and 0 elsewhere, for $k = 1, 2, \dots, n$. If a denominator equals zero, then the summand is assumed to be zero. For such a relation $\mathfrak{g}$-module, we will use the name *strongly tame* Γ-Gelfand–Tsetlin $\mathfrak{g}$-module.

Classification of highest weight strongly tame modules

The following assertion classifies all simple highest weight strongly tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-module, where $\mathfrak{b}$ is the standard Borel subalgebra of $\mathfrak{g}$.

Theorem (Futorny, H. M. and Ramirez, 2021)

The simple highest weight module $L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ is a strongly tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-module if and only if one of the following conditions holds:

- a) $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$, *for all* $\alpha \in \Delta_+ \setminus \{\alpha_{k,n} \mid k = 1, \dots, n\}$.
- b) *There exist unique* i, j *with* $1 \leq i \leq j < n$ *such that:*
  - i) $\langle \lambda + \rho, \alpha_k^\vee \rangle \in \mathbb{Z}_{>0}$ *for each* $k > j$,
  - ii) $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ *for all* $\alpha \in \Delta_+ \setminus \{\alpha_{i,k} \mid k \geq j\}$,
  - iii) $\langle \lambda + \rho, \alpha_{i,n}^\vee \rangle \in \mathbb{Z}_{\leq 0}$.

where $\alpha_{r,s} := \alpha_r + \ldots + \alpha_s$ *for* $1 \leq r \leq s \leq n$.

Realization of admissible highest weight modules

Corollary (Futorny, H. M. and Křižka, 2021)

If $\lambda \in \mathfrak{h}^*$ dominant integral or dominant regular, then $L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ is a strongly tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-module.

Let k be admissible, i.e., $k + n + 1 = \frac{p}{q}$ with $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$, $p > n$ and $\overline{\mathrm{Pr}}_k$ the set of admissible weight of $\mathfrak{g}$ (Arakawa, 2015). Given that $\lambda \in \overline{\mathrm{Pr}}_k$ implies that λ is dominant regular, we have that

Corollary (Futorny, H. M. and Křižka, 2021)

For $\lambda \in \overline{\mathrm{Pr}}_k$, the simple $\mathfrak{g}$-module $L^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$ is a strongly tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-module.

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Remark

All simple finite-dimensional admissible $\mathfrak{g}$-module of level k is a strongly tame Gelfand-Tsetlin module in the zero nilpotent orbit.

Remark

For $n = 1$. We have:

- All simple $\mathfrak{g}$-module is bounded.
- All simple $\mathfrak{g}$-module is strongly tame Gelfand-Tsetlin bounded module.
- All simple infinite-dimensional admissible $\mathfrak{g}$-module of level k is in the minimal nilpotent orbit. In this case, miminal orbit is the principal orbit.

For now suppose that $n > 1$ and M is a simple infinite-dimensional bounded $\mathfrak{g}$-modules.

Classification of bounded admissible modules

Proposition (Futorny, H. M. and Křižka, 2022)

Let M be a simple infinite-dimensional bounded $\mathfrak{g}$-module. M is admissible if and only $M \simeq D_F^{\mathbf{x}} L(\lambda)$ for some admissible highest weight bounded $\mathfrak{g}$-module $L(\lambda)$.

Theorem (Futorny, H. M. and Křižka, 2022)

If M is simple dense admissible bounded $\mathfrak{g}$-module then $M \simeq D_F^{\mathbf{x}} L(\lambda)$ such that $L(\lambda)$ and $D_F L(\lambda)$ are Γ_{st}-Gelfand–Tsetlin strongly tame $\mathfrak{g}$-module.

The question now is when M is a Γ_{st}-Gelfand–Tsetlin strongly tame $\mathfrak{g}$-module?

Cuspidal admissible strongly tame modules

Theorem (Futorny, H. M. and Křižka, 2022)

Let $L(\lambda)$ be an admissible simple highest weight bounded $\mathfrak{g}$-module, such that $\langle \lambda, \alpha_1^\vee \rangle \notin \mathbb{Z}$, $F = \{f_{\alpha_{1j}} \mid j = 1, 2, \ldots, n\}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for some set of complex numbers $\{x_i \mid i = 1, 2, \ldots, n\}$, then

- (a) *$D_F^{\mathbf{x}} L(\lambda)$ is strongly tame Γ_{st}-Gelfand–Tsetlin $\mathfrak{g}$-module if and only if*
$$\sum_{j=i}^{n} x_j + \langle \lambda, \alpha_1^\vee \rangle \notin \mathbb{Z}, \text{ for all } i = 2, \ldots, n;$$
- (b) *The $\mathfrak{g}$-module $D_F^{\mathbf{x}} L(\lambda)$ is simple strongly tame Γ_{st}-Gelfand–Tsetlin $\mathfrak{g}$-module if and only if $x_i \notin \mathbb{Z}$ for all $i = 1, 2, \ldots, n$ and*
$$\sum_{j=i}^{n} x_j + \langle \lambda, \alpha_1^\vee \rangle \notin \mathbb{Z}, \text{ for all } i = 1, 2, \ldots, n.$$

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References

- V. Futorny, O. A. Hernández Morales and L. Križka. Admissible representations of simple affine vertex algebras. arXiv:2107.11128, 2021
- V. Futorny, O. A. Hernández Morales and L. Križka. Admissible Bounded Modules. In progress.
- V. Futorny, O. A. Hernández Morales and L. E. Ramirez. Simple modules for affine vertex algebras in the minimal nilpotent orbit. IMRN, 2021.
- V. Futorny, L. E. Ramirez, and J. Zhang, *Combinatorial construction of Gelfand–Tsetlin modules for* $\mathfrak{gl}_n$, Adv. Math., 343 (2019), 681–711.
- K. Kawasetsu and D. Ridout, *Relaxed highest weight modules II: Classification for affine vertex algebras*, Commun. Contemp. Math. **0** (2021), 1–43.