Categorification of some integer sequences and Brauer configuration algebras

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Ukrainian People

Figure: Dedicated to the Ukrainian people

Road Map

Motivation

Motivation

- P. Fahr and C.M. Ringel: *"A Partition Formula for Fibonacci Numbers "*, 2008.

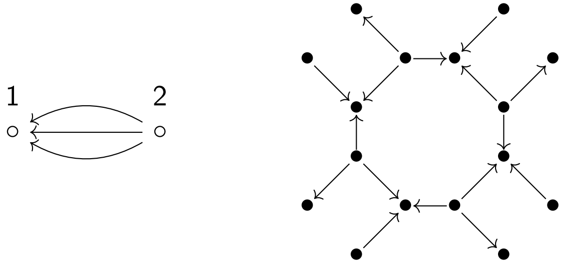


Figure: The 3-Kronecker quiver and an illustration of its corresponding universal covering.

- New sequence in the OEIS A132262.
- P. Fahr and C.M. Ringel: *Categorification of the Fibonacci Numbers Using Representations of Quivers*, 2012.

What is a categorification of a sequence?

According to Ringel and Fahr a categorification of a set of numbers means to consider instead of these numbers suitable objects in a category (here representations of quivers), so that the numbers in question occur as invariants of the objects.

Equality of numbers may be visualized by isomorphisms of objects, functional relations by functorial ties.

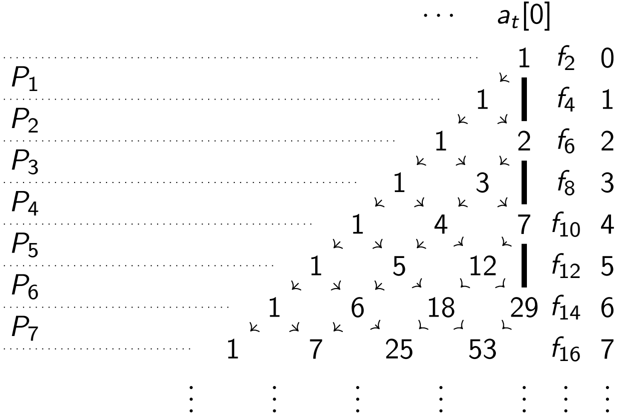
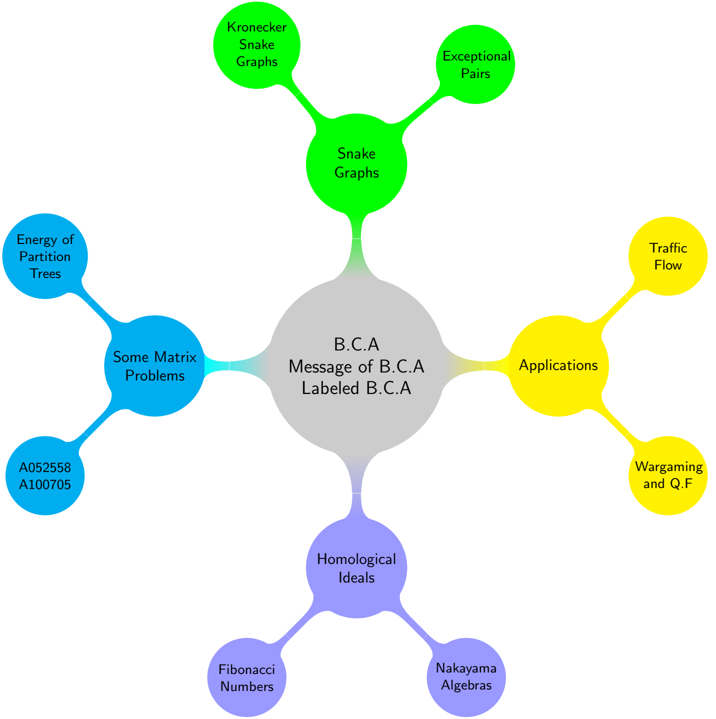


Figure: The even-index Fibonacci partition triangle.

For example for $t = 3$ and $t = 4$, we compute f_8 and f_{10} as follows;

$$21 = f_8 = 0 + 3(3 \cdot 2^0) + 0 + 1(3 \cdot 2^2),$$

$$55 = f_{10} = 1 \cdot 7 + 0 + 4(3 \cdot 2^1) + 0 + 1(3 \cdot 2^3).$$


B.C.A
Message of B.C.A
Labeled B.C.A
Snake Graphs
Kronecker Snake Graphs
Exceptional Pairs
Applications
Traffic Flow
Wargaming and Q.F
Homological Ideals
Fibonacci Numbers
Nakayama Algebras
Some Matrix Problems
Energy of Partition Trees
A052558
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Aims and Scope of this Talk

In this talk, homological ideals associated with some Nakayama algebras are characterized and enumerated via integer specializations of some suitable Brauer configuration algebras. Besides, it is shown how the number of such homological ideals can be connected with the categorification process of Fibonacci numbers defined by Ringel and Fahr.

Brauer Configuration Algebras

Brauer Configuration

These algebras were introduced by Green and Schroll as a way to deal with the research of algebras of wild representation type in 2017.

A *Brauer configuration* Γ is a quadruple of the form $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ where:

1. Γ_0 is a finite set whose elements are called *vertices*,
2. Γ_1 is a finite collection of multisets called *polygons*. In this case, if $V \in \Gamma_1$ then the elements of V are vertices possibly with repetitions, $\mathrm{occ}(\alpha, V)$ denotes the frequency of the vertex α in the polygon V and the *valency* of α denoted *val*(α) is defined in such a way that:

$$\mathit{val}(\alpha) = \sum_{V \in \Gamma_1} \mathrm{occ}(\alpha, V). \tag{1}$$

1. μ is an integer valued function such that $\mu : \Gamma_0 \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers, it is called the *multiplicity function*,
2. $\mathcal{O}$ denotes an orientation defined on Γ_1 which is a choice, for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which α occurs as a vertex, including repetitions, we denote S_α such collection of polygons.

   The set $(S_\alpha, \leq)$ is called the *successor sequence* at the vertex α.

1. Every vertex in Γ_0 is a vertex in at least one polygon in Γ_1,
2. Every polygon has at least two vertices,
3. Every polygon in Γ_1 has at least one vertex α such that $\mu(\alpha)\mathit{val}(\alpha) > 1$.

A vertex $\alpha \in \Gamma_0$ is said to be *truncated* if $\mathit{val}(\alpha)\mu(\alpha) = 1$, that is, α is truncated if it occurs exactly once in exactly one $V \in \Gamma_1$ and $\mu(\alpha) = 1$. A vertex is *non-truncated* if it is not truncated.

The Quiver of a Brauer Configuration Algebra

The quiver $Q_\Gamma = ((Q_\Gamma)_0, (Q_\Gamma)_1)$ of a Brauer configuration algebra is defined in such a way that the vertex set $(Q_\Gamma)_0 = \{v_1, v_2, \ldots, v_m\}$ of Q_Γ is in correspondence with the set of polygons $\{V_1, V_2, \ldots, V_m\}$ in Γ_1, noting that there is one vertex in $(Q_\Gamma)_0$ for every polygon in Γ_1.

Arrows in Q_Γ are defined by the successor sequences. That is, there is an arrow $v_i \xrightarrow{s_i} v_{i+1} \in (Q_\Gamma)_1$ provided that $V_i \leq V_{i+1}$ in $(S_\alpha, \leq) \cup \{V_t \leq V_1\}$ for some non-truncated vertex $\alpha \in \Gamma_0$. In other words, for each non-truncated vertex $\alpha \in \Gamma_0$ and each successor V' of V at α, there is an arrow from v to v' in Q_Γ where v and v' are the vertices in Q_Γ associated with the polygons V and V' in Γ_1, respectively.

The Ideal of Relations and Definition of a Brauer Configuration Algebra

Fix a polygon $V \in \Gamma_1$ and suppose that $\mathrm{occ}(\alpha, V) = t \geq 1$ then there are t indices $i_1, \ldots, i_t$ such that $V = V_{i_j}$. Then the *special* α*-cycles* at v are the cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_t}$ where v is the vertex in the quiver of Q_Γ associated with the polygon V. If α occurs only once in V and $\mu(\alpha) = 1$ then there is only one special α-cycle at v.

Let k be a field and Γ a Brauer configuration. The *Brauer configuration algebra associated with* Γ is defined to be the bounded path algebra $\Lambda_\Gamma = kQ_\Gamma/I_\Gamma$, where Q_Γ is the quiver associated with Γ and I_Γ is the *ideal* in kQ_Γ generated by the following set of relations ρ_Γ of type I, II and III.

1. **Relations of type I**. For each polygon $V = \{\alpha_1, \ldots, \alpha_m\} \in \Gamma_1$ and each pair of non-truncated vertices α_i and α_j in V, the set of relations ρ_Γ contains all relations of the form $C^{\mu(\alpha_i)} - C'^{\mu(\alpha_j)}$ where C is a special α_i-cycle and C' is a special α_j-cycle.
2. **Relations of type II**. Relations of type II are all paths of the form $C^{\mu(\alpha)}a$ where C is a special α-cycle and a is the first arrow in C.
3. **Relations of type III**. These relations are quadratic monomial relations of the form ab in kQ_Γ where ab is not a subpath of any special cycle unless $a = b$ and a is a loop associated with a vertex of valency 1 and $\mu(\alpha) > 1$.

Example

As an example, for $n \geq 4$ fixed, we consider a Brauer configuration $\Gamma_n = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ such that:

1. $\Gamma_0 = \{n - k - 1 \in \mathbb{N} \mid 2 \leq k \leq n - 1\} \cup \{n - 2\}$,
2. $\Gamma_1 = \{U_k = \{n - 2, n - k - 1\} \mid 2 \leq k \leq n - 1\}$.
3. The orientation $\mathcal{O}$ is defined in such a way that
   - Vertex $n - 2$ has associated the successor sequence $U_2 < U_3 < \cdots < U_{n-1}$, in this case, $val(n - 2) = n - 2$,
   - If $2 \leq k \leq n - 1$ then at vertex $n - k - 1$, it holds that the corresponding successor sequence consists only of U_k, and for each k, $val(n - k - 1) = 1$.
4. $\mu(n - 2) = 1$,
5. $\mu(n - k - 1) = n - 2, \quad 2 \leq k \leq n - 1$.

Example

The following figure shows the quiver Q_{Γ_n} associated with this configuration.

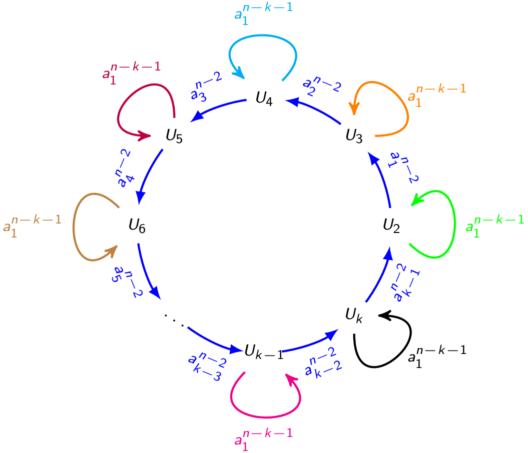

Figure: The quiver Q_{Γ_n} defined by the Brauer configuration Γ_n.

Example

The ideal I_{Γ_n} of the corresponding Brauer configuration algebra Λ_{Γ_n} is generated by the following relations, for which it is assumed the following notation for the special cycles:

$$C^{U_k}_{n-2} = \begin{cases} a_1^{n-2} a_2^{n-2} \cdots a_{k-1}^{n-2}, & \text{if } k = 2, \\ a_{k-1}^{n-2} a_k^{n-2} \cdots a_{k-2}^{n-2}, & \text{otherwise}, \end{cases} \tag{2}$$
$$C^{U_k}_{n-k-1} = a_1^{n-k-1}.$$

1. $a_i^h a_r^s$, if $h \neq s$, for all possible values of i and r unless for the loops associated with the vertices $n-k-1$,
2. $C^{U_k}_{n-2} - \left(C^{U_k}_{n-k-1}\right)^{n-2}$, for all possible values of k,
3. $C^{U_k}_{n-2} a$ with a being the first arrow of $C^{U_k}_{n-2}$ for all k,
4. $\left(C^{U_k}_{n-k-1}\right)^{n-2} a'$ with a' being the first arrow of $C^{U_k}_{n-k-1}$ for all k.

Message of a Brauer Configuration

The concept of the message of a Brauer configuration is helpful to categorify some integer sequences in the sense of Ringel and Fahr.

Let $\Gamma = \{\Gamma_0, \Gamma_1, \mu, \mathcal{O}\}$ be a Brauer configuration and let $U \in \Gamma_1$ be a polygon such that $U = \{\alpha_1^{f_1}, \alpha_2^{f_2}, \ldots, \alpha_n^{f_n}\}$, where $f_i = \mathrm{occ}(\alpha_i, U)$. The term

$$w(U) = \alpha_1^{f_1}\alpha_2^{f_2}\ldots\alpha_n^{f_n} \tag{3}$$

is said to be the *word associated with U*. The sum

$$M(\Gamma) = \sum_{U\in\Gamma_1} w(U) \tag{4}$$

is said to be the *message of the Brauer configuration* Γ.

Integer specialization

An *integer specialization* of a Brauer configuration Γ is a Brauer configuration $\Gamma^e = (\Gamma_0^e, \Gamma_1^e, \mu^e, \mathcal{O}^e)$ endowed with a preserving orientation map $e : \Gamma_0 \to \mathbb{N}$, such that

$$\begin{aligned} \Gamma_0^e &= \mathrm{Img}\; e \subset \mathbb{N}, \\ \Gamma_1^e &= e(\Gamma_1), \quad \text{if } H \in \Gamma_1 \text{ then } e(H) = \{e(\alpha_i) \mid \alpha_i \in H\} \in e(\Gamma_1), \\ \mu^e(e(\alpha)) &= \mu(\alpha), \text{ for any } \alpha \in \Gamma_0. \end{aligned} \tag{5}$$

Besides $e(U) \preceq e(V)$ in Γ_1^e provided that $U \preceq V$ in Γ_1.

We let $w^e(U) = (e(\alpha_1))^{f_1}(e(\alpha_2))^{f_2}\ldots(e(\alpha_n))^{f_n}$ denote the specialization under e of a word $w(U)$. In such a case, $M(\Gamma^e) = \sum\limits_{U \in \Gamma_1^e} w^e(U)$ is the *specialized message* of the Brauer configuration Γ with the usual integer sum and product (in general with the sum and product associated with $\mathrm{Img}\; e$).

Example

For the Brauer configuration Γ_n in the Example we define the specialization $e(\alpha) = 2^{\alpha}$, $\alpha \in \Gamma_0$ with the concatenation in each word given by the difference of the specializations of the vertices belonging to a determined polygon, in such a case for n fixed, we have:

$$\begin{aligned} w(U_k) &= (n-2)(n-k-1), \ \textit{for } 2 \leq k \leq n-1, \\ w^e(U_k) &= 2^{n-2} - 2^{n-k-1}, \ \textit{for } 2 \leq k \leq n-1, \\ M(\Gamma_n^e) &= \sum_{U_k \in \Gamma_1} w^e(U_k) = \sum_{k=1}^{n-1} 2^{n-2} - 2^{n-k-1}. \end{aligned} \tag{6}$$

Homological Ideals

An epimorphism of algebras $\phi : A \to B$ is called *homological epimorphism* if it induces a full and faithful functor

$$D^b(\phi^*) : D^b(B) \to D^b(A).$$

Let I be a two sided ideal of A. Since the quotient map $\pi : A \to A/I$ is an epimorphism then the induced functor $\pi^* : mod(A/I) \to mod(A)$ is full and faithful.

A two sided ideal I of A is *homological* if the quotient map $\pi : A \to A/I$ is an homological epimorphism.

The following results characterize homological ideals:

Proposition

Let I be an ideal of A, then

1. *I is an homological ideal of A if and only if $\mathrm{Tor}_n^A(I, A/I) = 0$ for all $n \geq 0$. In this case, I is idempotent.*
2. *If I is idempotent and A-projective, then I is homological.*
3. *If I is idempotent then I is homological if and only if $\mathrm{Ext}_A^n(I, A/I) = 0$ for all $n \geq 0$.*

We denote the *trace* of an A-module M in an A-module N as

$$tr_M(N) := \sum_{f \in Hom_A(M,N)} Im(f) \subset N.$$

Remark

We recall that according to Auslander et al., if P is an A-projective module then $tr_P(A)$ is an idempotent ideal of A and one obtains all the idempotent ideals of A this way.

Remark

Note that, since the homological ideals are idempotent ideals and the idempotent ideals are traces of projective modules over A then there is always a finite number of homological ideals.

Following the assumption that A is a bounded quiver algebra of the form kQ/I and the number of vertices of Q are finite for every subset $\{a_1, ..., a_m\} \subset Q_0$, we will assume the following notation for every idempotent ideal generated by the trace of $P(a_1) \oplus \cdots \oplus P(a_m)$ in A:

$$I_{a_1,...,a_m} = tr_{\left(P(a_1)\oplus\cdots\oplus P(a_m)\right)}(A). \tag{7}$$

Nakayama Algebras

Let Q be either a linearly oriented quiver with underlying graph $\mathbb{A}_n$ or a cycle $\widetilde{\mathbb{A}}_n$ with cyclic orientation. That is, Q is one of the following quivers

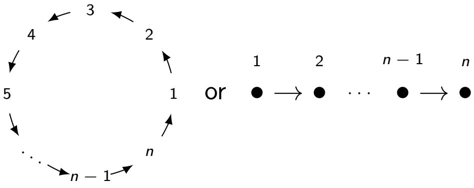

Figure: Quiver $\widetilde{\mathbb{A}}_n$ with cyclic orientation and Dynkin diagram $\mathbb{A}_n$ linearly oriented.

A quotient A of kQ by an admissible ideal I is called a *Nakayama algebra*.

In this work, for $n \geq 3$ fixed, we consider the algebras $A_{R_{(i,j,k)}} = kQ/I$ where Q is a Dynkin diagram of type $\mathbb{A}_n$ linearly oriented and I is an admissible ideal generated by one relation $R_{(i,j,k)}$ of length k starting at a vertex i and ending at a vertex j of the given quiver, $1 \leq i < j \leq n$.

The following picture shows the general structure of quivers Q which we are focused in this paper.

$$\mathbb{A}_n = 1 \to \cdots \to i \to i{+}1 \to \cdots \to i{+}k = j \to j{+}1 \to \cdots \to n{-}1 \to n.$$

Lemma

Every idempotent ideal I_r *of an algebra* $A_{R_{(i,j,k)}}$ *with* $j \leq r$ *or* $r \leq i$ *is homological.*

Lemma

Every idempotent ideal I_t *of an algebra* $A_{R_{(i,j,k)}}$*, with* $i+1 \leq t \leq j-1$ *is not homological.*

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Lemma

If each idempotent ideal I_{α_w} of an algebra $A_{R_{(i,j,k)}}$ is not homological then every idempotent ideal of the form $I_{\alpha_1,\ldots,\alpha_l}$ is not homological for $2 \leq l \leq k-1$.

Lemma

For l fixed, if each idempotent ideal I_{α_w} of an algebra $A_{R_{(i,j,k)}}$ with $1 \leq w \leq l$ is homological then every idempotent ideal of the form $I_{\alpha_1,\ldots,\alpha_l}$ is also homological.

Lemma

Every ideal $I_{i,t}$ or $I_{t,j}$ of an algebra $A_{R_{(i,j,k)}}$ is homological.

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Remark

If the non homological ideal I_t has the form $I_{t_1,\ldots,t_n}$ the previous Lemma also holds.

Lemma

For $1 \leq h \leq i-1$, $1 \leq l \leq k-1$ and $1 \leq m \leq n-j$ fixed. Every idempotent ideal of the form $I_{z_1,\ldots,z_h,t_1,\ldots,t_l,y_1,\ldots,y_m}$ of an algebra $A_{R_{(i,j,k)}}$, where $z_a \in [1, i-1]$, $t_b \in [i+1, j-1]$, $y_c \in [j+1, n]$ is not homological.

Lemma

For $1 \leq h \leq i-1$, $1 \leq l \leq k-1$ and $1 \leq m \leq n-j$ fixed. The idempotent ideals $I_{z_1,\ldots,z_h,t_1,\ldots,t_l}$ and $I_{t_1,\ldots,t_l,y_1,\ldots,y_m}$ of an algebra $A_{R_{(i,j,k)}}$ where $z_a \in [1, i-1]$, $t_b \in [i+1, j-1]$, $y_c \in [j+1, n]$ are not homological.

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The following results allow us to compute the number of homological and non homological ideals in a bounded algebra $A_{R_{(i,j,k)}}$ by using the integer specialization e of the Brauer configuration Γ_n introduced in the Example.

Theorem

For $n \geq 4$ fixed and $2 \leq k \leq n-1$ the number $|\mathbb{NHI}_n^k|$ of non homological ideals of an algebra $A_{R_{(i,j,k)}}$ is given by the identity $|\mathbb{NHI}_n^k| = w^e(U_k)$.

Corollary

For $n \geq 4$ *fixed and* $2 \leq k \leq n-1$ *the number of homological ideals* $|\mathbb{HI}_n^k|$ *of an algebra* $A_{R_{(i,j,k)}}$ *is given by the identity* $|\mathbb{HI}_n^k| = 2^n - w^e(U_k) = 3 \cdot 2^{n-2} + 2^{n-k-1}$.

The formula obtained in the last Theorem induces the following triangle:

Non homological triangle $\mathbb{NHIT}$

n/k	2	3	4	5	6	7	8	$\cdots$
3	1	-	-	-	-	-	-	-
4	2	3	-	-	-	-	-	-
5	4	6	7	-	-	-	-	-
6	8	12	14	15	-	-	-	-
7	16	24	28	30	31	-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Entries $|\mathbb{NHI}_n^k|$ of triangle $\mathbb{NHIT}$ can be calculated inductively as follows: we start by defining $|\mathbb{NHI}_n^2| = 2^{n-3}$ for all $n \geq 3$. Now, we assume that $|\mathbb{NHI}_n^k| = 0$ with $k \leq 1$ and for the sake of clarity we denote the specialization under e of a word $w(U_k)$ of the polygon U_k in the Brauer configuration Γ_n as $w^e(U_k^n)$. Then, for $k \geq 3$:

$$w^e(U_k) = w^e(U_k^n) = (w^e(U_{k-1}^n) + w^e(U_{k-1}^{n-1})) - w^e(U_{k-2}^{n-1}).$$

or equivalently,

$$|\mathbb{NHI}_n^k| = (|\mathbb{NHI}_n^{k-1}| + |\mathbb{NHI}_{n-1}^{k-1}|) - |\mathbb{NHI}_{n-1}^{k-2}|.$$

These arguments prove the following proposition.

Proposition

$M(\Gamma_n^e)$ equals the sum of the elements in the n-th row of the non homological triangle $\mathbb{NHIT}$.

Remark

The integer sequence generated by $M(\Gamma_n^e) = \sum_{k=1}^{n-1} 2^{n-2} - 2^{n-k-1} = \{1, 5, 17, 49, 129, 321, 769, 1793, 4097, 9217, \ldots\}$ is encoded A000337 in the OEIS. Elements of the sequence A000337 also correspond to the sums of the elements of the rows of the Reinhard Zumkeller triangle.

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Similarly, for the homological ideals last Corollary induces the following triangle:

Homological triangle $\mathbb{HIT}$.

n/k	2	3	4	5	6	7	8	$\cdots$
3	7	-	-	-	-	-	-	-
4	14	13	-	-	-	-	-	-
5	28	26	25	-	-	-	-	-
6	56	52	50	49	-	-	-	-
7	112	104	100	98	97	-	-	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The elements of the homological triangle are closely related with the research of categorification of integer sequences. Particularly, these numbers deal with the work of Ringel and Fahr regarding categorification of Fibonacci numbers.

Note that to each entry $d_{i,i-j}$ it is possible to assign a weight $w_{i,i-j}$ by using the numbers in the homological triangle $\mathbb{HIT}$ as follows:

$$w_{i,i-j} = \begin{cases} |\mathbb{HI}^k_{2s+2}| - 2^{2\cdot s-k+1}, & \text{if } j \text{ is even}, i \text{ is odd and } i \neq j+1, \\ |\mathbb{HI}^k_{2s+1}| - 2^{2\cdot s-k}, & \text{if } j \text{ is even}, i \text{ is even}, \\ 3, & \text{if } i \text{ odd}, j \text{ even and } i = j+1, \\ 1, & \text{if } i = j = 2h \text{ for some } h \geq 0, \\ 0, & \text{if } j \text{ is odd}, \ i \neq j. \end{cases}$$

Where $s = \lfloor \frac{i-j}{2} \rfloor$ and $\lfloor x \rfloor$ is the greatest integer number less than x. If we consider the multiplication of the entry $d_{i,i-j}$ with its corresponding weight $w_{i,i-j}$ we can define a partition formula for even-index Fibonacci numbers in the following form:

$$f_{2i+2} = \sum_{j=0}^{i} (w_{i,i-j})(d_{i,i-j}), \tag{8}$$

The following result give us a relationship between the number of homological ideals and Fibonacci numbers:

Theorem

$$\begin{aligned} \sum_{j=0}^{2t}(w_{2t,2t-j})(d_{2t,2t-j}) &= \sum_{r even}|T_r|\cdot a_t[r], \quad t\geq 0 \\ \sum_{j=0}^{2t-1}(w_{2t-1,2t-1-j})(d_{2t-1,2t-1-j}) &= \sum_{r odd}|T_r|\cdot a_t[r], \quad t\geq 1. \end{aligned} \tag{9}$$

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Thank You.