Beyond the 10-fold way: 13 associative $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superdivision algebras

Zhanna Kuznetsova

CMCC - Federal University of ABC
Santo Andre - SP, Brazil

Workshop on Representation Theory and Applications
ICTP - SAIFR, 25-29th April 2022
São Paulo, Brazil

10-fold way

Altland and Zirnbauer discovered in 1996 that substances can be divides in 10 kinds.

The basic idea:

- some substances have time reversal symmetry T (described by this ainit-unitary operator).
  There are three possibilities $T^2 = 1, -1$ or $T = 0$;
- some substances have charge conjugation symmetry C.
  Possibilities $C^2 = 1, -1$ or $C = 0$;
- $\Rightarrow$ there are 9 klasses of matter. Where is the 10th kind?
- If $T = 0$ and $C = 0$ (no time reversal and charge conjugation symmetries) but a substance can be symmetric under $S = T \circ C$ symmetry. This ($S = 0, S = 1$) gives the 10th class of matter.

It happened that the 10-fold way is in a strong connection with the classification of super-division algebras.

On topological phases of condensed matter systems

Altland - Zirnbauer classes

quadratic fermionic Hamiltonians

Ex. $H = -i\hbar v_F \psi^\dagger(\sigma_x \tau_z \partial_x + \sigma_y \partial_y)\psi$.

two **anti-unitary** + one **unitary** symmetry

Symmetry	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
T	0	0	1	1	0	-1	-1	-1	0	1
C	0	0	0	1	1	1	0	-1	-1	-1
S	0	1	0	1	0	1	0	1	0	1

There is a *bijection* between the known universality classes of disordered single-particle systems and the large families of symmetric spaces (Cartan's classification).

Graded division algebras

An associative unital algebra $\mathbb{D}$ over a field k is called a **division algebra** if any nonzero element of $\mathbb{D}$ is invertible.

A graded algebra $\mathbb{D}$ over k is a **graded division algebra** is any nonzero *homogeneous* element is invertible.

In this talk we consider $\mathbb{Z}_2$-graded and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division algebras over a field $\mathbb{R}$ of real numbers.

$\mathbb{Z}_2$-graded (or *superdivision*) algebras:
Wall, Graded Brauer Groups, 1964;
Trimble, a paper on superdivision algebras, 2005

There are exactly 10 superdivision algebras:

3 of them are purely even - $\mathbb{R}, \mathbb{C}, \mathbb{H}$;

7 of them have nonzero odd sector.

Superdivision algebras

We consider $\mathbb{Z}_2$ grading: $\mathbb{D} = \mathbb{D}_0 \oplus \mathbb{D}_1$.

If $\mathbb{D}_1 = \{0\}$, it gives the options:
$\mathbb{D}_0 = \mathbb{R}$, $\mathbb{D}_1 = \{0\}$;
$\mathbb{D}_0 = \mathbb{C}$, $\mathbb{D}_1 = \{0\}$;
$\mathbb{D}_0 = \mathbb{H}$, $\mathbb{D}_1 = \{0\}$.

If $\mathbb{D}_1 \neq \{0\}$ and $\mathbb{D}_0 = \mathbb{R}$. We choose a nonzero element $e \in \mathbb{D}_1$. We can rescale e to obtain $e^2 = 1$ or $e^2 = -1$.
$\mathbb{D}_0 \cong \mathbb{D}_1 \cong \mathbb{R}$ with an odd element e such that $e^2 = 1$;
$\mathbb{D}_0 \cong \mathbb{D}_1 \cong \mathbb{R}$ with an odd element e such that $e^2 = -1$.

If $\mathbb{D}_0 = \mathbb{C}$, than the map $a \mapsto eae^{-1}$ defines am authomorfism of $\mathbb{D}_0$ which can be or identity, or complex conjugation: $eae^{-1} = a$ or $eae^{-1} = \bar{a}$. It gives three options:
$\mathbb{D}_0 \cong \mathbb{D}_1 \cong \mathbb{C}$, odd element e with $ei = ie$ and $e^2 = 1$;
$\mathbb{D}_0 \cong \mathbb{D}_1 \cong \mathbb{C}$, odd element e with $ei = -ie$ and $e^2 = 1$;
$\mathbb{D}_0 \cong \mathbb{D}_1 \cong \mathbb{C}$, odd element e with $ei = -ie$ and $e^2 = -1$.

Superdivision algebras

In the case $\mathbb{D}_0 = \mathbb{H}$ we have two cases:
$\mathbb{D}_0 \cong \mathbb{D}_1 \cong \mathbb{H}$, the element e commutes with $\mathbb{D}_0$ and $e^2 = 1$;
$\mathbb{D}_0 \cong \mathbb{D}_1 \cong \mathbb{H}$, the element e commutes with $\mathbb{D}_0$ and $e^2 = -1$.

We see here 10 different $\mathbb{Z}_2$-graded division algebras.

In this talk we discuss $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded division algebras

obtained by use of the alphabetic presentation of Clifford algebras.

$\mathbb{Z}_2 \times \mathbb{Z}_2$ grading

We will consider the grading labeled by the $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\mathbb{D} = \mathbb{D}_{00} \oplus \mathbb{D}_{01} \oplus \mathbb{D}_{10} \oplus \mathbb{D}_{11}$$

A homogeneous element g belongs to one of the graded sectors $g \in \mathbb{D}_{ij}$.

The graded multiplication results in the sector by the rule

$$g_{ij} g_{kl} \subset \mathbb{D}_{i+k,j+l}$$

the sum in the indexes $i + k$ and $j + l$ is understood as mod 2.

The procedure is similar to this of superdivision algebras, we begin with real, or complex numbers, or quaternions and construct a graded spaces.
It comes that alphabetic presentation is very much useful for the classification.

Alphabetic presentation

We denote by letters the following real 2×2 matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A tensor product of n matrices we write as a n-letter word.
Example $AXY \equiv A \otimes X \otimes Y$.

Properties:

1. if the word begins with I or X, the matrix M is block-diagonal ;
   If the word begins with Y or A, the matrix is block-antidiagonal;
2. if a matrix M includes even number of the letter A, than $M^2 = Id$;
   If M contains an odd number of A, than $M^2 = -Id$.

For example, the 8×8 matrix $M = AXA$ is block-antidiagonal and $M^2 = Id$.

Example: quaternions

The three imaginary quaternions q_i ($i = 1, 2, 3$) satisfy

$$q_i q_j = -\delta_{ij} + \varepsilon_{ijk} q_k$$

(where $\varepsilon_{123} = 1$.)

The alphabetic presentation is given by

$$\overline{q}_1 = IA, \qquad \overline{q}_2 = AY, \qquad \overline{q}_3 = AX,$$

or by

$$\widetilde{q}_1 = AI, \qquad \widetilde{q}_2 = YA, \qquad \widetilde{q}_3 = XA.$$

The 4×4 real matrices for $\bar{q}_i$ are given by

$$\overline{q}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \overline{q}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{q}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Division algebras

In the alphabetic presentation of not-graded division algebras

$\mathbb{R}$: I;

$\mathbb{C}$: I, A

$\mathbb{H}$: II, IA, AX, AY.

Graded division algebras

A homogeneous element $g \in \mathbb{D}$ is represented by a matrix $g = M \otimes N$.

The matrix M encodes the information of the grading: $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The matrix N is for the real, complex or quaternionic structure.

The matrix size for M is

$$\mathbb{Z}_2\text{-grading}: \ (2 \times 2); \qquad \mathbb{Z}_2 \times \mathbb{Z}_2\text{-grading}: \ (4 \times 4).$$

The matrix size for N is

$$\mathbb{R}\text{-series}: \ (1 \times 1); \qquad \mathbb{C}\text{-series}: \ (2 \times 2); \qquad \mathbb{H}\text{-series}: \ (4 \times 4).$$

Structure

$\mathbb{Z}_2$ grading: the even (odd) sector is denoted as M_0 (M_1).

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \in M_0\,, \qquad \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \in M_1\,.$$

$$M_0 : \quad I,\ X; \qquad M_1 : \quad Y,\ A.$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-grading:

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \in M_{00}\,, \qquad \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix} \in M_{01}\,,$$

$$\begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix} \in M_{10}\,, \qquad \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} \in M_{11}\,.$$

$$M_{00} : \quad II,\ IX,\ XI,\ XX; \qquad M_{01} : \quad IA,\ IY,\ XA,\ XY;$$
$$M_{10} : \quad AI,\ AX,\ YI,\ YX; \qquad M_{11} : \quad AA,\ AY,\ YA,\ YY.$$

$\mathbb{Z}_2$-graded superdivision algebras

Without loss of generality (up to similarity transformations) the even sector $\mathbb{D}_0$ can be expressed as

$$\mathbb{R}\text{-series}: \ I; \qquad \mathbb{C}\text{-series}: \ II, \ IA; \qquad \mathbb{H}\text{-series}: \ III, \ IIA, \ IAX, \ IAY.$$

Real series:

$$\begin{array}{lllll}
\mathbb{D}_{\mathbb{R};1}^{[1]} & : & I \in \mathbb{D}_0^{[1]}, & A \in \mathbb{D}_1^{[1]}, & e = A, e^2 = -1 \\
\mathbb{D}_{\mathbb{R};2}^{[1]} & : & I \in \mathbb{D}_0^{[1]}, & Y \in \mathbb{D}_1^{[1]} & e = Y, e^2 = 1.
\end{array}$$

Complex series ($i = IA$):

$$\begin{array}{lllll}
\mathbb{D}_{\mathbb{C};1}^{[1]} & : & II, \ IA \in \mathbb{D}_0^{[1]}, & AX, \ AY \in \mathbb{D}_1^{[1]}, & e = AX, ei = -ie, e^2 = -1 \\
\mathbb{D}_{\mathbb{C};2}^{[1]} & : & II, \ IA \in \mathbb{D}_0^{[1]}, & YX, \ YY \in \mathbb{D}_1^{[1]}, & e = YX, ei = -ie, e^2 = 1 \\
\mathbb{D}_{\mathbb{C};3}^{[1]} & : & II, \ IA \in \mathbb{D}_0^{[1]}, & AI, \ AA \in \mathbb{D}_1^{[1]} & e = IA, ie = ei, e^2 = 1
\end{array}$$

Quaternionic series:

$$\begin{array}{lllll}
\mathbb{D}_{\mathbb{H};1}^{[1]} & : & III, IIA, IAY, IAX \in \mathbb{D}_0^{[1]}, & AII, AIA, AAY, AAX \in \mathbb{D}_1^{[1]}, & e = AII, e^2 = -1 \\
\mathbb{D}_{\mathbb{H};2}^{[1]} & : & III, IIA, IAY, IAX \in \mathbb{D}_0^{[1]}, & YII, YIA, YAY, YAX \in \mathbb{D}_1^{[1]} & e = YII, e^2 = 1
\end{array}$$

Associated superdivision algebras

Symmetry	A	AIII	AI	BDI	D	DIII	AII	CII	C	CI
T	0	0	1	1	0	-1	-1	-1	0	1
C	0	0	0	1	1	1	0	-1	-1	-1
S	0	1	0	1	0	1	0	1	0	1
division	C3	$\mathbb{C}$	C2	R2	$\mathbb{R}$	R1	H2	C1	$\mathbb{H}$	H1

$\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division algebras: the structure

The alphabetic presentation is extended to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ superdivision algebras by taking into account that:

- up to similarity transformations the even sector $\mathbb{D}_{00}$ can be expressed as

$$\mathbb{R}\text{-series}: \; II; \quad \mathbb{C}\text{-series}: \; III, \; IIA; \quad \mathbb{H}\text{-series}: \; IIII, \; IIIA, \; IIAX, \; IIAY; \tag{2}$$

- each one of the three subalgebras $\mathbb{S}_{10}, \mathbb{S}_{01}, \mathbb{S}_{11} \subset \mathbb{D}$, given by the direct sums

$$\mathbb{S}_{01} := \mathbb{D}_{00} \oplus \mathbb{D}_{01} \qquad \mathbb{S}_{10} := \mathbb{D}_{00} \oplus \mathbb{D}_{10}, \qquad \mathbb{S}_{11} := \mathbb{D}_{00} \oplus \mathbb{D}_{11}, \tag{3}$$

  is isomorphic to one (of the seven) $\mathbb{Z}_2$-graded superdivision algebra;
- the alphabetic presentation can be assumed for $\mathbb{S}_{01}$ and, since the second $\mathbb{Z}_2$ grading is independent from the first one, $\mathbb{S}_{10}$. The closure under multiplication for any $g \in \mathbb{D}_{01}, \; g' \in \mathbb{D}_{10}$ implies that $gg' \in \mathbb{D}_{11}$ is alphabetically presented.

The 13 cases are split into $13 = 4 + 5 + 4$ subcases; 4 -from the real series, 5 - from the complex series, 4 - from the quaternionic series.

The sectors $01, 10, 11$ are on equal footing, the $\mathbb{Z}_2$-graded subalgebra projections of a superdivision algebra can be characterized by the triple

$$(\mathbb{S}_\alpha / \mathbb{S}_\beta / \mathbb{S}_\gamma), \tag{4}$$

where the order of the subalgebras is inessential.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division algebras: **real** series

The matrix representatives of ”real” division algebras are expressed in the table below:

	00	01	10	11
$\mathbb{D}^{[2]}_{\mathbb{R};1}$:	II	IA	AX	AY
$\mathbb{D}_{\mathbb{R};2}$:	II	IA	YX	YY
$\mathbb{D}_{\mathbb{R};3}$:	II	IA	AI	AA
$\mathbb{D}^{[2]}_{\mathbb{R};4}$:	II	IY	YI	YY

Comment 1 - squaring the matrices entering the 01, 10, 11 sectors gives the signs

$$\mathbb{D}_{\mathbb{R};1}\ :\ ---;\qquad \mathbb{D}^{[2]}_{\mathbb{R};2}\ :\ ++-;\qquad \mathbb{D}^{[2]}_{\mathbb{R};3}\ :\ +--;\qquad \mathbb{D}_{\mathbb{R};4}\ :\ +++.$$

Comment 2 - the projections to the $\mathbb{Z}_2$-graded subalgebras, see formula (4), are given by

$$\mathbb{D}_{\mathbb{R};1}\ :\ (\underline{1}/\underline{1}/\underline{1});\qquad \mathbb{D}_{\mathbb{R};2}\ :\ (\underline{1}/\underline{2}/\underline{2});\qquad \mathbb{D}_{\mathbb{R};3}\ :\ (\underline{1}/\underline{1}/\underline{2});\qquad \mathbb{D}_{\mathbb{R};4}\ :\ (\underline{2}/\underline{2}/\underline{2}),$$

where $\underline{1} := \mathbb{D}^{[1]}_{\mathbb{R};1}$ and $\underline{2} := \mathbb{D}^{[1]}_{\mathbb{R};2}$.

Comment 3 - the $\mathbb{D}^{[2]}_{\mathbb{R};1}$ superdivision algebra is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ gradation of the quaternions $\mathbb{H}$, the $\mathbb{D}^{[2]}_{\mathbb{R};2}$ superdivision algebra is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ gradation of the split-quaternions $\widetilde{\mathbb{H}}$, the superdivision algebras $\mathbb{D}^{[2]}_{\mathbb{R};3}$ and $\mathbb{D}^{[2]}_{\mathbb{R};4}$ are commutative.

The complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division algebras

	00	01	10	11
$\mathbb{D}^{[2]}_{\mathbb{C};1}$:	$III,\ IIA$	$IAX,\ IAY$	$AIX,\ AIY$	$AAI,\ AAA$
$\mathbb{D}^{[2]}_{\mathbb{C};2}$:	$III,\ IIA$	$AIX,\ AIY$	$IYX,\ IYY$	$AYI,\ AYA$
$\mathbb{D}^{[2]}_{\mathbb{C};3}$:	$III,\ IIA$	$YIX,\ YIY$	$IYX,\ IYY$	$YYI,\ YYA$
$\mathbb{D}^{[2]}_{\mathbb{C};4}$:	$III,\ IIA$	$YII,\ YIA$	$XYI,\ XYA$	$AYI,\ AYA$
$\mathbb{D}^{[2]}_{\mathbb{C};5}$:	$III,\ IIA$	$YII,\ YIA$	$IYI,\ IYA$	$YYI,\ YYA$

We denote here the three complex $\mathbb{Z}_2$-graded superalgebras as $\underline{1} := \mathbb{D}^{[1]}_{\mathbb{C};1}$, $\underline{2} := \mathbb{D}^{[1]}_{\mathbb{C};2}$ and $\underline{3} := \mathbb{D}^{[1]}_{\mathbb{C};3}$.
The table below (where the first underlined number denotes $\mathbb{S}_{01}$, the second number $\mathbb{S}_{10}$ and the arrow gives the $\mathbb{S}_{11}$ output:

$$
\begin{array}{lll}
\underline{1}\times\underline{1}\rightarrow\underline{3}, & \underline{1}\times\underline{2}\rightarrow\underline{3}, & \underline{1}\times\underline{3}\Rightarrow \begin{smallmatrix}\nearrow \underline{1}\\ \searrow \underline{2}\end{smallmatrix}, \\
\underline{2}\times\underline{1}\rightarrow\underline{3}, & \underline{2}\times\underline{2}\rightarrow\underline{3}, & \underline{2}\times\underline{3}\Rightarrow \begin{smallmatrix}\nearrow \underline{1}\\ \searrow \underline{2}\end{smallmatrix}, \\
\underline{3}\times\underline{1}\Rightarrow \begin{smallmatrix}\nearrow \underline{1}\\ \searrow \underline{2}\end{smallmatrix}, & \underline{3}\times\underline{2}\Rightarrow \begin{smallmatrix}\nearrow \underline{1}\\ \searrow \underline{2}\end{smallmatrix}, & \underline{3}\times\underline{3}\Rightarrow\underline{3}^{(*)}.
\end{array}
$$

$\mathbb{D}^{[2]}_{\mathbb{C};1}$: $(\underline{1}/\underline{1}/\underline{3})$; $\mathbb{D}^{[2]}_{\mathbb{C};2}$: $(\underline{1}/\underline{2}/\underline{3})$; $\mathbb{D}^{[2]}_{\mathbb{C};3}$: $(\underline{2}/\underline{2}/\underline{3})$; $\mathbb{D}^{[2]}_{\mathbb{C};4}$: $(\underline{3}/\underline{3}/\underline{3})$; $\mathbb{D}^{[2]}_{\mathbb{C};5}$: $(\underline{3}/\underline{3}/\underline{3})$.

In $\mathbb{D}_4$ generators in different sectors anticommute; In $\mathbb{D}_5$ generators in different sectors commute.

Quaternionic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division algebras

The generators of the 00 sector are chosen as

$$00 \quad : \quad IIII, \; IIIA, \; IIAX, \; IIAY.$$

The generators of the 01, 10, 11 sectors are expressed as

	01	10	11
$\mathbb{D}^{[2]}_{\mathbb{H};1}$:	$IAII, \; IAIA, \; IAAX, \; IAAY$	$AXII, \; AXIA, \; AXAX, \; AXAY$	$AYII, \; AYIA, \; AYAX, \; AYAY$
$\mathbb{D}^{[2]}_{\mathbb{H};2}$:	$IAII, \; IAIA, \; IAAX, \; IAAY$	$AIII, \; AIIA, \; AIAX, \; AIAY$	$AAII, \; AAIA, \; AAAX, \; AAAY$
$\mathbb{D}^{[2]}_{\mathbb{H};3}$:	$IAII, \; IAIA, \; IAAX, \; IAAY$	$YXII, \; YXIA, \; YXAX, \; YXAY$	$YYII, \; YYIA, \; YYAX, \; YYAY$
$\mathbb{D}^{[2]}_{\mathbb{H};4}$:	$IYII, \; IYIA, \; IYAX, \; IYAY$	$YIII, \; YIIA, \; YIAX, \; YIAY$	$YYII, \; YYIA, \; YYAX, \; YYAY$

$\underline{1} := \mathbb{D}^{[1]}_{\mathbb{H};1}$ and $\underline{2} := \mathbb{D}^{[1]}_{\mathbb{H};2}$.

The quaternionic projections are then given by

$D^{[2]}_{\mathbb{H};1} : (\underline{1}/\underline{1}/\underline{1})$; $\quad \mathbb{D}^{[2]}_{\mathbb{H};2} : (\underline{1}/\underline{1}/\underline{2})$; $\quad \mathbb{D}^{[2]}_{\mathbb{H};3} : (\underline{1}/\underline{2}/\underline{2})$; $\quad \mathbb{D}^{[2]}_{\mathbb{H};4} : (\underline{2}/\underline{2}/\underline{2})$.

Some considerations

- This classification is done in terms os matrix representation of Clifford algebras expressed by alphabetic presentation. It can be easily expressed in terms of graded elements $e_i, i = 1, 2, 3$ together with the description of admissible properties as it was done for super division algebras.
- It is interesting to discuss the possibilities to "refine" the periodic table of condensed matter using the extra grading. Parafermionic sistems? What are their properties? Where they can be found?
- The division algebra are: 3 + 7 (superdivision) + 13 ($\mathbb{Z}_2 \times \mathbb{Z}_2$-division). What is the 20-fold way?

Thank you for the attention