Representations of Jordan algebras and superalgebras

Iryna Kashuba

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Let **k** be an algebraically closed field of char 0.

Definition

A **Jordan algebra** is a commutative **k**-algebra $(J, \cdot)$ satisfying

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x) \quad x, y \in J.$$

Any associative **k**-algebra A gives rise to a Jordan algebra A^+ under symmetric multiplication

$$x \cdot y = \frac{1}{2}(xy + yx)$$

A Jordan algebra is called **special** if it can be realized as a Jordan subalgebra of some A^+.

Example (Two types of special algebras)

- Let $(A, \star)$ be an associative algebra with involution. The subspace of hermitian elements $H(A, \star) = \{x^\star = x \,|\, x \in A\}$ forms a special Jordan algebra.
- Let f be symmetric bilinear form on vector space V $\dim V \geq 2$. The space $J(V, f) = \mathbf{k}1 \oplus V$ becomes a Jordan algebra (called **Jordan algebra of Clifford type**) by making 1 act as a unit and defining $v \cdot w = f(v, w)1$, $v, w \in V$. $J(V, f)$ is special since we have one-to-one mapping

$$J(V, f) \to C(V, f)$$
$$\alpha 1 + v = \alpha 1 + v + R,$$

where $C(V, f) = T(V)/R$ is the Clifford algebra of V relative to f, $R = \langle v \otimes v - f(v, v)1 \,|\, v \in V \rangle$.

Suppose M is a $\mathbf{k}$-vector space with $l : (a, m) \to am$, $r : (a, m) \to ma$, define a product on $\Omega = J \oplus M$

$$(a_1 + m_1) \circ (a_2 + m_2) = a_1 \cdot a_2 + a_1 m_2 + m_1 a_2.$$

Definition

M is a **Jordan bimodule** for $J \Leftrightarrow \Omega = (\Omega, \circ)$ is a Jordan algebra. Equivalently a linear map $\rho : J \to \mathrm{End}_k M$, $\rho(a)m = am$ defines a **(bi)representation** if for all $a, b \in J$

$$\begin{aligned} [\rho(a), \rho(a \cdot a)] &= 0, \\ 2\rho(a)\rho(b)\rho(a) + \rho(a^{\cdot 2} \cdot b) &= 2\rho(a)\rho(a \cdot b) + \rho(b)\rho(a^{\cdot 2}) \end{aligned} \tag{1}$$

J-bimod is equivalent to U-mod, where $U = U(J)$, the **universal multiplication envelope**. $U(J) = T(J)/R$, where R is an ideal generated by (1).

Jacobson has shown the following

1. If $\dim_{\mathbf{k}} J < \infty \;\Rightarrow\; \dim_{\mathbf{k}} U(J) < \infty$.
2. For any finite-dimensional simple J its $U(J)$ is finite-dimensional semi-simple.
3. if J has an identity element e

$$U(J) = \mathbf{k} \oplus S_1(J) \oplus U_1(J),$$

where $\mathbf{k} \oplus S_1(J) = S(J) = T(J)/\langle ab + ba - 2a \cdot b\rangle$ the **special universal envelope** of J.

$$\begin{array}{ccccccc} J\text{-mod} & \simeq & J\text{-mod}_0 & \oplus & J\text{-mod}_{\frac{1}{2}} & \oplus & J\text{-mod}_1, \\ e \text{ acts as} & & 0 & & \frac{1}{2} & & 1 \end{array}$$

$$J\text{-mod}_{\frac{1}{2}} \simeq S(J)\text{-mod}, \qquad J \subset S(J)^+ \iff J \text{ is special}$$

Albert's classification of simple finite-dimensional Jordan algebras:

$deg = 1$ $\mathbf{k}$

$deg = 2$ $J(f,n) := J(V,f)$, where f is non-degenerate.

$deg \geq 3$ $H_n(C)$, $n \geq 3$, (C,τ) composition algebra of dimension 1, 2, 4 for $n \geq 4$, and 1, 2, 4, 8 for $n = 3$.

$deg = 2$: $S_1(J) \simeq C(V,f)$ and $U_1(J) \simeq T(V)/R_M$,
$R_M = \langle u \otimes v \otimes u - f(u,v)u \mid u,v \in V \rangle$,

J	$S_1(J)$	$U_1(J)$
$J(f,n)$ n is even	M_{2^n}	$\bigoplus_s M_s$ $s = \binom{n+1}{1}, \binom{n+1}{3}, \ldots, \binom{n+1}{n+1}$
$J(f,n)$ $n = 2\nu - 1$	$M_{2^{n-1}} + M_{2^{n-1}}$	$M_{\frac{1}{2}\binom{n+1}{\nu}} \oplus M_{\frac{1}{2}\binom{n+1}{\nu}} \bigoplus_s M_s$ $s = \binom{n+1}{0}, \binom{n+1}{1}, \ldots, \binom{n+1}{\nu-1}$

$deg \geq 3$: If C is associative $S_1(H_n(C)) \simeq M_n(C)$.

There is a functor $\mathcal{H}_n$: (C,τ)-bimod $\to H_n(C)$-mod.

J	$S_1(J)$	$U_1(J)$
$H_n(\mathbf{k})$	M_n	$M_{\frac{n(n+1)}{2}} \oplus M_{\frac{n(n-1)}{2}}$
$H_n(\mathbf{k}+\mathbf{k})$	$M_n \oplus M_n$	$M_{n^2} \oplus M_{\frac{n(n+1)}{2}} \oplus M_{\frac{n(n+1)}{2}}$ $\oplus M_{\frac{n(n-1)}{2}} \oplus M_{\frac{n(n-1)}{2}}$
$H_n(M_2(\mathbf{k}))$	M_{2n}	$M_{n(2n-1)} \oplus M_{n(2n+1)}$ *
$\mathcal{A}$	0	M_{27}

Case of $Rad^2 J = 0$

$S_1(J)$ for $degJ \geq 3$ were described in *K., Ovsienko S., Shestakov I., Representation type of Jordan algebras, 2011*

Quivers $Q(S_1(J))$ were constructed using generalization of functor $\mathcal{H}_n$. It was shown that

$$Rad^2 S_1(J) = 0.$$

For finite dimensional associative A, $Rad^2 A = 0$, A is of finite (tame) representation type $\Longleftrightarrow$ the quiver double $D(Q(A))$ is a disjoint union of oriented Dynkin diagrams (extended Dynkin diagrams).

A **short grading** of $\mathfrak{g}$ is a $\mathbb{Z}$-grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.
Let P be the commutative bilinear map on J: $P(x, y) = x \cdot y$.
We associate to J a Lie algebra with short grading

$$Lie(J) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Put $\mathfrak{g}_{-1} = J$, $\mathfrak{g}_0 = \langle L_a, [L_a, L_b] | a, b \in J\rangle$, $\mathfrak{g}_1 = \langle P, [L_a, P] | a \in J\rangle$

- $[L, x] = L(x)$ for $x \in \mathfrak{g}_{-1}$, $L \in \mathfrak{g}_0$;
- $[B, x](y) = B(x, y)$ for $B \in \mathfrak{g}_1$ and $x, y \in \mathfrak{g}_{-1}$;
- $[L, B](x, y) = L(B(x, y)) - B(L(x), y) - B(x, L(y))$ for any $B \in \mathfrak{g}_1$, $L \in \mathfrak{g}_0$ and $x, y \in \mathfrak{g}_{-1}$.

Then $\mathfrak{g} = Lie(J)$ is Lie algebra and is called the **Tits-Kantor-Koecher (TKK) construction** for J.

A **short subalgebra** of $\mathfrak{g}$ is an $\mathfrak{sl}_2$ subalgebra spanned by e, h, f such that the eigenspace decomposition of $ad\, h$ defines a short grading on $\mathfrak{g}$.

For any J with identity e consider in $Lie(J)$

$$h_J = -L_e, \quad f_J = P, \quad \text{then } \alpha_J = \langle e, h_J, f_J \rangle$$

defines short subalgebra of $Lie(J)$.

Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the $\mathbb{Z}_2$-graded Lie algebra, $p \in \mathfrak{g}_1$. For any $x, y \in \mathfrak{g}_{-1}$ set

$$x \cdot y = [[p, x], y]$$

then $Jor(\mathfrak{g}) := (\mathfrak{g}_{-1}, \cdot)$ is a Jordan algebra.

Relations between J-mod and $\mathfrak{g} = TKK(J)$-modules?
We define two adjoint functors *Jor* and *Lie* between J-mod and $\mathfrak{g}$-modules admitting a short grading.

Not every J-module can be obtained from a $\mathfrak{g}$-module by application of *Jor*: one has to consider $\hat{\mathfrak{g}}$ the universal central extension of $\mathfrak{g}$.

Let $\mathcal{S}$ (resp. $\mathcal{S}_{\frac{1}{2}}$) be the category of $\hat{\mathfrak{g}}$-modules M such that the action of α_J induces a short grading on M (resp. a grading of length 2, namely $M_{-\frac{1}{2}} \oplus M_{\frac{1}{2}}$).

$$J\text{-mod}_{\frac{1}{2}} \simeq \mathcal{S}_{\frac{1}{2}}$$
$$J\text{-mod}_0 \oplus J\text{-mod}_1 \leftrightarrow \mathcal{S}$$

Representation table for $TKK(J)$

J	$\mathfrak{g}$	$\mathcal{S}_{\frac{1}{2}}$	$\mathcal{S}$
$H_n(\mathbf{k})$	$\mathfrak{sp}_{2n}$	V	$ad,\ \Lambda^2 V$
$H_n(\mathbf{k}+\mathbf{k})$	$\mathfrak{sl}_{2n}$	$V,\ V^*$	$ad,\ S^2(V),\ S^2(V^*),\ \Lambda^2(V),\ \Lambda^2(V^*)$
$H_n(M_2(\mathbf{k}))$	$\mathfrak{so}_{4n}$	V	$ad,\ S^2(V)$
$\mathcal{A}$	E_7		ad
$J(f,n)$ $n=2\nu$	$\mathfrak{so}_{n+3}$	Γ spinor	$\Lambda^i(V),\ i=1,\dots,\nu+1$
$J(f,n)$ $n=2\nu-1$	$\mathfrak{so}_{n+3}$	$\Gamma^+,\ \Gamma^-$ spinor	$\Lambda^i(V),\ i=1,\dots,\nu$ $\Lambda^{\nu+1}(V)^{\pm}$

Quiver of an abelian category

Let $\mathcal{C}$ be an abelian category with finitely many simple modules such that every object has finite length and every simple object has a projective cover.
Then $\mathcal{C}$ is equivalent to the category of finite-dimensional A-modules. If $L_1, \ldots, L_r$ is the set of all up to isomorphism simple objects in $\mathcal{C}$ and $P_1, \ldots, P_r$ are their projective covers, then A is a pointed algebra which is usually realized as the path algebra of a certain quiver Q with relations.
The vertices

$$Q_0 = \{\text{simple modules } L_1, \ldots, L_r\}$$

$$Q_1 = \{\#\text{arrows from vertex } L_i \text{ to vertex } L_j \text{ is } \dim \operatorname{Ext}^1(L_j, L_i)\}$$

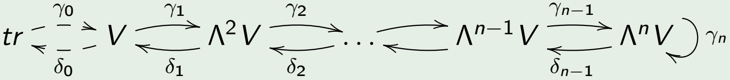
$Q(\mathcal{S})$ and $Q(\mathcal{S}_{\frac{1}{2}})$ are now straightforward:

Lemma

Let $\mathfrak{g} = \mathfrak{g}_s + R$ be the Levi decomposition of $\mathfrak{g}$. Denote by $\mathfrak{r} = R/R^2$. Let L and L' be two simple $\mathfrak{g}_s$-modules then $\dim \operatorname{Ext}^1(L, L')$ equals the multiplicity of L' in $L \otimes \mathfrak{r}$.

Example (Category $\mathcal{S}$)

Let $J = J(2n-2, f) + V_1$ then $Lie(J) = \mathfrak{g} = \mathfrak{so}_{2n+1} \oplus V$, $n \geq 3$



with the relations

$$\gamma_{r-1}\gamma_r = \delta_r\delta_{r-1} = 0,\ \gamma_{r-1}\delta_{r-1} = \delta_r\gamma_r,$$
$$\gamma_{n-1}\delta_{n-1} = \gamma_n^2,\ \text{for } r = 1, \ldots, n-1.$$

Tame and finite J-mod$_1$, $degJ = 2$ were determined in *K., Serganova V., On the Tits-Kantor-Koecher construction of unital Jordan bimodules, 2017.*

Category $\mathcal{S}_{\frac{1}{2}}$

It is completely different picture for $\mathcal{S}_{\frac{1}{2}}$.

Example

Let $J = J(f, 2n-2) + V_1$ then $Lie(J) = \mathfrak{g} = \mathfrak{so}_{2n+1} \oplus V,\ n \geq 3$

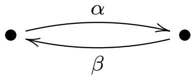
$\alpha^2 = 0.$

Theorem

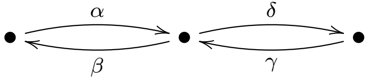
The category $\mathcal{S}_{\frac{1}{2}}$ is equivalent to the category of representations of a finite-dimensional graded quadratic algebra

Quiver blocks corresponding to simple modules in $\mathcal{S}_{\frac{1}{2}}$:

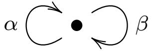
- Each block has either one, two, three or four vertices;
- Each simple block has either one or two arrows;
- Quivers with relations are the following quivers

1. 

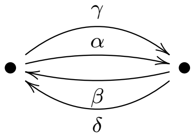
$\alpha\beta \neq 0$

2. 

$\alpha\beta = \gamma\delta$

3. 

$\alpha\beta = \beta\alpha$

4. 

$\beta\alpha = -\delta\gamma \quad \delta\alpha = -\beta\gamma$

Case of superalgebras

Theorem

Every finite dimensional simple Jordan superalgebra over an algebraically closed field of characteristic 0 *is isomorphic to*

$$Q_n\,(n \geq 2),\ \ M^{(+)}_{n|m},\ \ JP(n)\,(n \geq 2),\ \ Josp(n, 2m),\ \ J(m+2n),\\ K_3,\ \ D_t,\ \ K_{10}, Kan(n).$$

Example

$$TKK(M^{(+)}_{n|m}) = \mathfrak{sl}(2m, 2n) \qquad TKK(JP(n)) = P(2n-1)$$
$$TKK(Q(n)) = Q(2n) \qquad TKK(D_t) = D(2, 1; t)$$
$$TKK(J(m+2n)) = Osp(m+3, 2n) \qquad TKK(K_{10}) = F(4)$$
$$TKK(Josp(n, 2m)) = Osp(2n, 2m) \qquad TKK(Kan(n)) = H(n+3)$$

Irreducible modules

- All irreducible bimodules over finite dimensional simple Jordan superalgebras over an algebraically closed field of characteristic 0 are classified.
- A.Shtern: K_{10} and $Kan(n)$ have only regular irreducible supermodules. (He did not consider central extension).
- C.Martinez and E.Zelmanov proved that if J is finite-dimensional simple Jordan superalgebra such that J_0 is of rank ≥ 3 then $U(J)$ is finite-dimensional and semisimple ($Q(n)$, $JP(n)$, $(n \geq 3)$; $M_{n|m}^{(+)}$, $(n+m \geq 3)$; $Josp(n, 2m)$; K_{10}).
- Analogously to algebra case superalgebras $J(m+2n)$ have the finite number of irreducible finite-dimensional bimodule, the same is true for $Q(2)$. Moreover $Q(2)$ is completely reducible.
- C.Martinez, E.Zelmanov, I.Shestakov proved that the superalgebras K_3, D_t, $M_{1|1}^{(+)}$, $JP(2)$ have infinite number of irreducible superbimodules.

Indecomposable modules

Simple Jordan superalgebras which are not completely reducible: K_3, D_t, $JP(2)$, $Kan(n)$, $M_{1|1}^{(+)}$.

C.Martinez and E.Zelmanov have described the indecomposable modules over K_3 and D_t.

The indecomposable modules over $JP(2)$ and $Kan(n)$ and $M_{1|1}^{(+)}$ were described in
K., Serganova V., Representations of simple Jordan superalgebras, 2020.

Recall that $JP(n)$ is the superalgebra of symmetric elements of $M_{n+n}(F)$ with respect to the superinvolution

$$* : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} D^t & -B^t \\ C^t & A^t \end{pmatrix},$$

$$JP(n) = \left\{ \begin{bmatrix} A & B \\ C & A^T \end{bmatrix} \mid A,\ B,\ C \in M_n(\mathbf{k}),\ B^T = -B,\ C^T = C \right\}.$$

$TKK(JP(n)) = P(2n-1)$.
The Lie superalgebra $P(n)$, $n \geq 2$ a subalgebra of $\mathfrak{sl}_{n+1,n+1}$ consisting of all matrices of the form

$$\begin{bmatrix} A & B \\ C & -A^t \end{bmatrix}, \quad trA = 0,\ B^t = B,\ C^t = -C$$

For $JP(n)$, Martinez and Zelmanov presented four unital irreducible bimodules: regular and $P(n-1)$ together with their opposites.

MZ showed that $Lie(M)$ decomposes into a direct sum of eigenspaces with respect to certain Cartan subalgebra and using weight arguments they proved that for $n \geq 3$ there are at most four irreducible modules in $\mathcal{S}$.

$\widehat{P(3)} = P(3) + \mathbf{k}$.

For $JP(2)$ again using Lie algebra arguments MZ showed that for an arbitrary $t \in \mathbf{k}$ there are at most four (two + opposite) non-isomorphic unital irreducible finite-dimensional modules of level t (of central charge t, the central element z acts as t on $Lie(M)$).

Finally, MZ gave the explicit realization of two $JP(2)$-bimodules of level t as submodules of $M_{2,2}(W)^+$, where $W = \sum_{i\geq 0}(\mathbf{k}1 + \mathbf{k}a)d^i = \mathbf{k}[d] + a\mathbf{k}[d]$, the Weyl algebra of the differential algebra $(\mathbf{k}1 + \mathbf{k}a, d)$, $da - ad = d(a) = ta$.

Let $\mathfrak{g} = \hat{P}(3)$ be the central extension of the simple Lie superalgebra $P(3)$. There is a consistent (with $\mathbb{Z}_2$-grading) $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{-2}$ is a one-dimensional center, $\mathfrak{g}_0$ is isomorphic to $\mathfrak{so}_6$ and $\mathfrak{g}_{-1}$ is the standard $\mathfrak{so}_6$-module.

Fix $z \in \mathfrak{g}_{-2}$. On $V = \mathbf{k}^{4|4}$ define $\rho_t : \mathfrak{g} \to \mathrm{End}_{\mathbf{k}}(V)$ by

$$\rho_t \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} := \begin{bmatrix} A & B + tC^* \\ C & -A^t \end{bmatrix}, \quad \rho(z) := t,$$

where $c^*_{ij} = (-1)^\sigma c_{kl}$ for $\sigma = \{1,2,3,4\} \to \{i,j,k,l\}$. Denote the corresponding $\mathfrak{g}$-module by $V(t)$. When $t = 0$ this module coincides with the standard $\mathfrak{g}$-module.

Proposition

- *Any simple object in the category* $\mathcal{S}_{\frac{1}{2}}$ *is isomorphic to* $V(t)$ *or* $V(t)^{op}$ *for some* $t \in \mathbb{C}$.
- *Every block in the category* $\mathcal{S}_{\frac{1}{2}}$ *has a unique simple object.*
- *The category* $\mathcal{S}^t_{\frac{1}{2}}$ *is equivalent to the direct sum of two copies of the category of finite-dimensional representations of the polynomial ring* $\mathbb{C}[x]$.

$$V(t/2) \otimes V(t/2) = S^2 V(t/2) \oplus \Lambda^2 V(t/2).$$

Then clearly both $S^2 V(t/2)$ and $\Lambda^2 V(t/2)$ are objects in $\mathcal{S}$ and have central charge t.

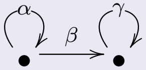
$t \neq 0$

If $t \neq 0$ then $S^2V(t/2)$ and $\Lambda^2V(t/2)$ are simple.
Let $\mathfrak{g} - \text{mod}_1^t$ be the full subcategory of $\mathcal{S}$ consisting of modules on which z acts with eigenvalue t.

Theorem

- *The category $\mathfrak{g} - mod_1^t$ has two equivalent blocks Ω_t^+ and Ω_t^-. Ω_t^+ has two simple objects $S^2V(t/2)$ and $\Lambda^2V(t/2)$.*
- *For every block $\Omega_t^{\pm}$ two corresponding projective modules are consructed.*
- *The category Ω_t^+ is equivalent to the category of nilpotent representations of the quiver Q with relation $\beta\alpha = \gamma\beta$*

Q :

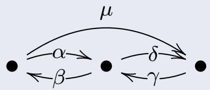


If $t = 0$ both $S^2V(t/2)$ and $\Lambda^2V(t/2)$ have simple submodules of codimension 1, denote them $\Lambda^{\pm}(0)$.

Theorem

- *The category* $\mathfrak{g} - mod_1^0$ *has two equivalent blocks* Ω_0^+ *and* Ω_0^-. Ω_0^+ *has three simple objects* $\Lambda^{\pm}(0)$ *and* $\mathbf{k}^{op}$.
- *The category* Ω_0^+ *is equivalent to the category of nilpotent representations of the quiver* Q

Q :



modulo some relations. These relations include $\delta\alpha = \beta\gamma = 0$, $\mu\beta\alpha = \delta\gamma\mu$.