Minicourse: Universality in the ϵ-expansion

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Universality in the ϵ-expansion: overview

- The primary goal of RG analysis is the study of **universality classes** (UC) in arbitrary dimension
- **IR**: **classify phase transition** and dictate the structure of phase diagrams
- **UV**: needed to construct continuum limits, **asymptotic freedom and safety**
- Universal quantitative properties: **CFT data** (most UCs are CFTs!)
- In general a universality class (UC) is characterized by an **upper critical dimension** d_c where it is described by mean field theory (MFT) and close to which we can perform the **ϵ-expansion**
- Most UCs have **fractional** d_c and appear for the first time at a loop order $L_{\rm LO}$ **generally bigger than one... many UCs have been missed!**
- Lesson from Non-Perturbative RG (NPRG): **be functional**! Smart way to access **CFT data in the ϵ-expansion**
- **Multicomponent world still mostly uncharted!**

Loop Expansion for the Effective Action

- Action for a general **single component scalar theory** in d-dimensions

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + V(\phi) \right\}$$

- Effective action in the **loop expansion**

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log S^{(2)} + \frac{1}{8} S^{(4)}_{xyzw} G_{xy} G_{zw} - \frac{1}{12} S^{(3)}_{xyz} S^{(3)}_{abc} G_{xa} G_{yb} G_{zc} + \cdots$$

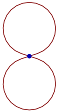
  where $G_{xy} \equiv (S^{(2)})^{-1}_{xy}$

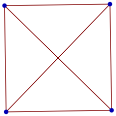
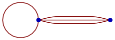
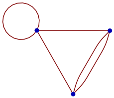
- We will use **dimensional regularization** $\overline{\mathrm{MS}}$ scheme in

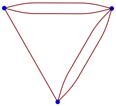
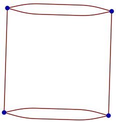
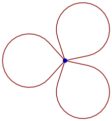
$$d = d_c - \epsilon$$

  In single scalar perturbation theory d_c labels `universality classes`

- Work with **renormalized loop expansion** to account for counter terms and sub-divergencies

$\Gamma_2 = \frac{1}{8}$  $-\frac{1}{12}$ 

$\Gamma_3 = -\frac{1}{24}$  $+\frac{1}{8}$  $+\frac{1}{8}$  $-\frac{1}{48}$ 

$+\frac{1}{8}$  $-\frac{1}{16}$  $+\frac{1}{48}$  $-\frac{1}{16}$ 

(EXERCISE Derive the three loop diagrams with their coefficients)

Beta Functionals: definition

- **Beta functionals** β_V, β_Z describing the RG flow of the potential V and of the wave function renormalization function Z can be computed from the $\frac{1}{\epsilon}$-poles of the effective action

$$\Gamma^{(L)}_{\mathrm{div}} = -\frac{1/L}{\epsilon} \int \mathrm{d}^{d_c} x \left\{ \beta_V^{(L)}(\phi) + \beta_Z^{(L)}(\phi) \frac{1}{2} (\partial\phi)^2 + \cdots \right\} + O\left(\frac{1}{\epsilon^2}\right)$$

- The complete beta functionals are the sum of all the loop contributions

$$\beta_V = \sum_L \beta_V^{(L)} \qquad\qquad \beta_Z = \sum_L \beta_Z^{(L)}$$

- We will see that the coefficients of the LO and NLO beta functionals are universal and constitutes the *universal data* of the critical QFT.
- We focus on the **relevant + marginal** sector of the theory so that β_V and β_Z are composed solely with the potential V
- For a general universality class the LO and NLO contributions to β_V do not coincide with the one and two loop contributions (`Ising` and `Lee-Yang` are special in this respect) and and are **hidden in the loop expansion...** We need to determine the order L_{LO} where the first contribution appears. L_{LO} represents also the periodicity of the contributions since $L_{\mathrm{NLO}} = 2L_{\mathrm{LO}}$, $L_{\mathrm{NNLO}} = 3L_{\mathrm{LO}}$ and so on.

Which are the possible d_c?

- When the background field is taken constant this gives a loop expansion for the effective potential and the coefficients of the $\frac{1}{\epsilon}$ poles are in on-to-one correspondence with the loop contributions to beta functional β_V
- Since all diagrams are vacuum diagrams we must have (remember we are looking for $\frac{1}{\epsilon}$ poles)

$$d_c L = 2P + 2k$$

where P is the number of propagators and $k = 0, 1, 2, 3, ...$ is the number of $V^{(2)}$ insertions (since each increases by one the number of propagators). Thus $d_c = 2(P + k)/L$. The minimum number of propagators is $P_{\rm min} = L + 1$ (coming from the L-sun diagram) apart the one loop case where $P_{\rm min} = 1$. Thus we have

$$d_c = 2 + \frac{2}{L} k \qquad\qquad k = 0, 1, 2, 3, ...$$

where the $k = 0$ case is present only when $L = 1$

- Solving for L shows that a universality class receives contributions at every loop order

$$L = \frac{2k}{d_c - 2}$$

for which the rhs is an integer

- Solutions are multiples of $L_{\rm LO}$, where $L_{\rm LO}$ is the critical loop order at which a universality class first appears

Which are the possible d_c?

$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
2					
4	3	$\frac{8}{3}$	$\frac{5}{2}$	$\frac{12}{5}$	$\frac{7}{3}$
6	4	$\frac{10}{3}$	3	$\frac{14}{5}$	$\frac{8}{3}$
	5	4	$\frac{7}{2}$	$\frac{16}{5}$	3
	6	$\frac{14}{3}$	4	$\frac{18}{5}$	$\frac{10}{3}$
		$\frac{16}{3}$	$\frac{9}{2}$	4	$\frac{11}{3}$
		6	5	$\frac{22}{5}$	4
			$\frac{11}{2}$	$\frac{24}{5}$	$\frac{13}{3}$
			6	$\frac{26}{5}$	$\frac{14}{3}$
				$\frac{28}{5}$	5
				6	$\frac{16}{3}$
					$\frac{17}{3}$
					6

Landau-Ginzburg potentials

$L=1$					
φ^{∞}	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
φ^4	φ^6	φ^8	φ^{10}	φ^{12}	φ^{14}
φ^3	φ^4	φ^5	φ^6	φ^7	φ^8
		φ^4			φ^6
	φ^3		φ^4		φ^5
				φ^4	
		φ^3			φ^4
			φ^3		
				φ^3	
					φ^3

Multicritical models

- Landau-Ginzburg lagrangian for multicritical models

$$S[\phi] = \int \mathrm{d}^{d_c} x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{g}{m!} \phi^m \right\}$$

  for m a natural number $m \geq 3$

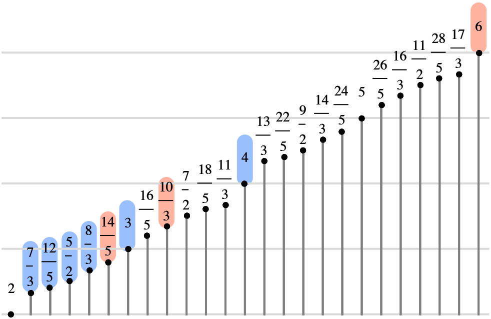
- **Even models** $m = 2n$: they are the so-called unitary multicritical models which are protected by a $\mathbb{Z}_2$ parity ($\phi \to -\phi$) and include both the `Ising` ($m = 4$) and `Tricritical` ($m = 6$) universality classes as the first special cases

- **Odd models** $m = 2n + 1$: represents a sequence of multicritical non-unitary theories which are protected by a generalization of parity $\mathcal{PT} : S[\phi] \to S[-\phi]^\star$ and include the `Lee-Yang` universality class ($m = 3$) as first example

- Even models are known to interpolate in $d = 2$ with the unitary minimal CFTs $\mathcal{M}_{p,p+1}$ for $p = 1 + m/2$, which arise from the representations of the infinite dimensional Virasoro algebra

- Similarly, there are speculations pointing at the fact that the non-unitary models might interpolate with the sequence of minimal non-unitary multicritical theories $\mathcal{M}_{2,m+2}$. This is established for the `Lee-Yang` case $m = 3$ (Cardy)

Minimal models and their range of existence



Allowed $2 \leq d_c \leq 6$ with integer LG lagrangian with $L_{\mathrm{LO}} \leq 6$: unitary and non-unitary minimal models

- A UC is generally present in all dimensions $2 \leq d \leq d_c$ and in particular for all integer dimensions it the range (caveat: if it does not become first order...)
- The only non-trivial UCs in $d = 3$ are $d_c = 6, 4, \frac{10}{3}$ while the only non-trivial in $d = 4$ is $d_c = 6$
- Universal quantities $U(d)$ are such in all the range $2 \leq d \leq d_c$. $U(2)$ is known from CFT, $U(d_c)$ from MFT (i.e. `Gaussian`), $U'(d_c)$ and in some cases $U''(d_c)$ from ϵ-expansion. General form of $U(d)$ demands non-perturbative methods: **NPRG** or **Conformal Bootstrap**

Beta functionals: functional form of β_V

- **We can determine the functional form of the beta functional β_V given d_c**
- We first determine the *vertex set* at the given loop order (i.e. the set of possible vertices products contributing a $\frac{1}{\epsilon}$-pole) The L–loop contribution to the functional β_V is a monomial built out of the possible vertices in the vertex set of order L
- The two loop vertex set is $\{(V^{(3)})^2\}$ while at three loop it is $\{(V^{(3)})^4, (V^{(3)})^2 V^{(4)}, (V^{(4)})^2, V^{(5)} V^{(3)}\}$
- Note that the information included in the vertex set is smaller than the information implied by the knowledge of the diagrams, since topologically different diagrams can have the same vertex set
- At a given loop order L the full vertex set is

$$\left\{(V^{(3)})^{2(L-1)}, \ldots, (V^{(L+1)})^2\right\}$$

since a given graph must satisfy

$$V - P + L = 1 \quad \Rightarrow \quad V = P - (L-1)$$

due to the fact that they are all closed vacuum diagrams, and topology demands them to have Euler character one

- We thus need to consider the partitions of $2P$ of length V for $P_{\min} = L+1 \leq P \leq P_{\max} = 3(L-1)$
- It's easy in this way to construct the $L = 4$ vertex set

$$\left\{(V^{(3)})^6, (V^{(3)})^4 V^{(4)}, (V^{(3)})^2 (V^{(4)})^2, (V^{(3)})^3 (V^{(5)}), (V^{(4)})^3, V^{(3)} V^{(4)} V^{(5)}, (V^{(3)})^2 (V^{(6)}), (V^{(5)})^2, V^{(7)} V^{(3)}\right\}$$

as one can also check explicitly from the knowledge of four loop diagrams
(**EXERCISE**: draw all four loop diagrams)

- In dimensional regularization diagrams with bubbles do not contribute. We also remove irrelevant contributions

Beta functionals: functional form of β_V

- Once the vertex set is known we can determine which monomials (with additional $(V^{(2)})^k$ insertions) contribute to β_V for given d_c and L since the dimension of a vertex is

$$[V^{(n)}] = [V][\phi]^{-n} = d - n\left(\frac{d}{2} - 1\right) = d\left(1 - \frac{n}{2}\right) + n$$

and each monomial must match the dimension of β_V which is obviously d_c

- Clearly at one loop the beta functional can only be of the form $(V^{(2)})^{\frac{d_c}{2}}$ and can be present only when d_c is even. At two loop it must be of the form $(V^{(2)})^k(V^{(3)})^2$ with k integer solution of

$$2\left[d_c\left(1 - \frac{3}{2}\right) + 3\right] + 2k = d_c$$

At three loop the vertex set contains three elements. One consider the following relations

$$
\begin{array}{ll}
(V^{(2)})^k(V^{(3)})^4 & 2k + 4\left[d_c\left(1 - \frac{3}{2}\right) + 3\right] = d_c \\
(V^{(2)})^k(V^{(3)})^2 V^{(4)} & 2k + 2\left[d_c\left(1 - \frac{3}{2}\right) + 3\right] + \left[d_c\left(1 - \frac{4}{2}\right) + 4\right] = d_c \\
(V^{(2)})^k(V^{(4)})^2 & 2k + 2\left[d_c\left(1 - \frac{4}{2}\right) + 4\right] = d_c
\end{array}
$$

and looks for solutions with $k \in \mathbb{N}$

Beta functionals: functional form of β_V

- `Ising` $(d_c = 4)$

$$\beta_V = \underbrace{a(V^{(2)})^2}_{L=1} + \underbrace{bV^{(2)}(V^{(3)})^2}_{L=2} + \underbrace{c_1(V^{(3)})^4 + c_2V^{(2)}(V^{(3)})^2V^{(4)} + c_3(V^{(2)})^2(V^{(4)})^2}_{L=3} + \dots$$

- `Lee-Yang` $(d_c = 6)$

$$\beta_V = \underbrace{a(V^{(2)})^3}_{L=1} + \underbrace{b(V^{(2)})^3(V^{(3)})^2}_{L=2} + \underbrace{c_1(V^{(2)})^3(V^{(3)})^4 + c_2(V^{(2)})^4(V^{(3)})^2V^{(4)} + c_3(V^{(2)})^5(V^{(4)})^2}_{L=3} + \dots$$

- `Tri-Lee-Yang` $(d_c = \frac{10}{3})$

$$\beta_V = \underbrace{A_1(V^{(3)})^2V^{(4)} + A_2V^{(2)}(V^{(4)})^2}_{L=3} + \dots$$

- `Tricritical` $(d_c = 3)$

$$\beta_V = \underbrace{a(V^{(3)})^2}_{L=2} + \underbrace{b_1V^{(3)}V^{(4)}V^{(5)} + b_2(V^{(4)})^3 + b_3V^{(2)}(V^{(5)})^2 + b_4V^{(3)}V^{(7)}}_{L=4} + \dots$$

- Similar expression can be easily derived for any d_c and for any order N^{k}LO (**EXERCISE**: determine the form of LO and NLO β_V for $d_c = \frac{8}{3}, \frac{5}{2}$)

Beta functionals: functional form of β_Z

- S induces a running of the couplings of the operators $\phi^k(\partial\phi)^2$ or, equivalently, a running of the anomalous dimensional functional $Z(\phi)$ described by β_Z
- The analysis of the functional form of β_Z proceeds as for β_V apart the fact that the vertex set is expanded, since we now look at the two point function. At two loop the procedure gives

$$(V^{(3)})^2 \qquad \to \qquad \{(V^{(4)})^2, V^{(3)}V^{(5)}, (V^{(3)})^2V^{(4)}, (V^{(3)})^4\}$$

  Clearly, the dimension of the monomials must now be zero. The factors $(V^{(2)})^k$ are not present in the anomalous dimension beta functional
- `Ising`

$$\beta_Z = \underbrace{c(V^{(4)})^2}_{L=2} + \dots$$

- `Lee-Yang`

$$\beta_Z = \underbrace{c(V^{(3)})^2}_{L=1} + \underbrace{d(V^{(3)})^4}_{L=2} + \dots$$

- `Tricritical`

$$\beta_Z = \underbrace{c(V^{(6)})^2}_{L=4} + \dots$$

The Atlas of Universality Classes for $N = 1$ (β_V)

$(4\pi)^{\frac{d_c}{2}L}\beta_V$	d_c	$L=1$	$L=2$	$L=3$
`Sine-Gordon`	2	$-V^{(2)}$	0	0
`Tetracritical`	$\frac{8}{3}$	0	0	$\frac{1}{8}\Gamma(\frac{1}{3})^3(V^{(4)})^2$
`Tricritical`	3	0	$\frac{1}{3}\Gamma(\frac{1}{2})^2(V^{(3)})^2$	0
`Tri-Lee-Yang`	$\frac{10}{3}$	0	0	$\frac{\Gamma(\frac{1}{2})^4\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})^2}[\frac{1}{9}V^{(2)}(V^{(4)})^2 - \frac{1}{2}(V^{(3)})^2V^{(4)}]$
`Ising`	4	$\frac{1}{2}(V^{(2)})^2$	$-\frac{1}{2}V^{(2)}(V^{(3)})^2$	$\frac{1}{16}(V^{(2)})^2(V^{(4)})^2+\frac{7}{4}V^{(2)}(V^{(3)})^2V^{(4)}-\frac{1-4\zeta_3}{8}(V^{(3)})^4$
`Lee-Yang`	6	$-\frac{1}{6}(V^{(2)})^3$	$-\frac{23}{144}(V^{(2)})^3(V^{(3)})^2$	$-\frac{1595+432\zeta_3}{5184}(V^{(2)})^3(V^{(3)})^4$

$(4\pi)^{\frac{d_c}{2}L}\,\beta_Z$	d_c	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
Sine-Gordon	2	0	0	0	0	0
Tetracritical	$\frac{8}{3}$	0	0	0	0	$-\frac{\Gamma(\frac{1}{3})^6}{1120}(V^{(8)})^4$
Tricritical	3	0	0	0	$-\frac{\Gamma(\frac{1}{2})^4}{45}(V^{(6)})^4$	0
Tri-Lee-Yang	$\frac{10}{3}$	0	0	$-\frac{3}{40}\Gamma(\frac{2}{3})^3(V^{(5)})^4$	0	$\dots$
Ising	4	0	$-\frac{1}{6}(V^{(4)})^2$	$\frac{1}{8}(V^{(4)})^3$	$-\frac{65}{96}(V^{(4)})^4$	$\dots$
Lee-Yang	6	$-\frac{1}{6}(V^{(3)})^2$	$-\frac{13}{216}(V^{(3)})^4$	$-\frac{5195-2592\zeta_3}{31104}(V^{(3)})^6$	$\dots$	$\dots$

Beta functionals: determining the LO and NLO coefficients

- **The LO and NLO order contributions are scheme independent** To see this we just take $V(\phi) = \frac{g}{m!}\phi^m$ so that the beta functionals are projected to the beta functions of the respective critical coupling which we already know has LO and NLO universal beta function coefficients.
- **These coefficients are the *universal ϵ-expansion data* of the theory**
- **Example**: computing `Ising` LO and NLO universal coefficients

$$\beta_V = a(V'')^2 + bV''(V''')^2$$

Even if they can be easily determined by matching with standard perturbative RG here we re-compute them in a different way offered by the functional constraint which does not involve the critical `Ising` coupling $\frac{\lambda_4}{4!}\varphi^4$

1. The LO contribution can be extracted from the free theory alone $V = \frac{\lambda_2}{2}\varphi^2$ if we look at the flow of the vacuum energy since $\beta_0 = a\lambda_2^2$. This is trivial since the one loop contribution to the vacuum energy is just one half the number of degrees of freedom $a = \frac{1}{2}\frac{1}{(4\pi)^2}$
2. The NLO coefficient can be determined considering $V = \frac{1}{2}\lambda_2\varphi^2 + \frac{1}{3!}\lambda_3\varphi^3$ from $\beta_0 = a\lambda_2^2 + b\lambda_2\lambda_3^2$ which comes from the sunset diagram

$$\mathrm{sunset} = -\frac{1}{12}\frac{\lambda_3^2}{(4\pi)^d}\mathrm{Vol}\int_0^\infty \frac{ds_1\,ds_2\,ds_3}{(s_1s_2+s_1s_3+s_2s_3)^{\frac{d}{2}}}e^{-\lambda_2(s_1+s_2+s_3)} \to \frac{1}{4\epsilon}\frac{\lambda_2\lambda_3^2}{(4\pi)^4}\mathrm{Vol}$$

This corresponds to a two loop vacuum renormalization $\beta_0^{II} = -\frac{1}{2}\frac{\lambda_2\lambda_3^2}{(4\pi)^4}$ and thus $b = -\frac{1}{2}\frac{1}{(4\pi)^4}$

- **EXERCISE**: do the sunset integral

Computing the anomalous dimension

- The simplest way to compute c uses its direct relation with the anomalous dimension $\eta = -\beta_Z = -c\lambda_4^2$ when the potential contains only the marginal coupling $V = \frac{1}{4!}\lambda_4\phi^4$
- The method is the SDE+CFT even if in this simple application we don't use any CFT property but just scale invariance at the fixed point
- The two point function at the fixed point in $d = 4$ has the form

$$\langle\phi(x)\phi(y)\rangle = \frac{\alpha}{|x-y|^{2+\eta}} \tag{1}$$

  where α is a non-trivial constant in the interacting theory
- We will act with two Laplacians on (1) and treat the rhs directly while the lhs with SDE. The rhs gives

$$\partial_x^2\partial_y^2 \frac{\alpha}{|x-y|^{2+\eta}} = \frac{\alpha(2+\eta)^2(4\eta+\eta^2)}{|x-y|^{6+\eta}} \to 16\eta\frac{\alpha_{\rm G}}{|x-y|^6} + O(\eta^2) \tag{2}$$

  where $\alpha_{\rm G} = \frac{1}{4\pi^2}$ is the Gaussian value
- Using the EoM $\partial^2\phi = \frac{\lambda_4}{3!}\phi^3$ in the lhs of (1) after the action of the two Laplacian gives instead

$$\left\langle\partial_x^2\phi(x)\partial_y^2\phi(y)\right\rangle = \left(\frac{\lambda_4}{3!}\right)^2\left\langle\phi^3(x)\phi^3(y)\right\rangle \to \frac{\lambda_4^2}{3!}\frac{\alpha_{\rm G}^3}{|x-y|^6} \tag{3}$$

  We used Wick's theorem: $\left\langle\phi^3(x)\phi^3(y)\right\rangle = \langle\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\rangle = 3!(\langle\phi(x)\phi(y)\rangle)^3$
- Consistency between (2) and (3) demands $16\,\alpha_{\rm G}\,\eta = \frac{1}{6}\alpha_{\rm G}^3\lambda_4^2$ which gives $\eta = \frac{\lambda_4^2}{6(4\pi)^4}$ and thus $c = -\frac{1}{6}\frac{1}{(4\pi)^4}$
- **EXERCISE**: check the calculation

Beta functions from Beta functionals

- Ising dimensionful beta functionals

$$
\begin{aligned}
\beta_V &= \frac{1}{(4\pi)^2}\frac{1}{2}(V^{(2)})^2 - \frac{1}{(4\pi)^4}\frac{1}{2}V^{(2)}(V^{(3)})^2 + \cdots \\
\beta_Z &= -\frac{1}{(4\pi)^4}\frac{1}{6}(V^{(4)})^2 + \cdots
\end{aligned}
$$

- Dimensionless variables: $v(\varphi) \equiv (4\pi)^2\mu^{-(4-\epsilon)}V(\varphi\mu^{(2-\epsilon+\eta)/2})$ and $z(\varphi) = \mu^\eta Z(\varphi\mu^{(2-\epsilon+\eta)/2})$

$$
\beta_v = -4v + \varphi v' + \epsilon\left(v - \frac{1}{2}\varphi v'\right) + \frac{1}{2}\eta\varphi v' + \frac{1}{2}(v'')^2 - \frac{1}{2}v''(v''')^2
$$

$$
\beta_z = \eta z + \varphi z' - \frac{\epsilon}{2}\varphi z' + \frac{1}{2}\eta\varphi z' - \frac{1}{6}(v^{(4)})^2
$$

Note: this is the step where we introduce ϵ!

- Expand the potential in Taylor series ($\mathbb{Z}_2$ even and odd operators)

$$
v(\varphi) = \sum_{n=1}^{\infty}\frac{\lambda_n}{n!}\varphi^n
$$

then the beta functional β_V becomes *generating function of beta functions*

- The anomalous dimension obtains from the normalization $z(0) = 1$ and is $\eta = \frac{\lambda_4^2}{6}$.

Beta functions

- Beta function system

$$
\begin{aligned}
\beta_1 &= -\left(3-\frac{\epsilon}{2}\right)\lambda_1+\lambda_2\lambda_3-\frac{1}{2}\lambda_3^3-\lambda_2\lambda_3\lambda_4+\frac{1}{12}\lambda_1\lambda_4^2\\
\beta_2 &= -2\lambda_2+\lambda_4\lambda_2+\lambda_3^2-\frac{5}{2}\lambda_3^2\lambda_4-\frac{5}{6}\lambda_2\lambda_4^2-\lambda_2\lambda_3\lambda_5\\
\beta_3 &= -\left(1+\frac{\epsilon}{2}\right)\lambda_3+3\lambda_4\lambda_3-\frac{23}{4}\lambda_3\lambda_4^2+\lambda_2\lambda_5-\frac{7}{2}\lambda_3^2\lambda_5-3\lambda_2\lambda_4\lambda_5-\lambda_2\lambda_3\lambda_6\\
\beta_4 &= -\epsilon\lambda_4+3\lambda_4^2-\frac{17}{3}\lambda_4^3+4\lambda_3\lambda_5-3\lambda_2\lambda_5^2+\lambda_2\lambda_6-\frac{9}{2}\lambda_3^2\lambda_6-22\lambda_3\lambda_4\lambda_5\\
&\quad -4\lambda_2\lambda_4\lambda_6-\lambda_2\lambda_3\lambda_7\\
\beta_5 &= \ldots
\end{aligned}
$$

- $\frac{1}{4!}\lambda_4\phi^4$ two loop beta function recovered
- In dimensional regularization the fixed point is very simple:

$$
\beta_i=0 \qquad \Rightarrow \qquad \lambda_i^*=\left(\frac{\epsilon}{3}+\frac{17\epsilon^2}{81}\right)\delta_{i,4}
$$

Beta functions

- Expand around the fixed point: $\lambda_i = \lambda_i^* + \delta\lambda_i$ with $\Lambda_i \equiv i!\,\delta\lambda_i$

$$
\begin{aligned}
\partial_t \Lambda_1 &= -\left(3 - \frac{\epsilon}{2} - \frac{\epsilon^2}{108}\right)\Lambda_1 + 12\left(1 - \frac{\epsilon}{3}\right)\Lambda_2\Lambda_3 + \frac{4}{3}\epsilon\,\Lambda_1\Lambda_4 + \dots \\
\partial_t \Lambda_2 &= -\left(2 - \frac{\epsilon}{3} - \frac{19\epsilon^2}{162}\right)\Lambda_2 + 24\left(1 - \frac{5}{9}\epsilon\right)\Lambda_2\Lambda_4 + 18\left(1 - \frac{5}{6}\epsilon\right)\Lambda_3^2 + \dots \\
\partial_t \Lambda_3 &= -\left(1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{108}\right)\Lambda_3 + 72\left(1 - \frac{23}{18}\epsilon\right)\Lambda_3\Lambda_4 + \dots \\
\partial_t \Lambda_4 &= -\left(-\epsilon + \frac{17\epsilon^2}{27}\right)\Lambda_4 + 72\left(1 - \frac{17}{9}\epsilon\right)\Lambda_4^2 + \dots
\end{aligned}
$$

- CFT data encoded in the beta functions:
  1. RG eigenvalues θ_i Related to the scaling dimension $\Delta_i = d - \theta_i$ of the composite operators ϕ^i
  2. `Gaussian` OPE coefficients Obtainable from Wick's theorem at $d = d_c$
  3. RG OPE coefficients ${c^a}_{bc}$ Note: $O(\epsilon^2)$ terms receive contributions from NNLO and are omitted

CFT data in the ϵ-expansion

- General form of beta functions around a FP in CFT perturbation theory

$$\partial_t \Lambda^a = -(d-\Delta_a)\Lambda^a + \sum_{b,c} C^a{}_{bc}\, \Lambda^b \Lambda^c + O(\Lambda^3)$$

transforms under a change of variables $\tilde{\Lambda}_i = \Lambda_i + \cdots$ as:

1. The spectrum is universal

$$\tilde{\Delta}_a = \Delta_a$$

   and up to $O(\epsilon^2)$ is scheme independent
2. The $c^i{}_{jk}$ are not universal away from d_c

$$\tilde{c}^c{}_{ab} = c^c{}_{ab} + \frac{1}{2}\left(\theta_c - \theta_a - \theta_b\right) \frac{\partial^2 \Lambda_c}{\partial \tilde{\Lambda}_p \partial \tilde{\Lambda}_q} \frac{\partial \tilde{\Lambda}_p}{\partial \Lambda_a} \frac{\partial \tilde{\Lambda}_q}{\partial \Lambda_b}$$

   but are scheme independent up to order $O(\epsilon)$

- LO and NLO beta functionals are scheme independent (i.e. a, b, c are scheme independent)

$$\beta_V = a(V^{(2)})^2 + bV^{(2)}(V^{(3)})^2 + \cdots \qquad \beta_Z = c(V^{(4)})^2 + \cdots$$

- $\Rightarrow$ θ_i are scheme independent up to order $O(\epsilon^2)$
- $\Rightarrow$ $c^i{}_{jk}$ are scheme independent up to order $O(\epsilon)$ (since NLO terms enter at order $O(\epsilon^2)$)

SDE+CFT

- FPRG gives a coherent description of multicritical models in the ϵ-expansion (Codello+Safari+Vacca+Zanusso)[arXiv:1705.05558]
- RG OPE coefficients are scheme independent at order $O(\epsilon)$ ...
- **... can we compare with other analytical approaches?**
- $\Rightarrow$ **use SDE+CFT to study multicritical models**
  (Codello+Safari+Vacca+Zanusso)[arXiv:1703.04830;1809.05071]
  We generalize the arguments made for `Tricritical` ($m = 3$) in (Nii)(Hasegawa+Nakayama), for `Ising` ($m = 4$) in (Rychkov+Tan)(Nakayama)(Nii) and for `Lee-Yang` ($m = 6$) in (Basu+Krishnan)(Nii), and assume that for each value of m the multicritical models at the critical point are CFTs for any dimension $2 \leq d \leq d_c$
- **SDE+CFT is an elegant method for the computation of CFT data at LO in the ϵ-expansion**
- The key idea is that all the CFT data must interpolate with that of the `Gaussian` theory in the limit $\epsilon \to 0$.
- Achieve consistency between conformal symmetry and the equations of motion through the use of Schwinger-Dyson equations

RG OPE $\cap$ CFT OPE

- `Ising`

  Relevant:

$$c^1{}_{23} = 6 - 2\epsilon \qquad c^2{}_{33} = 18 - 15\epsilon$$

  Marginal:

$$c^1{}_{14} = \frac{2}{3}\epsilon$$

- `Tricritical`

  Relevant:

$$c^1{}_{25} = 6\epsilon \qquad c^1{}_{34} = 24 - \frac{72}{5}\epsilon \qquad c^2{}_{35} = 60 - 90\epsilon$$

$$c^2{}_{44} = 96 - \frac{18}{5}(32 + 3\pi^2)\epsilon \qquad c^3{}_{45} = 240 - 6(98 + 9\pi^2)\epsilon \qquad c^4{}_{55} = 600 - 15(167 + 18\pi^2)\epsilon$$

  Marginal:

$$c^1{}_{16} = \frac{6}{5}\epsilon \qquad c^2{}_{26} = \frac{192}{5}\epsilon$$

- `Tetracritical`

  Relevant:

$$c^1{}_{27} = \frac{36}{5}\epsilon \qquad c^1{}_{36} = \frac{972}{35}\epsilon \qquad c^1{}_{45} = 120 - \frac{432}{7}\epsilon \qquad \cdots$$

  Marginal:

$$c^1{}_{18} = \frac{72}{35}\epsilon \qquad c^2{}_{28} = \frac{432}{7}\epsilon \qquad c^3{}_{38} = \frac{60696}{35}\epsilon$$

- **OPE coefficients computed with RG and SDE+CFT agree in all cases considered!**

Lee-Yang $[d_c = 6\ ;\ L_{\mathrm{LO}} = 1]$

- **Only non-trivial single scalar UC in $d = 4$... but is non-unitary** $\lambda_3^* \sim \sqrt{-\epsilon}$
- Beta functionals (LO+NLO)

$$\beta_V = -\frac{1}{6}\frac{(V^{(2)})^3}{(4\pi)^3} - \frac{23}{144}\frac{(V^{(2)})^3(V^{(3)})^2}{(4\pi)^6} \qquad \beta_Z = -\frac{1}{6}\frac{(V^{(3)})^2}{(4\pi)^3} - \frac{13}{216}\frac{(V^{(3)})^4}{(4\pi)^6}$$

- Spectrum (LO+NLO)

$$\gamma_i = \frac{1}{18} i(6i-7)\epsilon - \frac{1}{2916} i(414i^2 - 1371i + 1043)\epsilon^2 - \left(\frac{\epsilon}{3} + \frac{47}{486}\epsilon^2\right)\delta_{i,3}$$

- RG OPE coefficients

$$c^k{}_{ij} = \sqrt{\frac{2}{3}}\sqrt{-\epsilon}\, i(i-1)j(j-1)\delta_{i+j,k+3} + \sqrt{-6\epsilon}\,\delta_{i,3}\delta_{j,3}\delta_{k,3} + \sqrt{\frac{2}{3}}\sqrt{-\epsilon}\,(i\delta_{j,3}\delta_{i,k} + j\delta_{i,3}\delta_{j,k})$$

- First two scheme independent RG OPE coefficients agree with SDE+CFT

$$c^1{}_{2\,2} = -4\sqrt{\frac{2}{3}}\sqrt{-\epsilon} \qquad c^1{}_{1\,3} = \sqrt{\frac{2}{3}}\sqrt{-\epsilon}$$

Tri-Lee-Yang $[d_c = \frac{10}{3}\ ;\ L_{\mathrm{LO}} = 3]$

- **New UC never considered in perturbation theory!**
- Together with Ising and Lee-Yang only non-trivial UC in $d = 3$
- Only requires $\epsilon = \frac{1}{3}$ to reach the nearest physical dimensions. Example of well behaved ϵ-expansion?
- Beta functionals (LO)

$$\beta_V = \frac{1}{(4\pi)^5} \frac{\Gamma(\frac{1}{2})^4 \Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})^2} \left\{ \frac{1}{9} V^{(2)} (V^{(4)})^2 - \frac{1}{2} (V^{(3)})^2 V^{(4)} \right\}$$

$$\beta_Z = -\frac{1}{(4\pi)^5} \frac{3\Gamma(\frac{2}{3})^3}{40} (V^{(5)})^2$$

- Spectrum (LO)

$$\gamma_i = \frac{\epsilon}{153} \left(\frac{52}{5} i - \frac{139}{12} i^2 - \frac{1}{2} i^3 + \frac{19}{12} i^4 - \delta_{i,5} \right) \qquad \frac{\gamma_2}{\gamma_1} = 42 + O(\epsilon)$$

- CFT data at LO agrees with SDE+CFT (these ratios do not depend on $g(\epsilon)$)

$$\frac{c^1{}_{15}}{\sqrt{\gamma_1}} = 4\sqrt{15} + O(\epsilon) \qquad \frac{c^1{}_{24}}{\sqrt{\gamma_1}} = 32\sqrt{15} + O(\epsilon) \qquad \frac{c^1{}_{33}}{\sqrt{\gamma_1}} = -108\sqrt{15} + O(\epsilon)$$

- For more details and numerical estimates (Codello+Safari+Vacca+Zanusso)[arXiv:1706.06887]

- Beta functionals (LO+NLO) (Osborn+Dolan)

$$\begin{aligned}\beta_V &= a_n(V^{(n)})^2 - b_n\left\{\frac{1}{3}\sum_{r+s+t=2n}\frac{K_{rst}}{r!s!t!}\,V^{(r+s)}V^{(s+t)}V^{(t+r)} + \sum_{s+t=n}\frac{L_{st}}{s!t!}\,V^{(n)}V^{(n+s)}V^{(n+t)}\right\}\\ \beta_Z &= c_n(V^{(2n)})^2\end{aligned}$$

where

$$a_n = \frac{1}{(4\pi)^n}\frac{n-1}{n!}\Gamma\left(\frac{1}{n-1}\right)^{n-1} \qquad b_n = \frac{(n-1)^2}{(4\pi)^{2n}}\Gamma\left(\frac{1}{n-1}\right)^{2(n-1)} \qquad c_n = -\frac{4(n-1)^2}{(4\pi)^{2n}(2n)!}\Gamma\left(\frac{1}{n-1}\right)^{2(n-1)}$$

and

$$K_{rst} = \frac{\Gamma(\frac{n-r}{n-1})\Gamma(\frac{n-s}{n-1})\Gamma(\frac{n-t}{n-1})}{\Gamma(\frac{r}{n-1})\Gamma(\frac{s}{n-1})\Gamma(\frac{t}{n-1})} \qquad L_{st} = n-1+\chi\left(\frac{1}{n-1}\right)-\chi\left(\frac{s}{n-1}\right)-\chi\left(\frac{t}{n-1}\right)+\chi(1)$$

- Spectrum (LO+NLO) ($\theta_i = d - \Delta_i = d - i\frac{d-2}{2} - \gamma_i$)

$$\begin{aligned}\gamma_1 &= \frac{2(n-1)^2 n!^6}{(2n)!^3}\epsilon^2\\ \gamma_2 &= \frac{8(n+1)(n-1)^3 n!^6}{(n-2)(2n)!^3}\epsilon^2 && n>2\\ \gamma_i &= \frac{2(n-1)n!}{(2n)!}\frac{i!}{(i-n)!}\epsilon && i\geq n\end{aligned}$$

$\texttt{Sine-Gordon}$ $[d_c = 2 \; ; \; L_{\mathrm{LO}} = 1]$

- $d_c = 2$ corresponds to the Sine-Gordon universality class. To *all orders* the beta functionals are

$$\beta_V = -\frac{1}{4\pi} V'' \qquad\qquad \beta_Z = 0$$

- Flow of the dimensionless potential in $d = 2$ is

$$\beta_v = -2v(\varphi) - \frac{1}{4\pi} v''(\varphi)$$

- Because the field is canonically dimensionless in $d = 2$ fluctuations do not generate a nonzero anomalous dimension
- Interestingly, the fixed point solution of the Sine-Gordon UC can be obtained by direct integration. Using $v''(0) = \sigma$ as boundary condition we obtain

$$v(\varphi) = -\frac{\sigma}{8\pi} \cos(\sqrt{8\pi}\varphi)$$

  in which we can recognize the well-known Coleman phase $\sqrt{8\pi}$

Relation with Non-Perturbative RG (NPRG)

- Exact RG flow equation for the generator of the irreducible 1PI diagrams (Wetterich) (Morris) (Polchinski)

$$\partial_t \Gamma_\mu = \frac{1}{2} \mathrm{Tr} \left(\Gamma_\mu^{(2)} + R_\mu \right) \partial_t R_\mu$$

- We can compute the flow of the effective potential (LPA)

$$\beta_V = \partial_t V = c_d \frac{\mu^{d+2}}{\mu^2 + V''} \qquad\qquad c_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}$$

- Let us expand β_V in powers of V''

$$\beta_V = c_d \left\{ \mu^d - \mu^{d-2} V'' + \mu^{d-4} (V'')^2 - \mu^{d-6} (V'')^3 + \dots \right\}.$$

- Terms independent of μ correspond to the $\frac{1}{\epsilon}$ poles of dimensional regularization

$$\begin{aligned}
\beta_V &= -\, c_2 V'' = -\frac{1}{4\pi} V'' && d = 2 \\
\beta_V &= c_4 (V'')^2 = \frac{1}{2(4\pi)^2} (V'')^2 && d = 4 \\
\beta_V &= -\, c_6 (V'')^3 = -\frac{1}{6(4\pi)^3} (V'')^3 && d = 6
\end{aligned}$$

- **We see** `Ising`, `Lee-Yang` **and** `Sine-Gordon` **but what about all other with $L_{\mathrm{LO}} \geq 2$?**

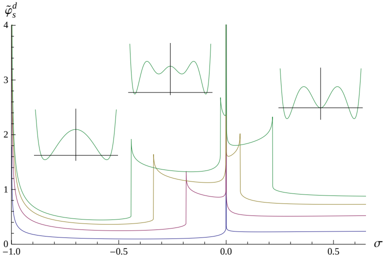
Multicritical models in the LPA

- Scaling solutions are given by the ODE

$$0 = -dv + \left(\frac{d}{2} - 1 + \frac{\eta}{2}\right)\varphi v' + c_d \frac{1 - \frac{\eta}{d+2}}{1 + v''}$$

with initial condition $v'(0) = 0$ and $v''(0) = \sigma$ that encode the $\mathbb{Z}_2$ symmetry

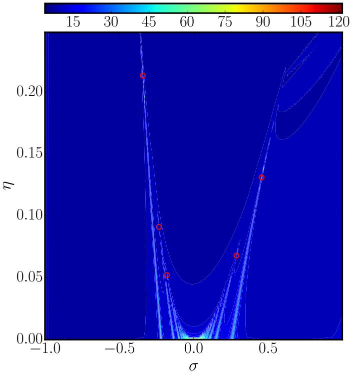
- Spike plot (Morris) (Codello)

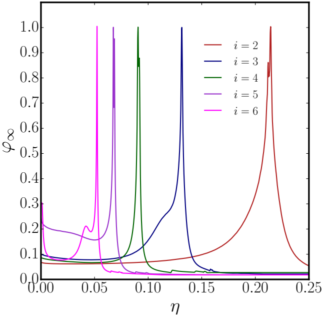


Spike plot for unitary models in $2 \le d < \infty$: `Ising`, `Tricritical`, `Tetracritical`, ...

- `SineGordon`: when $d = 2$ and $\eta = 0$ scaling solutions are periodic functions with Coleman phase $\sqrt{8\pi}$

Multicritical models in the LPA





$d = 2$ unitary models are seen using $O(\partial^2)$ derivative expansion (Morris) (Defenu+Codello)[arXiv:1711.01809]

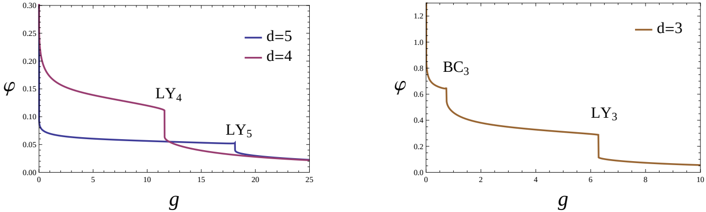
Lee-Yang and Blume Capel in the LPA

- $\mathcal{PT}$ symmetry $v(\varphi) \to v(-\varphi)^*$
- Scaling solutions for complex potential $v(\varphi) = ih(\varphi)$

$$0 = -dh + \left(\frac{d}{2} - 1 + \frac{\eta}{2}\right)\varphi h' + c_d h'' \frac{1 - \frac{\eta}{d+2}}{1 + (h'')^2}$$

with initial conditions $h(0) = 0$ and $h'''(0) = g$

- Spike plot (Zambelli+Zanusso)



Spike plot for non-unitary models in $d = 5, 4, 3$

- **We see all UCs: the LPA really captures information from every loop order!**

Multicomponent world

- Multicomponent world (Osborn+Stergiou)

  "hardly a *terra incognita*, nevertheless there is as yet no *mappa mundi*"

- FPRG is the most agile tool for charting of this vast landscape of theories
- N component scalar order parameter

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2} \partial\phi_i \cdot \partial\phi_i + V(\phi_1, ..., \phi_N) \right\}$$

- **The magic of the functional constraint: (almost) no new computation needed!**
- The universal LO and NLO coefficients are the $N = 1$ in $d_c = 4, 3, \frac{8}{3}, \ldots$ (unitary family)

$$(V^{(2)})^2 \to V_{ij} V_{ij} \qquad\qquad V^{(2)}(V^{(3)})^2 \to V_{ij} V_{iab} V_{jab}$$

- The universal LO coefficients are the $N = 1$ in $d_c = 6, \frac{10}{3}, \ldots$ (non-unitary family). For the NLO coefficients we have only a constraint

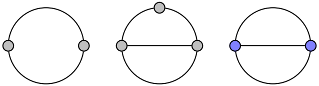
$$(V^{(2)})^3 (V^{(3)})^2 \to \alpha V_{ik} V_{kl} V_{lj} V_{iab} V_{jab} + \beta V_{ik} V_{kj} V_{lm} V_{ija} V_{lmb} + \gamma V_{ia} V_{jb} V_{kc} V_{abc} V_{ijk}$$

  with $\alpha + \beta + \gamma = 1$

- **RG information and group information factorize at LO (and at NLO for unitary family)!**

Master functionals

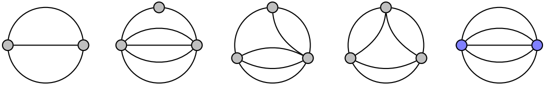
$$d_c = 4$$



$$\beta_V = \frac{1}{2} V_{a_1 a_2} V_{a_1 a_2} - \frac{1}{2} V_{a_1 a_2} V_{a_1 a_3 a_4} V_{a_2 a_3 a_4}$$

$$(\beta_Z)_{a_1 a_2} = -\frac{1}{6} V_{a_1 a_3 a_4 a_5} V_{a_2 a_3 a_4 a_5}$$

$$d_c = 3$$

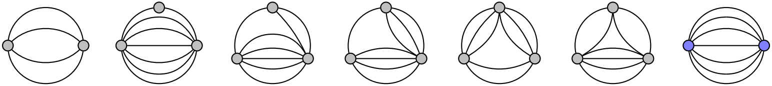


$$\begin{aligned}\beta_V = {} & \frac{1}{3} V_{a_1 a_2 a_3} V_{a_1 a_2 a_3} + \frac{1}{6} V_{a_1 a_2} V_{a_1 a_3 a_4 a_5 a_6} V_{a_2 a_3 a_4 a_5 a_6} \\ & - \frac{4}{3} V_{a_1 a_2 a_3} V_{a_3 a_4 a_5 a_6} V_{a_1 a_2 a_4 a_5 a_6} - \frac{\pi^2}{12} V_{a_1 a_2 a_3 a_4} V_{a_3 a_4 a_5 a_6} V_{a_1 a_2 a_5 a_6}\end{aligned}$$

$$(\beta_Z)_{a_1 a_2} = -\frac{1}{45} V_{a_1 a_3 a_4 a_5 a_6 a_7} V_{a_2 a_3 a_4 a_5 a_6 a_7}$$

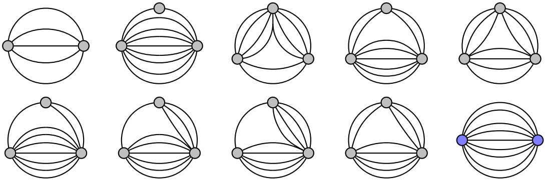
Master functionals

$$d_c = \frac{8}{3}$$



$$\begin{aligned}
\beta_V = {}& \frac{1}{8} V_{a_1 a_2 a_3 a_4} V_{a_1 a_2 a_3 a_4} + \frac{1}{160} V_{a_1 a_2} V_{a_1 a_3 a_4 a_5 a_6 a_7 a_8} V_{a_2 a_3 a_4 a_5 a_6 a_7 a_8} \\
&+ \frac{9}{80} V_{a_1 a_2 a_3} V_{a_3 a_4 a_5 a_6 a_7 a_8} V_{a_1 a_2 a_4 a_5 a_6 a_7 a_8} \\
&- \frac{3}{8} V_{a_1 a_2 a_3 a_4} V_{a_2 a_3 a_4 a_5 a_6 a_7 a_8} V_{a_1 a_5 a_6 a_7 a_8} \\
&- \frac{\Gamma(1/3)^3}{24} V_{a_1 a_2 a_3 a_4 a_5 a_6} V_{a_1 a_2 a_3 a_7 a_8} V_{a_4 a_5 a_6 a_7 a_8} \\
&+ \frac{3}{64} \left[\sqrt{3}\pi - 3(2 + \log 3)\right] V_{a_1 a_2 a_3 a_4} V_{a_1 a_2 a_5 a_6 a_7 a_8} V_{a_3 a_4 a_5 a_6 a_7 a_8} \\
(\beta_Z)_{a_1 a_2} = {}& -\frac{1}{1120} V_{a_1 a_3 a_4 a_5 a_6 a_7 a_8 a_9} V_{a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9}
\end{aligned}$$

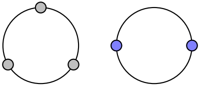
Master functionals: $d_c = \frac{5}{2}$ is 8-loops!



$$\beta_V = \frac{1}{30} V_{a_1 a_2 a_3 a_4 a_5} V_{a_1 a_2 a_3 a_4 a_5} + \frac{1}{7560} V_{a_1 a_2} V_{a_1 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}} V_{a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}$$
$$- \frac{1}{144}\sqrt{\frac{\pi}{2}}\Gamma(1/4)^2\, V_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8} V_{a_1 a_2 a_3 a_4 a_9 a_{10}} V_{a_5 a_6 a_7 a_8 a_9 a_{10}}$$
$$+ \frac{\Gamma(3/4)^2}{45\sqrt{2\pi}} V_{a_1 a_2 a_3 a_4} V_{a_1 a_2 a_5 a_6 a_7 a_8 a_9 a_{10}} V_{a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}$$
$$- \frac{\pi\Gamma(1/4)^2}{216\Gamma(3/4)^2} V_{a_1 a_2 a_3 a_4 a_5 a_6} V_{a_1 a_2 a_3 a_7 a_8 a_9 a_{10}} V_{a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}$$
$$+ \frac{2}{945} V_{a_1 a_2 a_3} V_{a_1 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}} V_{a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}$$
$$+ \frac{2}{135} V_{a_1 a_2 a_3 a_4} V_{a_1 a_5 a_6 a_7 a_8 a_9 a_{10}} V_{a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}$$
$$- \frac{2}{45} V_{a_1 a_2 a_3 a_4 a_5} V_{a_1 a_6 a_7 a_8 a_9 a_{10}} V_{a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}$$
$$+ \frac{1}{45}[-4 + \pi - \log(4)]\, V_{a_1 a_2 a_3 a_4 a_5} V_{a_1 a_2 a_6 a_7 a_8 a_9 a_{10}} V_{a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}$$

Master functionals

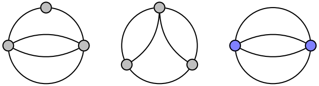
$$d_c = 6$$



$$\beta_V = -\frac{1}{6} V_{a_1 a_2} V_{a_2 a_3} V_{a_3 a_1}$$

$$(\beta_Z)_{a_1 a_2} = -\frac{1}{6} V_{a_1 a_3 a_4} V_{a_2 a_3 a_4}$$

$$d_c = \frac{10}{3}$$



$$\beta_V = \frac{3}{4} V_{a_1 a_2} V_{a_1 a_3 a_4 a_5} V_{a_2 a_3 a_4 a_5} - \frac{27}{8} V_{a_1 a_2 a_3} V_{a_1 a_4 a_5} V_{a_2 a_3 a_4 a_5}$$

$$(\beta_Z)_{a_1 a_2} = -\frac{3}{40} V_{a_1 a_3 a_4 a_5 a_6} V_{a_2 a_3 a_4 a_5 a_6}$$

O(N) $[d_c = 4 \; ; \; L_{\rm LO} = 1]$

- Maximal symmetry of kinetic term and one invariant $\rho = \frac{1}{2}\varphi_i\varphi_i$

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2}\partial\phi_i \cdot \partial\phi_i + U(\rho) \right\}$$

- LO and NLO beta functional in $d_c = 4$ are

$$\beta_V = \frac{1}{2}\frac{1}{(4\pi)^2} V_{ij}V_{ij} - \frac{1}{2}\frac{1}{(4\pi)^2} V_{ij}V_{iab}V_{jab}$$

- The only invariant is $\rho = \frac{1}{2}\varphi_i\varphi_i$ and the potential is a function of it $U(\rho) \equiv V(\varphi_i)$
- The LO and NLO beta functional is

$$\begin{aligned}\beta_U \quad &= \quad (N-1)\left\{\frac{1}{2}(U')^2 - 3\rho(U'')^2\left(U' + \frac{2}{3}\rho U''\right)\right\} \\ &\quad + \frac{1}{2}\left(U' + 2\rho U''\right)^2 - 9\rho\left(U' + 2\rho U''\right)\left(U'' + \frac{2}{3}\rho U'''\right)^2\end{aligned}$$

with $\eta = \frac{1}{6N} V_{ijkl}V_{ijkl} = 3N(N+2)(U'')^2 + \ldots$

- Reproduces all O(N) universal beta functions and critical exponents at LO and NLO
- LO β_U agrees with expansion of $O(N)$ LPA

- The Hessian is

$$U_i \equiv \frac{\partial U}{\partial \varphi_i} = U' \partial_i \rho = U' \varphi_i \qquad U_{ij} \equiv \frac{\partial^2 U}{\partial \varphi_i \partial \varphi_j} = U' \delta_{ij} + U'' \varphi_i \varphi_j$$

- Introducing the projector $P_{ij} \equiv \varphi_i \varphi_j / \varphi^2$ we can write the Hessain as

$$U_{ij} = U'(1-P)_{ij} + (U' + 2\rho U'')P_{ij}$$

- The one loop monomial is easily computed

$$\begin{aligned} U_{ij} U_{ji} &= (U')^2 (1-P)_{ii} + (U' + 2\rho U'')^2 P_{ii} \\ &= (U')^2 (N-1) + (U' + 2\rho U'')^2 \end{aligned}$$

- The three vertex is

$$U_{ijk} = (\delta_{ij}\varphi_k + \delta_{ik}\varphi_j + \delta_{jk}\varphi_i) U'' + \varphi_i \varphi_j \varphi_k U'''$$

which we can rewrite as

$$\begin{aligned} \delta_{ij}\varphi_k + \delta_{ik}\varphi_j + \delta_{jk}\varphi_i &= (1-P)_{ij}\varphi_k + P_{ij}\varphi_k + (1-P)_{ik}\varphi_j + P_{ik}\varphi_j \\ &\quad + (1-P)_{jk}\varphi_i + P_{jk}\varphi_i \\ \varphi_i \varphi_j \varphi_k &= \frac{1}{3}\varphi_i\varphi_j\varphi_k + \frac{1}{3}\varphi_i\varphi_j\varphi_k + \frac{1}{3}\varphi_i\varphi_j\varphi_k \\ &= \frac{2}{3}\rho \left(P_{ij}\varphi_k + P_{ik}\varphi_i + P_{jk}\varphi_i \right) \end{aligned}$$

- We obtian

$$
\begin{aligned}
U_{ijk} &= U''\left[(1-P)_{ij}\varphi_k + (1-P)_{ik}\varphi_j + (1-P)_{jk}\varphi_i\right] \\
&\quad + \left(U'' + \frac{2}{3}\rho U'''\right)\left(P_{ij}\varphi_k + P_{ik}\varphi_i + P_{jk}\varphi_i\right)
\end{aligned}
$$

- The four vertex is

$$
\begin{aligned}
U_{ijkl} &= (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})U'' + (\delta_{ij}\varphi_k\varphi_l + \delta_{ik}\varphi_j\varphi_l + \delta_{jk}\varphi_i\varphi_l \\
&\quad + \delta_{il}\varphi_j\varphi_k + \delta_{jl}\varphi_i\varphi_k + \delta_{kl}\varphi_i\varphi_j)U''' + \varphi_i\varphi_j\varphi_k\varphi_l U''''
\end{aligned}
$$

- A little algebra gives

$$
\begin{aligned}
U_{ij}U_{iab}U_{jab} &= -3(N-1)\rho(U'')^2\left(U' + \frac{2}{3}\rho U''\right) \\
&\quad -9\rho\left(U' + 2\rho U''\right)\left(U'' + \frac{2}{3}\rho U'''\right)^2
\end{aligned}
$$

- Inserting into the $d_c = 4$ general beta functionals gives

$$
\begin{aligned}
\beta_U &= (N-1)\left\{\frac{1}{2}(U')^2 - 3\rho(U'')^2\left(U' + \frac{2}{3}\rho U''\right)\right\} \\
&\quad + \frac{1}{2}\left(U' + 2\rho U''\right)^2 - 9\rho\left(U' + 2\rho U''\right)\left(U'' + \frac{2}{3}\rho U'''\right)^2
\end{aligned}
$$

- In terms of dimensionless variables $U(\rho) = \mu^d u(\chi)$ we find

$$\begin{aligned}\beta_u \quad = \quad & -du + (d-2+\eta)\rho u' + (N-1)\left\{\frac{1}{2}(u')^2 - 3\rho(u'')^2\left(u' + \frac{2}{3}\rho u''\right)\right\} \\ & +\frac{1}{2}\left(u' + 2\rho u''\right)^2 - 9\rho\left(u' + 2\rho u''\right)\left(u'' + \frac{2}{3}\rho u'''\right)^2\end{aligned}$$

- The beta functions for the potential $u(\rho) = \lambda_2\rho + \frac{\lambda_4}{6}\rho^2$ are

$$\beta_2 = -2\lambda_2 + \frac{(N+2)}{3}\lambda_2\lambda_4 - \frac{5(N+2)}{18}\lambda_2\lambda_4^2$$

$$\beta_4 = -\epsilon\lambda_4 + \frac{N+8}{3}\lambda_4^2 - \frac{3N+14}{3}\lambda_4^3$$

We used the contraction

$$U_{ijkl}U_{ijkl} = 3N(N+2)(u'')^2 + ... = \frac{N(N+2)}{3}\lambda_4^2$$

from which we computed the anomalous dimension

$$\eta = -\beta_Z = \frac{1}{6N}U_{ijkl}U_{ijkl} = \frac{N+2}{18}\lambda_4^2$$

- The non trivial fixed point is

$$\lambda_4^* = \frac{3\epsilon}{N+8} + \frac{9(3N+14)}{(N+8)^3}\epsilon^2 + O(\epsilon^3)$$

- The critical exponents are

$$\begin{aligned}
\eta &= \frac{N+2}{2(N+8)^2}\epsilon^2 + O(\epsilon^4) \\
\nu &= \frac{1}{2} + \frac{N+2}{4(N+8)}\epsilon + \frac{N^3+25N^2+106N+120}{8(N+8)^3}\epsilon^2 + O(\epsilon^3)
\end{aligned}$$

and

$$\omega = \epsilon - \frac{3(3N+14)}{(N+8)^2}\epsilon^2 + O(\epsilon^3)$$

- Clearly in the case $N = 1$ we reproduce the `Ising` exponents
- In the limit $N \to \infty$ the critical exponents are

$$\begin{aligned}
\eta &= 0 + O(\epsilon^4) \\
1/\nu &= 2 - \epsilon + O(\epsilon^3) = d - 2 + O(\epsilon^3) \\
\omega &= \epsilon + O(\epsilon^3) = d - 4 + O(\epsilon^3)
\end{aligned}$$

which correctly agree with the `Spherical-Model` UC if the higher order contributions die out

- EXERCISE: compute the large N limit potential
- The limit $N \to 0$ represents the Self Avoiding Random Walk `SAW` UC

Tricritical-O(N) $[d_c = 3 \,;\, L_{\mathrm{LO}} = 2]$

- General beta functional in $d_c = 3$ at LO is

$$\beta_V = \frac{1}{3}\frac{1}{(4\pi)^3} V_{ijk} V_{ijk} + \ldots$$

- The Triciritcal-O(N) LO beta functional turns out to be

$$\beta_U \quad = \quad 2(N-1)\rho(U'')^2 + 6\rho\left(U'' + \frac{2}{3}\rho U'''\right)^2 + \ldots$$

- The beta functions for the potential $U = m^2\rho + \frac{1}{6}\lambda\rho^3$ are

$$\begin{aligned}\beta_{m^2} &= -2m^2 + \frac{16}{3}(N+2)(N+4)\lambda^2 m^2 \\ \beta_\lambda &= -2\epsilon\lambda + 4(22+3N)\lambda^2 - 4(3304+858N+53N^2)\lambda^3 \\ &\quad -\frac{\pi^2}{2}(2720+620N+34N^2+N^3)\lambda^3\end{aligned}$$

and the anomalous dimension is

$$\eta = \frac{1}{3}(N+2)(N+4)\lambda^2$$

Tricritical-O(N) $[d_c = 3\ ;\ L_{\mathrm{LO}} = 2]$

- The fixed point is

$$\lambda_* = \frac{\epsilon}{2(3N+22)}$$

  and vanishes as $N \to \infty$

- Critical exponents are

$$\begin{aligned} \eta &= \frac{(N+2)(N+4)}{3(6N+44)^2}\epsilon^2 + O(\epsilon^4) \\ \nu &= \frac{1}{2} + \frac{(N+2)(N+4)}{3(3N+22)^2}\epsilon^2 + O(\epsilon^3) \end{aligned}$$

  Note that there is no $O(\epsilon)$ term in the expansion for ν

- For $N = 1$ they reproduce the `Tricritical` exponents
- **The large N limit is non-trivial**

$$\begin{aligned} \eta &= \frac{\epsilon^2}{108} - \frac{13\epsilon^2}{162}\frac{1}{N} + \ldots \\ \nu &= \frac{1}{2} + \frac{\epsilon^2}{27} - \frac{26\epsilon^2}{81}\frac{1}{N} + \ldots \end{aligned}$$

- This is related to the Bardeen-Mosche-Bander line of fixed points

`Tetracritical-O(N)` $[d_c = \frac{8}{3}\ ;\ L_{\rm LO} = 3]$

- General beta functional in $d_c = \frac{8}{3}$ at LO is

$$\beta_V = \frac{1}{8}\frac{1}{(4\pi)^4} V_{ijkl} V_{ijkl} + \ldots$$

- The `Tetraciritcal-O(N)` LO beta functional turns out to be

$$\begin{aligned}\beta_U \quad &= \quad \frac{3}{8}(N-1)^2 (U'')^2 + \frac{3}{2}(N-1)\left[(U'')^2 + 2\rho U'' U''' + 2\rho^2 (U''')^2\right] \\ &\quad + \frac{1}{8}(3U'' + 12\rho U''' + 4\rho^2 U'''')^2\end{aligned}$$

- With the choice $U = m^2\rho + \frac{1}{24}\lambda\rho^4$ we find

$$\begin{aligned}\beta_{m^2} \quad &= \quad -2m^2 + \frac{45}{32}(N+2)(N+4)(N+6)\lambda^2 m^2 \\ \beta_\lambda \quad &= \quad -3\epsilon\lambda + \frac{3}{4}\left(3N^2 + 150N + 1072\right)\lambda^2 + \\ &\quad -6\Gamma(1/3)^3 \left(15N^3 + 636N^2 + 8360N + 33864\right)\lambda^3 \\ &\quad -\frac{27}{8}\left(3N^4 + 345N^3 + 12012N^2 + 139620N + 514000\right)\lambda^3 \\ &\quad +\frac{27}{16}(\sqrt{3}\pi - 3\log 3)\left(N^4 + 64N^3 + 2264N^2 + 26936N + 99360\right)\lambda^3\end{aligned}$$

Note that λ^3 and $\lambda^2 m^2$ terms are six-loop contributions!

Tetracritical-O(N) $[d_c = \frac{8}{3}\ ;\ L_{\mathrm{LO}} = 3]$

- The anomalous dimension is

$$\eta = \frac{3}{32}(N+2)(N+4)(N+6)\lambda^2$$

- The fixed point is

$$\lambda_* = \frac{4\epsilon}{3N^2 + 150N + 1072}$$

  and vanishes as $N \to \infty$

- Critical exponents turn out to be

$$\begin{aligned} \eta &= \frac{3(N+2)(N+4)(N+6)}{2\left(3N^2+150N+1072\right)^2}\epsilon^2 + O(\epsilon^4) \\ \nu &= \frac{1}{2} + \frac{45(N+2)(N+4)(N+6)}{8\left(3N^2+150N+1072\right)^2}\epsilon^2 + O(\epsilon^3) \end{aligned}$$

  Note that there is no $O(\epsilon)$ term in the expansion for ν

- In the case $N = 1$ this expression correctly agrees with `Tetracritical`
- Differently than the `Tricritical O(N)` case the large N limit of these expressions is `Gaussian` as for the standard `O(N)` class

Pentacritical-O(N) $[d_c = \frac{5}{2}\ ;\ L_{\rm LO} = 4]$

- Is there a patter to be uncovered? Let's try the next order!
- The fixed point is

$$\lambda_* = \frac{3\epsilon}{8\,(15N^2 + 490N + 3464)}$$

  and vanishes as $N \to \infty$
- Critical exponents turn out to be

$$\begin{aligned}
\eta &= \frac{(N+2)(N+4)(N+6)(N+8)}{15\,(15N^2+490N+3464)^2}\epsilon^2 + O(\epsilon^4) \\
\nu &= \frac{1}{2} + \frac{4(N+2)(N+4)(N+6)(N+8)}{15\,(15N^2+490N+3464)^2}\epsilon^2 + O(\epsilon^3)
\end{aligned}$$

- Remark: **we did implicitly an 8-loop calculation!**
- In the case $N = 1$ this expression correctly agrees with Pentacritical
- The large N limit is non-trivial:

$$\eta = \frac{\epsilon^2}{3375} + O\left(1/N\right) \qquad\qquad \nu = \frac{1}{2} + \frac{4\epsilon^2}{3375} + O\left(1/N\right)$$

  Alternating behaviour for large N limit!

Platonic Field Theories

	Polytope	Schläfli	$\mathcal{G}$
N=2	n-Polygon	$\{n\}$	$\mathbb{D}_n$
N=3	Tetrahedron	$\{3,3\}$	S_4
	Octahedron	$\{3,4\}$	$S_4 \times \mathbb{Z}_2$
	Cube	$\{4,3\}$	$S_4 \times \mathbb{Z}_2$
	Icosahedron	$\{3,5\}$	$A_5 \times \mathbb{Z}_2$
	Dodecahedron	$\{5,3\}$	$A_5 \times \mathbb{Z}_2$
N=4	5-cell	$\{3,3,3\}$	S_5
	16-cell	$\{3,3,4\}$	$(\mathbb{Z}_2)^4 \rtimes S_4$
	8-cell	$\{4,3,3\}$	$(\mathbb{Z}_2)^4 \rtimes S_4$
	24-cell	$\{3,4,3\}$	F_4
	120-cell	$\{3,3,5\}$	H_4
	600-cell	$\{5,3,3\}$	H_4

Platonic solids and their symmetry groups

- Explore scalar QFTs with the internal symmetries of Polygons ($N = 2$), Platonic solids ($N = 3$) and Hyper-Platonic solids ($N = 4$) [arXiv:1902.05328]
- New UC in $d = 3$ with good critical exponents: Pentagon. Simulation is progress!
- Many PFTs with $d_c < 3$ can lead to new (non-rational) CFTs in $d = 2$

Platonic Field Theories: Molien Series and d_c's

$N=2$

Polytope	Group	Molien Series	d_c
$\{3\}$	$\mathbb{D}_3$	$1+t^2+t^3+t^4+t^5+2t^6+t^7+2t^8+2t^9+2t^{10}+\cdots$	6
$\{4\}$	$\mathbb{D}_4$	$1+t^2+2t^4+2t^6+3t^8+3t^{10}+\cdots$	4
$\{5\}$	$\mathbb{D}_5$	$1+t^2+t^4+t^5+t^6+t^7+t^8+t^9+2t^{10}+\cdots$	$10/3$
$\{6\}$	$\mathbb{D}_6$	$1+t^2+t^4+2t^6+2t^8+2t^{10}+\cdots$	3
$\{7\}$	$\mathbb{D}_7$	$1+t^2+t^4+t^6+t^7+t^8+t^9+t^{10}+\cdots$	$14/5$
$\{8\}$	$\mathbb{D}_8$	$1+t^2+t^4+t^6+2t^8+2t^{10}+\cdots$	$8/3$
$\{9\}$	$\mathbb{D}_9$	$1+t^2+t^4+t^6+t^8+t^9+t^{10}+\cdots$	$18/7$
$\{10\}$	$\mathbb{D}_{10}$	$1+t^2+t^4+t^6+t^8+2t^{10}+\cdots$	$5/2$

$N=3$

Polytope	Group	Molien Series	d_c
$\{3,3\}$	S_4	$1+t^2+t^3+2t^4+t^5+3t^6+2t^7+4t^8+3t^9+5t^{10}+\cdots$	$6, 4$
$\{4,3\}$	$S_4\times\mathbb{Z}_2$	$1+t^2+2t^4+3t^6+4t^8+5t^{10}+\cdots$	$4, 3$
$\{5,3\}$	$A_4\times\mathbb{Z}_2$	$1+t^2+t^4+2t^6+2t^8+3t^{10}+\cdots$	$3, 8/3$

Platonic Field Theories: Invariants

- **How do we construct invariants?** $\mathcal{G}$-invariant polynomials of degree k

$$I^{(k)}_{i_1\cdots i_k} = \sum_\alpha e^\alpha_{i_1}\cdots e^\alpha_{i_k}$$

can be constructed geometrically taking advantage of the strong symmetry of regular polytopes. Here $\{e^\alpha\}_{\alpha=1,\dots,V}$ is the set of versors defining the V vertices of a given polytope $\mathcal{P}$

- Example: **Triangle** $\{3\}$ (blackboard & Mathematica)
- Example: **Icosahedron** $\{3,5\}$ - **Dodecahedron** $\{5,3\}$

$$\begin{aligned}
\rho_{\{3,5\}} &= 4(\phi_1^2+\phi_2^2+\phi_3^2)\\
\tau_{\{3,5\}} &= \frac{4}{25}(10\phi_1^6+6\phi_3\phi_1^5+15\left(2\phi_2^2+3\phi_3^2\right)\phi_1^4-60\phi_2^2\phi_3\phi_1^3\\
&\quad+15\left(2\phi_2^4+6\phi_3^2\phi_2^2+\phi_3^4\right)\phi_1^2+30\phi_2^4\phi_3\phi_1+10\phi_2^6+13\phi_3^6+15\phi_2^2\phi_3^4+45\phi_2^4\phi_3^2)\\
\sigma_{\{3,5\}} &= \frac{4}{625}(127\phi_1^{10}+360\phi_3\phi_1^9+45\left(13\phi_2^2+35\phi_3^2\right)\phi_1^8-120\left(24\phi_2^2\phi_3-7\phi_3^3\right)\phi_1^7\\
&\quad+210\left(7\phi_2^4+30\phi_3^2\phi_2^2+10\phi_3^4\right)\phi_1^6-252\left(-\phi_3^5+30\phi_2^2\phi_3^3+20\phi_2^4\phi_3\right)\phi_1^5\\
&\quad+210\left(5\phi_2^6+45\phi_3^2\phi_2^4+30\phi_3^4\phi_2^2+3\phi_3^6\right)\phi_1^4-840\left(3\phi_2^2\phi_3^5+5\phi_2^4\phi_3^3\right)\phi_1^3\\
&\quad+45\left(15\phi_2^8+140\phi_3^2\phi_2^6+140\phi_3^4\phi_2^4+28\phi_3^6\phi_2^2+\phi_3^8\right)\phi_1^2+60\left(30\phi_3\phi_2^8+70\phi_3^3\phi_2^6+21\phi_3^5\phi_2^4\right)\phi_1\\
&\quad+125\phi_2^{10}+313\phi_3^{10}+45\phi_2^2\phi_3^8+630\phi_2^4\phi_3^6+2100\phi_2^6\phi_3^4+1575\phi_2^8\phi_3^2)
\end{aligned}$$

- Since the field power of $\tau_{\{3,5\}}$ and $\sigma_{\{3,5\}}$ are respectively $k=6$ and $k=10$ the interesting upper critical dimensions are $d_c = 3, 8/3, 5/2$

Platonic Field Theories: Beta Functions

- Example: **Triangle** $\{3\}$ The $\mathbb{D}_3$ symmetry is encoded in the following two invariants

$$\begin{aligned} \rho_{\{3\}} &= \frac{3}{2}\left(\phi_1^2+\phi_2^2\right) \\ \tau_{\{3\}} &= \frac{3}{4}\phi_2\left(\phi_2^2-3\phi_1^2\right) \end{aligned}$$

Since the non-trivial invariant polynomial $\tau_{\{3\}}$ is of order $k=3$, we study the Triangle in $d_c=6$ and therefore we consider the following marginal potential

$$U(\tau)=\frac{1}{3!}X\,\tau_{\{3\}}$$

The beta function β_X and the anomalous dimension η are obtained from the general formulae and they read

$$\begin{aligned} \beta_X &= -\frac{1}{2}\epsilon X+\frac{9}{32}X^3 \\ \eta &= \frac{3}{16}X^2 \end{aligned}$$

We find that the universality class associated to the Triangle is the well known $\mathtt{Potts}_3$

Platonic Field Theories: Beta Functions

- Example: **Icosahedron** $\{3, 5\}$ - **Dodecahedron** $\{5, 3\}$ In $d_c = 3$ the marginal potential is

$$U(\rho, \tau) = \frac{1}{6!} \left(X \, \rho^3_{\{3,5\}} + Y \, \tau_{\{3,5\}} \right)$$

and the corresponding LO beta functions read

$$\begin{aligned} \beta_X &= -2\epsilon \, X + \frac{7936}{15} X^2 + \frac{96}{5} XY + \frac{3}{25} Y^2 \\ \beta_Y &= -2\epsilon \, Y + \frac{32}{3} Y^2 + \frac{1024}{3} XY \end{aligned}$$

with anomalous dimension

$$\eta = \frac{28672}{135} X^2 + \frac{512}{45} XY + \frac{208}{1125} Y^2$$

This system exhibits no other real fixed point than `Tri-O(3)`
- Also in $d_c = 8/3$ we find only `Tetra-O(3)`
- To find a real icosahedral fixed point we have to shift to the third possible upper critical dimension which is $d_c = 5/2$ for which the marginal potential assumes the following form

$$U(\rho, \tau, \sigma) = \frac{1}{10!} \left(X \, \rho^5_{\{3,5\}} + Y \, \rho^2_{\{3,5\}} \tau_{\{3,5\}} + Z \, \sigma_{\{3,5\}} \right)$$

The LO beta function system in this case reads

$$\begin{aligned} \beta_X &= -4\epsilon X + \frac{10381312}{945} X^2 + \frac{3968}{15} XY - \frac{896}{45} XZ + \frac{19601}{23625} Y^2 - \frac{1817}{4500} YZ + \frac{203}{200000} Z^2 \\ \beta_Y &= -4\epsilon Y + \frac{11429888}{945} XY + \frac{68864}{45} XZ + \frac{232928}{945} Y^2 + \frac{7024}{225} YZ - \frac{7}{250} Z^2 \\ \beta_Z &= -4\epsilon Z + \frac{32768}{15} XZ + \frac{13568}{21} Y^2 + \frac{3776}{15} YZ + \frac{84}{5} Z^2 \end{aligned}$$

with anomalous dimension

$$\eta = \frac{2883584}{42525} X^2 + \frac{360448}{99225} XY + \frac{2048}{14175} XZ + \frac{381952}{7441875} Y^2 + \frac{1664}{354375} YZ + \frac{1252}{8859375} Z^2$$

Apart from the `Penta-O(3)` FP there are **two pure icosahedral real FPs**

Platonic Field Theories: Fixed Points and Critical Exponents

	Polytope		d_c	Fixed Points
N=2		Triangle	6	$\texttt{Potts}_3$
		Square	4	2×Ising, O(2)
		Pentagon	10/3	Pentagon
		Hexagon	3	Tri-O(2)
		Heptagon	14/5	Heptagon
		Octagon	8/3	Tetra-O(2)
	⋮	⋮	⋮	⋮
N=3		Tetrahedron	6 4	No real FP 3×Ising, O(3), $\texttt{Cubic}_3$
		Octahedron	4 3	3×Ising, O(3), $\texttt{Cubic}_3$ 3×Tri-Ising, Tri-O(3), ϕ^6-$\texttt{Cubic}_3$
		Icosahedron	3 8/3 5/2	Tri-O(3) Tetra-O(3) Penta-O(3), $\texttt{Ico}_{1\leq i\leq 2}$
N=4		5-cell	6 4 10/3	No real FP O(4), $\texttt{Quartic-Potts}_5$ No real FP
		16-cell	4 3 8/3	4×Ising, O(4) 4×Tri-Ising, Tri-O(4), ϕ^6-$\texttt{Cubic}_4$ 4×Tetra-Ising, Tetra-O(4), ϕ^8-$\texttt{Cubic}_4$
		24-cell	3 8/3 5/2 12/5	Tri-O(4) Tetra-O(4) Penta-O(4), $\texttt{24-cell}_1$ Hexa-O(4), $\texttt{24-cell}_{1\leq i\leq 2}$
		600-cell	12/5 ⋮ 20/9 ⋮ 15/7	Hexa-O(4) ⋮ Deca-O(4) ⋮ Triaconta-O(4)

	Universality Class	d_c	η	ν
$N=1$	Ising	4	$\frac{1}{54}\epsilon^2$	$\frac{1}{2}+\frac{1}{12}\epsilon+\frac{7}{162}\epsilon^2$
	Tri-Ising	3	$\frac{1}{500}\epsilon^2$	$\frac{1}{2}+\frac{1}{125}\epsilon^2$
	Tetra-Ising	$\frac{8}{3}$	$\frac{9}{85750}\epsilon^2$	$\frac{1}{2}+\frac{27}{68600}\epsilon^2$
$N=2$	$\texttt{Potts}_3$	6	$\frac{1}{3}\epsilon$	$\frac{1}{2}-\frac{5}{12}\epsilon$
	O(2)	4	$\frac{1}{50}\epsilon^2$	$\frac{1}{2}+\frac{1}{10}\epsilon+\frac{11}{200}\epsilon^2$
	Pentagon	$\frac{10}{3}$	$\frac{3}{5}\epsilon$	$\frac{1}{2}+\frac{3}{20}\epsilon$
	Tri-O(2)	3	$\frac{1}{392}\epsilon^2$	$\frac{1}{2}+\frac{1}{98}\epsilon^2$
	Heptagon	$\frac{14}{5}$	$\frac{10}{7}\epsilon$	$\frac{1}{2}+\frac{5}{14}\epsilon$
	Tetra-O(2)	$\frac{8}{3}$	$\frac{9}{59858}\epsilon^2$	$\frac{1}{2}+\frac{135}{239432}\epsilon^2$
$N=3$	O(3)	4	$\frac{5}{242}\epsilon^2$	$\frac{1}{2}+\frac{5}{44}\epsilon+\frac{345}{5324}\epsilon^2$
	$\texttt{Cubic}_3$	4	$\frac{5}{243}\epsilon^2$	$\frac{1}{2}+\frac{1}{9}\epsilon+\frac{599}{8748}\epsilon^2$
	Tri-O(3)	3	$\frac{35}{11532}\epsilon^2$	$\frac{1}{2}+\frac{35}{2883}\epsilon^2$
	ϕ^6-$\texttt{Cubic}_3$	3	$0.00261529\,\epsilon^2$	$\frac{1}{2}+0.0104612\,\epsilon^2$
	Tetra-O(3)	$\frac{8}{3}$	$\frac{945}{4798802}\epsilon^2$	$\frac{1}{2}+\frac{14175}{19195208}\epsilon^2$
$N=4$	O(4)	4	$\frac{1}{48}\epsilon^2$	$\frac{1}{2}+\frac{1}{8}\epsilon+\frac{7}{96}\epsilon^2$
	$\texttt{Quartic-Potts}_5$	4	$\frac{55}{2646}\epsilon^2$	$\frac{1}{2}+\frac{5}{42}\epsilon+\frac{22465}{222264}\epsilon^2$
	Tri-O(4)	3	$\frac{1}{289}\epsilon^2$	$\frac{1}{2}+\frac{4}{289}\epsilon^2$
	ϕ^6-$\texttt{Cubic}_4$	3	$0.00322216\,\epsilon^2$	$\frac{1}{2}+0.0128886\,\epsilon^2$
	Tetra-O(4)	$\frac{8}{3}$	$\frac{9}{36980}\epsilon^2$	$\frac{1}{2}+\frac{27}{29584}\epsilon^2$
	ϕ^8-$\texttt{Cubic}_4$	$\frac{8}{3}$	$0.000196765\,\epsilon^2$	$\frac{1}{2}+0.000737867\,\epsilon^2$

- Permutation group S_{N+1} (symmetry of hyper-tetrahedron) in $d_c = 6$

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2} \partial\phi_i \cdot \partial\phi_i + U(\rho, \tau, \sigma, \ldots) \right\}$$

- **Random Cluster Model**: $\mathtt{Potts}_1 = \mathtt{Percolation}$ and $\mathtt{Potts}_0 = \mathtt{SpanningForest}$
- Number of invariants $= N \Rightarrow$ functional analysis possible only at fixed N
- General LO beta functional in $d_c = 6$

$$\beta_V = -\frac{1}{3} \frac{1}{(4\pi)^3} V_{ij} V_{jk} V_{ki} + \ldots \qquad\qquad \beta_{Z_{ij}} = -\frac{1}{6} \frac{1}{(4\pi)^3} V_{iab} V_{jab} + \ldots$$

- $N = 2$ LO beta functional ($\mathtt{Potts}_3$) is ($\rho = \phi_1^2 + \phi_2^2$, $\tau = \frac{3}{\sqrt{2}} \phi_2 (\phi_2^2 - 3\phi_1^2)$)

$$\begin{aligned}
\beta_U \quad = \quad & 9 \left(3 U_{\tau\tau} \left(2 \left(\rho^3 - 6\tau^2 \right) U_{\rho\rho} - 3\rho\tau U_\tau \right) - 6 \left(\rho^3 - 6\tau^2 \right) U_{\rho\tau}^2 \right. \\
& + U_\rho \left(3\rho^2 U_{\tau\tau} + 8\rho U_{\rho\rho} + 24\tau U_{\rho\tau} \right) - 12 U_\tau \left(\rho^2 U_{\rho\tau} + 2\tau U_{\rho\rho} \right) - 6\rho U_\tau^2 + 4 U_\rho^2 \Big) \\
& - \frac{3}{4} \left(3\rho^2 U_{\tau\tau} + 8 \left(\rho U_{\rho\rho} + U_\rho + 3\tau U_{\rho\tau} \right) \right)^2
\end{aligned}$$

- Reproduces all $\mathtt{Potts}_{N+1}$ universal beta functions and critical exponents at LO and NLO
- LO β_U agrees with expansion of LPA analysis (Ben Ali Zinati+Codello)[arXiv:1707.03410]

Multiciritical Potts Models [$d_c = \frac{10}{3}$; $L_{\rm LO} = 3$]

- Permutation group S_{N+1} in $d_c = \frac{10}{3}$ [2010.09757]

$$S = \int \mathrm{d}^d x \left\{ \frac{1}{2}\partial\phi_i \cdot \partial\phi_i + \frac{1}{5!}\left(\lambda_{5,1}\delta_{(i_1 i_2} I^{(3)}_{i_3 i_4 i_5)} + \lambda_{5,2} I^{(5)}_{i_1 i_2 i_3 i_4 i_5}\right)\phi_{i_1}\dots\phi_{i_5}\right\}$$

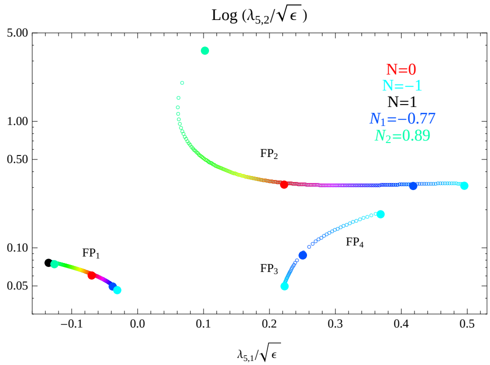
  arbitrary N invariants $I^{(k)}_{i_1\cdots i_k} = \sum_\alpha e^\alpha_{i_1}\cdots e^\alpha_{i_k}$

- General LO beta functionals are

$$\beta_V = \frac{1}{3}V_{ijkl}V_{ijkm}V_{lm} - \frac{3}{2}V_{ijk}V_{ilm}V_{jklm} \qquad \beta_{Z_{ij}} = -\frac{1}{30}V_{iklmn}V_{jklmn}$$

- Universal beta functions

$$\begin{aligned}
\beta_{5,1} &= -\frac{3\epsilon}{2}\lambda_{5,1} - \frac{3}{200}(17N^2+833N-3510)\lambda_{5,1}^3 + \frac{10}{3}(2N^2+25N-25)\lambda_{5,2}^3 \\
&\quad -\frac{1}{5}(97N^2-532N+432)\lambda_{5,1}^2\lambda_{5,2} - \frac{1}{4}(25N^3-181N^2+277N-673)\lambda_{5,1}\lambda_{5,2}^2 \\
\beta_{5,2} &= -\frac{3\epsilon}{2}\lambda_{5,2} - \frac{3}{25}(5N+144)\lambda_{5,1}^3 - \frac{1}{12}(459N^3-919N^2+1459N-919)\lambda_{5,2}^3 \\
&\quad -\frac{1}{40}(25N^2+3533N-4038)\lambda_{5,1}^2\lambda_{5,2} - \frac{1}{2}(217N^2-373N+300)\lambda_{5,1}\lambda_{5,2}^2
\end{aligned}$$



- No real fixed point in the $N \geq 2$. Consistent with the annihilation scenario
- Multicritical fixed points for $N = 1$ (multicritical `Percolation`) and $N = 0$ (multicritical `SpanningForest`)
- $N = 1$ is probably related to `CorrelatedPercolation` (percolation of Ising model clusters)
- NPRG analysis is work in progress

Multicritical hypercubic models

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ABSTRACT: We study renormalization group multicritical fixed points in the ϵ-expansion of scalar field theories characterized by the symmetry of the (hyper)cubic point group H_N. After reviewing the algebra of H_N-invariant polynomials and arguing that there can be an entire family of multicritical (hyper)cubic solutions with ϕ^{2n} interactions in $d = \frac{2n}{n-1} - \epsilon$ dimensions, we use the general multicomponent beta functionals formalism to study the special cases $d = 3 - \epsilon$ and $d = \frac{8}{3} - \epsilon$, deriving explicitly the beta functions describing the flow of three- and four-critical (hyper)cubic models. We perform a study of their fixed points, critical exponents and quadratic deformations for various values of N, including the limit $N = 0$, that was reported in another paper in relation to the randomly diluted single-spin models, and an analysis of the large N limit, which turns out to be particularly interesting since it depends on the specific multicriticality. We see that, in general, the continuation in N of the random solutions is different from the continuation coming from large-N, and only the latter interpolates with the physically interesting cases of low-N such as $N = 3$. Finally, we also include an analysis of a theory with quintic interactions in $d = \frac{10}{3} - \epsilon$ and, for completeness, the NNLO computations in $d = 4 - \epsilon$.