Double null coordinates and applications in Kerr spacetime

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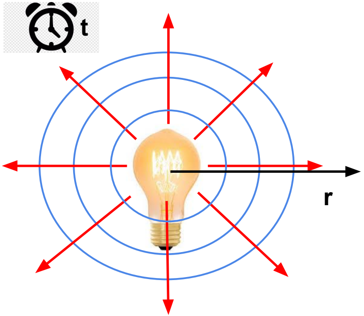
Little bit of intuition about null coordinates $\{u, v\}$

- What is the meaning of $u = constant$?
- Example: null coordinates in Minkowski spacetime:

$$u = ct - r = constant, \tag{1}$$

$$v = ct + r \tag{2}$$



The null coordinates $\{u, v\}$, are always related with a null congruence.

The principal null congruence

- In Schwarzschild spacetime (static and spherically symmetric): The **principal null congruence do not have twist**. One can easily build null coordinates adapted to those null directions.

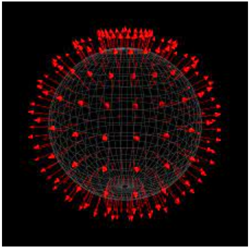


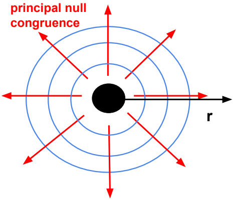
Figure: Schematic representation.

- In Kerr spacetime (stationary and axial symmetric): The **principal null congruence have twist**. Therefore it is impossible to define null coordinates adapted to those null directions. **We need another null congruence!**

Figure: Schematic representation.

Textbook's choice: using principal null congruence

- In Schwarzschild spacetime:



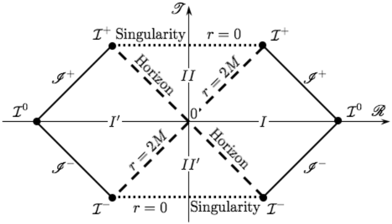
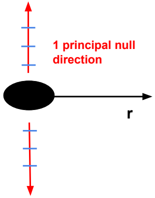


Figure: Complete $\forall\ (\theta, \phi)$.

- In Kerr spacetime:



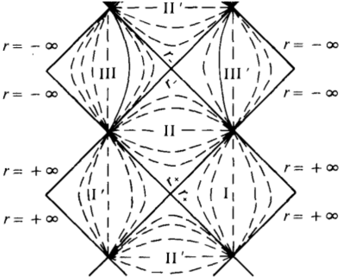


Figure: **Not global!** (only at $\theta = 0, \pi$) [Carter, 1968].

A very elusive subject: other attempts in literature

Previous definitions [Hayward, 2004, Fletcher and Lun, 2003, Bishop and Venter, 2006a, Pretorius and Israel, 1998a], have different types of divergent behavior at the axis of symmetry; as we have shown in [Argañaraz and Moreschi, 2021].:

- One of the most notable is [Hayward, 2004](Phys. Rev. Lett. 92, 191101), whose definition does not include the null geodesic along the symmetry axis. This introduces divergences in the scalar field equation.
- Our approach is more closely related to the work of Frans Pretorious and Werner Israel [Pretorius and Israel, 1998b], although their treatment only covers the northern hemisphere, their expressions also fail at the north pole and are difficult to calculate even numerically [Bishop and Venter, 2006b].

In order to solve the scalar field equation in double null coordinates, one needs a new definition. We call them *center of mass null coordinates.*

Searching the null congruence: from the beginning

The Kerr metric in [Boyer and Lindquist, 1967a] coordinates is

$$ds^2 = \left(1 - \frac{2mr}{\Sigma}\right) dt^2 + \frac{4amr}{\Sigma} \sin^2(\theta)\, dt\, d\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma\, d\theta^2 - \frac{\Upsilon}{\Sigma} \sin^2(\theta)\, d\phi^2;$$

$$\Sigma = r^2 + a^2 \cos(\theta)^2,\ \Delta = r^2 + a^2 - 2mr,\ \Upsilon = \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2(\theta) \geqq 0.$$

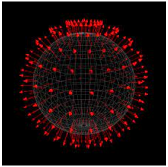
From [Carter, 1968], **the most general null geodesic congruence can be written as the one form**

$$V^a = \begin{pmatrix} \dot{t} \\ \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} \Longrightarrow V_a = g_{ab} V^b = E\, dt_a - \frac{\pm_{oi} \sqrt{[(r^2 + a^2)E - aL_z]^2 - K\Delta}}{\Delta} dr_a$$
$$- \left(\pm_h \sqrt{K - \left[a\, E \sin(\theta) - \frac{L_z}{\sin(\theta)}\right]^2} \right) d\theta_a - L_z d\phi_a. \tag{3}$$

where E, L_z and K (Carter Constant), are conserved quantities along each geodesic. In what follows, we will take ($E = 1$).

Congruence election: center of mass motivation

In our approach, we elect the congruence V^a which is orthogonal to an sphere at infinity $S_{t,r\to\infty}$



It means:

$$\lim_{r\longrightarrow\infty} g_{ab}V^a\left(\frac{\partial}{\partial\theta}\right)^b = -\left(\pm_{oi}\sqrt{K-\left[a\sin(\theta^*)-\frac{L_z}{\sin(\theta^*)}\right]^2}\right) = 0,$$
$$\lim_{r\longrightarrow\infty} g_{ab}V^a\left(\frac{\partial}{\partial\phi}\right)^b = L_z = 0. \tag{4}$$

This imposes a condition over the conserved quantities L_z and K

$$L_{z|_{r\to\infty}} = 0, \quad \text{and} \quad K_{|_{r\to\infty}} = a^2\sin(\theta^*)^2. \tag{5}$$

Note that $\pm_{oi}$ determines the congruence character. We will use ℓ_a (with $\pm_{oi}=+$) for the outgoing and n_a (with $\pm_{oi}=-$) for the ingoing

Congruence election + hypersurface orthogonal condition

Then we can define a null function u, such that

$$\ell_a = (du)_a. \tag{6}$$

Note that on $u =$ constant, for each (r,θ) one has $K = K(r,\theta)$. Then to find $K = K(r,\theta)$, recall that ℓ_a must be hypersurface orthogonal

$$d(\ell_a)_b = \left[\frac{1}{2\sqrt{(r^2+a^2)^2 - K\Delta}}\frac{\partial K}{\partial\theta}d\theta\wedge dr \pm|_h \frac{1}{2\sqrt{K - \left(a\sin(\theta)\right)^2}}\frac{\partial K}{\partial r}d\theta\wedge dr\right] = 0;$$

Then the differential equation for $K(r,\theta)$ is

$$\sqrt{(r^2+a^2)^2 - K(r,\theta)\Delta}\,\frac{\partial K(r,\theta)}{\partial r} \pm|_h \sqrt{K(r,\theta) - a^2\sin^2(\theta)}\,\frac{\partial K(r,\theta)}{\partial\theta} = 0,$$

where $K(r=\infty,\theta^*) = a^2\sin(\theta^*)^2$.

Solution of differential equation for $K(r, \theta)$

To solve it, is convenient to define

$$K(r, \theta) = a^2 \sin^2(\theta) + k^2(r, \theta) \tag{7}$$

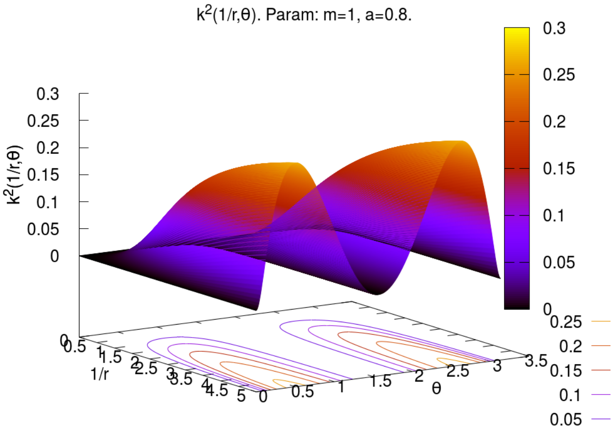


Figure: $k^2\left(\frac{1}{r}, \theta\right)$ con $\{r \in [\frac{r_-}{2},\ \infty)\}$

Final expressions for *center of mass* null coordinates $\{u, v\}$

Then with a solution of $K(r,\theta)$, and the expressions for du and dv

$$\ell_a = du_a = dt_a - \frac{\sqrt{(r^2+a^2)^2 - K(r,\theta)\Delta}}{\Delta} dr_a - k(r,\theta) d\theta_a \qquad (8)$$

$$n_a = dv_a = dt_a + \frac{\sqrt{(r^2+a^2)^2 - K(r,\theta)\Delta}}{\Delta} dr_a + k(r,\theta) d\theta_a \qquad (9)$$

we can integrate, to obtain a pair of null functions

$$u = t - r_s$$
$$v = t + r_s,$$

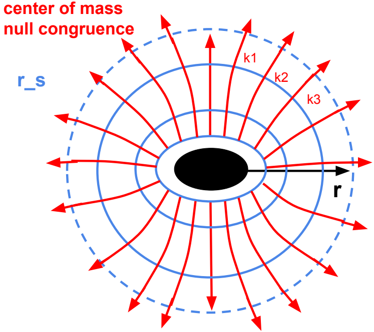
where

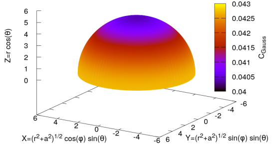
$$r_s = \left(r + \frac{2mr_+}{r_+ - r_-} \ln\left(\frac{r}{r_+} - 1\right) - \frac{2mr_-}{r_+ - r_-} \ln\left(\frac{r}{r_-} - 1\right) \right) + \int_0^\theta k(r,\theta')\, d\theta' \quad (10)$$

Note: $\lim_{a\to 0} u(t,r,\theta,\phi) = t - \left(r + r_+ \ln\left(\frac{r}{r_+} - 1\right) \right) = t - \left(r + 2m \ln\left(\frac{r}{2m} - 1\right) \right).$ (11)

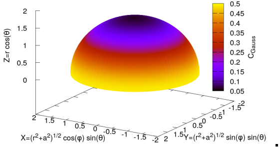
The null coordinates are

$$u = t - r_s,$$
$$v = t + r_s,$$

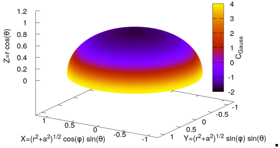




(a) $r_s(r,\theta) \approx 3r_+$



(b) $r_s(r,\theta) \approx r_+$



(c) $r_s(r,\theta) \approx 0.5r_+$

Kerr metric in extended *center of mass* null coordinates $\{U, V\}$

We can also define extended null functions and express the Kerr metric in extended null coordinates [Argañaraz and Moreschi, 2021]

$$U = -\,e^{-\kappa\, u}, \tag{12}$$

$$V = e^{\kappa\, v}, \tag{13}$$

where the inverse Kerr metric reads

$$\boxed{\begin{aligned}\left(\frac{\partial}{\partial s}\right)^2 =& -4\kappa^2 \frac{\Upsilon}{\Sigma\Delta} U\, V \left(\frac{\partial}{\partial U}\right)\left(\frac{\partial}{\partial V}\right) - \frac{1}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)^2 - \frac{1}{\Sigma \sin^2(\theta)}\left(\frac{\partial}{\partial \varphi}\right)^2 \\ & - 2\,\kappa\, U \left(\frac{\partial}{\partial U}\right)\left[\left(\frac{2amr}{\Sigma\Delta} - \pm_{io}\frac{a\sqrt{R}}{\Sigma\Delta}\right)\left(\frac{\partial}{\partial \varphi}\right) \pm_{|_h} \frac{\sqrt{\Theta}}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)\right] \\ & + 2\,\kappa\, V \left(\frac{\partial}{\partial V}\right)\left[\left(\frac{2amr}{\Sigma\Delta} \pm_{io}\frac{a\sqrt{R}}{\Sigma\Delta}\right)\left(\frac{\partial}{\partial \varphi}\right) - \pm_{|_h} \frac{\sqrt{\Theta}}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)\right].\end{aligned}} \tag{14}$$

Applications: Massless scalar field ($\nabla^a \nabla_a \Phi = 0$)

In [Boyer and Lindquist, 1967a] coordinates, the massless scalar field equation for Kerr [Teukolsky, 1972] is

$$(\nabla^a \nabla_a \Phi)\,\Sigma = \left[\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin(\theta)^2\right]\frac{\partial^2\Phi}{\partial t^2} + \frac{4amr}{\Delta}\frac{\partial^2\Phi}{\partial t\,\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin(\theta)^2}\right]\frac{\partial^2\Phi}{\partial\phi^2}$$
$$-\frac{\partial}{\partial r}\left(\Delta\frac{\partial\Phi}{\partial r}\right) - \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right) = 0.$$

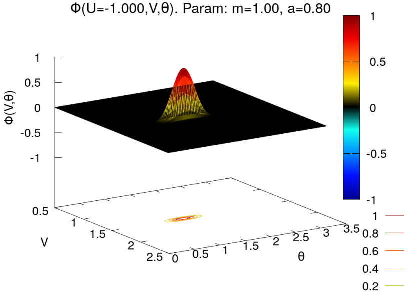
Using *extended center of mass* null coordinates $\{U, V, \theta, \varphi\}$

$$-4\kappa^2\frac{\Upsilon}{\Delta}UV\frac{\partial^2\Phi}{\partial V\partial U} - \kappa\left(V\frac{\partial\Phi}{\partial V} + U\frac{\partial\Phi}{\partial U}\right)\left[\partial_r\sqrt{\mathcal{R}} + \left(\partial_\theta k(r,\theta) + \frac{\cos(\theta)}{\sin(\theta)}k(r,\theta)\right)\right]$$
$$+\frac{2a\,\kappa V}{\Delta}\left(2mr \pm|_{oi}\sqrt{\mathcal{R}}\right)\left(\frac{\partial^2\Phi}{\partial V\partial\varphi}\right) - \frac{2a\,\kappa U}{\Delta}\left(2mr - \pm|_{oi}\sqrt{\mathcal{R}}\right)\left(\frac{\partial^2\Phi}{\partial U\partial\varphi}\right)$$
$$-2k(r,\theta)\,\kappa\left(V\frac{\partial^2\Phi}{\partial V\partial\theta} + U\frac{\partial^2\Phi}{\partial U\partial\theta}\right) - \frac{1}{\sin(\theta)^2}\left(\frac{\partial^2\Phi}{\partial\varphi^2}\right) - \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right) = 0,$$

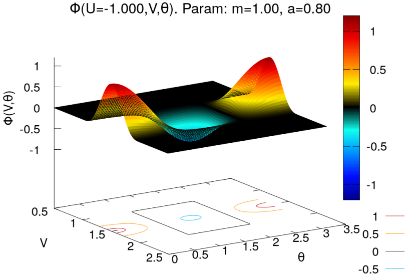
In [Argañaraz and Moreschi, 2022] it is proved that this equation is regular at both exterior horizons H_f (future horizon) and H_p (past horizon).

Axial symmetric initial data reduces the complexity, but keeping the non-trivial angular dependence on θ.

$$\frac{\partial^2\Phi}{\partial V\partial U} = \left[\frac{\Delta}{4\Upsilon\kappa^2 UV}\right]\Bigg\{-\kappa\left(V\frac{\partial\Phi}{\partial V}+U\frac{\partial\Phi}{\partial U}\right)\left(\partial_r\sqrt{\mathcal{R}}+\partial_\theta k(r,\theta)+\frac{\cos(\theta)}{\sin(\theta)}k(r,\theta)\right)$$
$$-2k(r,\theta)\,\kappa\left(V\frac{\partial^2\Phi}{\partial V\partial\theta}+U\frac{\partial^2\Phi}{\partial U\partial\theta}\right)-\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right)\Bigg\}.$$

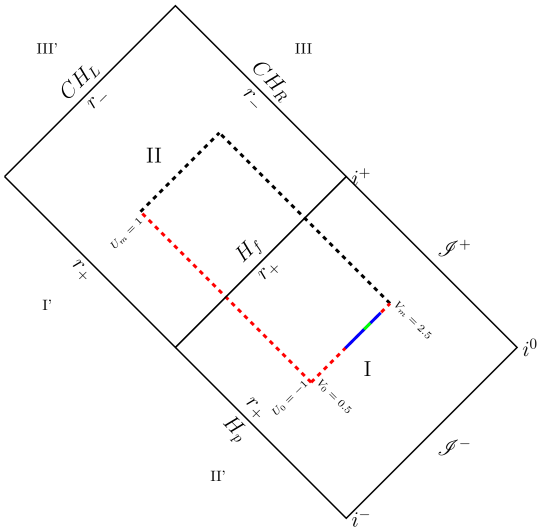


(d) Bell initial data



(e) Armonic initial data

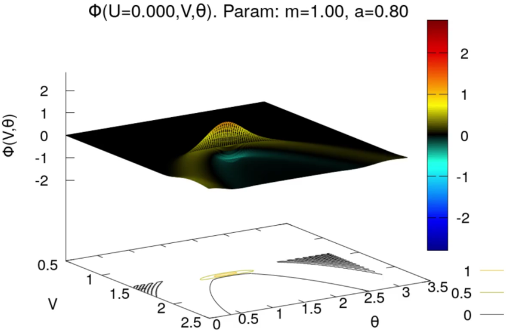
Numerical Evolution: Domain

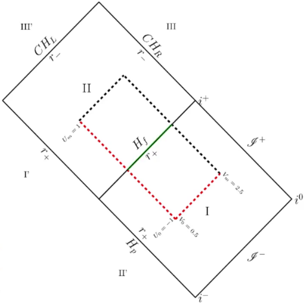


Conformal diagram of Kerr spacetime in *center of mass double null coordinates*. The initial data domain is drawn with dashed lines in red color. The initial data $\Phi(U_0 = -1, V, \theta)$ is non-zero over the blue line for *Armonic-init-data*, and non-zero over the green line for *Bell-init-data*. The whole domain of numerical evolution is the rectangle delimited by dashed lines (black and red)

Numerical Evolution (smooth at Horizon):

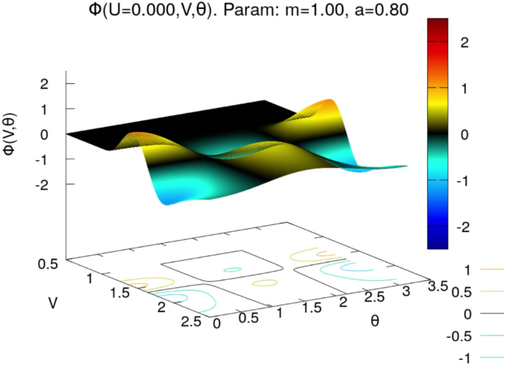
Wave type plot $\Phi(V,\theta)$ at $U = \mathit{constant}$. Bell initial data

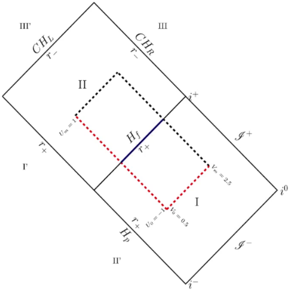




Numerical Evolution (smooth at Horizon):

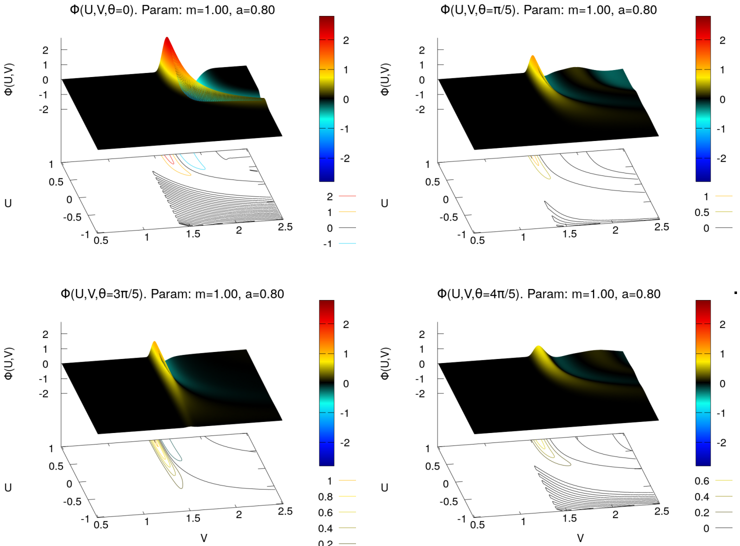
Wave type plot $\Phi(V, \theta)$ at $U = constant$. Armonic initial data





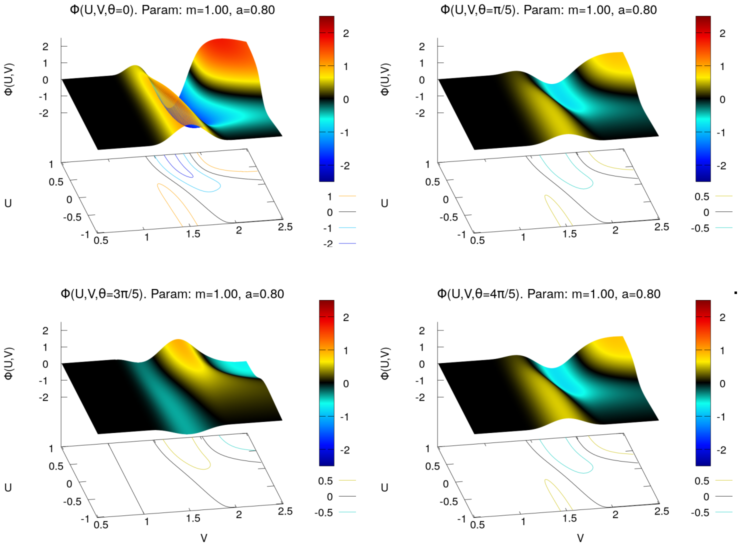
Numerical Evolution (smooth at Horizon):

Causal type plot $\Phi(U, V)$ at $\theta = constant$. Bell initial data.



Numerical Evolution (smooth at Horizon):

Causal type plot $\Phi(U, V)$ at $\theta = constant$. Armonic initial data.



Final Remarks

- This is the first time that a double null coordinate system was used to solve the scalar field equation in Kerr spacetime. The results provides a clear example of how useful can be the *center of mass double null coordinate system*, which is well behaved throughout the spacetime including the axis of symmetry, unlike previous attempts in the literature [Hayward, 2004, Fletcher and Lun, 2003, Bishop and Venter, 2006a, Pretorius and Israel, 1998a].
- Since the work of [Gundlach et al., 1994] (in Schwarzschild), such double null evolution couldn't be extended to Kerr spacetime. In this work, the numerical scheme and code development for Kerr spacetime, clearly establish the feasibility of solving these type of equations with non-trivial angular dependence, at second-order precision $\mathcal{O}\left[h^2\right]$.
- In this work we have shown that the scalar field equation is well behaved across future and past exterior event horizon H_f and H_p, from a fundamental analytical point of view. We have also shown that numerical solutions are well behaved across future exterior horizon H_f.
- The numerical results fidelity was tested with an independent procedure of energy conservation. The energy variation was less than 0.003% in one case and less than 0.005% in the other, which manifest the reliability and accuracy of the numerical evolution.

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