Example: BCFW shift

n-gluon ampl.

choose 2 legs k, l:

$$|\hat{k}] = |k] + z|l] \;; \quad |\hat{l}\rangle = |l\rangle - z|k\rangle$$

all others unchanged

(so the mom: $|\hat{k}]\langle\hat{k}| = (|k] + z|l])\langle k| \qquad r_k = z|l]\langle k|$

$(|\hat{l}]\langle\hat{l}| = |l]\langle l| - z|l]\langle k|) \qquad r_l = -r_k$)

and all other r's $= 0$

$$A = \sum_{\substack{\text{fact}\\ \text{chann.}\\ ij}} \hat{A}_L(z_{ij}) \frac{1}{P_{ij}^2} \tilde{A}_R(z_{ij})$$

which fact channels: only ones w/ z poles:

$\Rightarrow$ legs k, l on different sides

Ex: use this to prove Parke-Taylor $\to$ 86

$$A_n(1^- 2^- 3^+ \cdots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

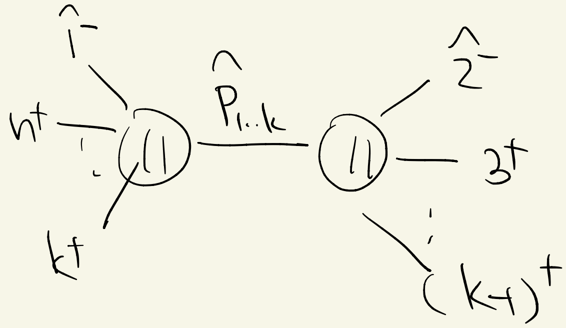
recur. relations Berends Giele 88

$n=3$ ✓

assume for $A_1 \cdots A_{n-1}$

BCFW shift $k=1$ $l=2$

$$A_n(1^- 2^- 3^+ \cdots n^+) = \sum_{k=4}^{n}$$



$$\hat{A}_L(\hat{1}^-; \hat{P}^h_{1..k}, k^+ \cdots n^+) \frac{1}{P^2_{1..k}} \hat{A}_R(-\hat{P}^{-h}_{1..k}, \hat{2}^-, 3^+ \cdots (k-1)^+)$$

$h=+$: $\hat{A}_L = 0$ unless 3pt $(k=n)$

$h=-$: $\hat{A}_R = 0$ — —

$\Rightarrow$ only $\hat{A}_3 \frac{1}{(p_1+p_n)^2} \hat{A}_{n-1} + \hat{A}_{n-1} \frac{1}{(p_2+p_3)^2} \hat{A}_3$

now just calculate.

(will need $|-p]$ $|-p\rangle$: just a phase (convention)

need $|-p]\langle -p| = -|p]\langle p|$

can choose for ex. $|-p\rangle = -|p\rangle$ $]$ unch.

Similarly: all n-gluon ampls.

beyond MHV: more fact. channels.

assumed $A(z) \xrightarrow[z\to\infty]{} 0$ (neglect Res at ∞)

different shifts work for diff. A's.

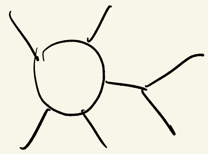
loop: generalized unitarity

construct loops w/out calculating?

tree: A_n^{tree} from lower order $A_{m<n}^{tree}$

based on poles.

loops: (1-loop): poles:

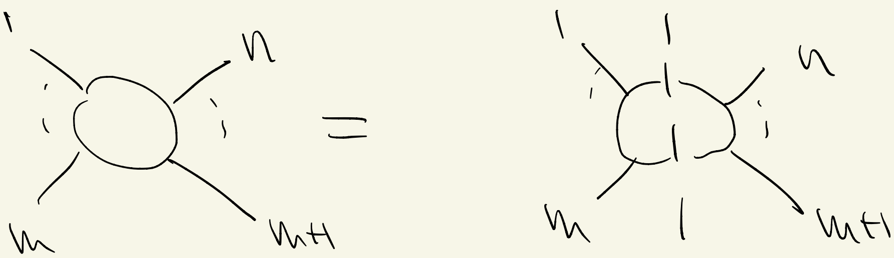


(1-prop on shell)

but also cuts: 2 prop's go on-shell

$$2\,\mathrm{Im}\,T = \mathrm{Disc}\,T = T^{\dagger}T \quad \leftarrow \text{unitarity}$$

Disc



In some cases: can construct A^{loop} from trees.

in others: construct some interesting part of A^{loop} from some A^{tree}'s this way

Why stop at 2 cut prop's? can generalize to larger # of "cut" propagators (props going on-shell) (d=4: 2,3,4) $\Rightarrow$ generalized unitarity.

Can write any loop inte. in terms of scalar integrals

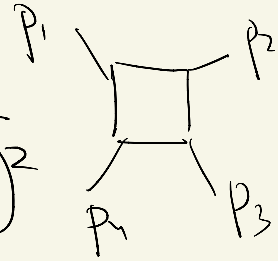
$$A^{1\text{-loop}} = \sum_i C_4^i I_4^i + \sum_j C_3^j I_3^j + \sum_k C_2^k I_2^k$$

Passarino Veltman reduction

Coeff (depend on kinematics

$$I_4(p_1,\ldots,p_4) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 (l-p_1)^2 (l-p_1-p_2)^2 (l-p_1-p_2-p_3)^2}$$

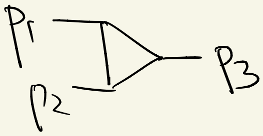
boxes

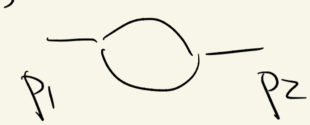


$$I_3(p_1,p_2,p_3) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 (l-p_1)^2 (l-p_1-p_2)^2 (l-p_1-p_2-p_3)^2}$$

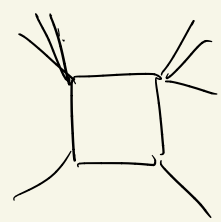
triangles

$$I_2(p_1,p_2) = \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 (l-p_1)^2}$$





p_i can be sum of several mom:



integrals are known; just need coeff C.

only bubbles C_2 are UV divergent! (RGE)

Why PV? $\int d^4 l \, \frac{l^\mu}{l^2 (l-p_1)^2 - (l-p_1-p_2-p_3)^2} = a_1 p_1^\mu + \ldots + a_3 p_3^\mu$

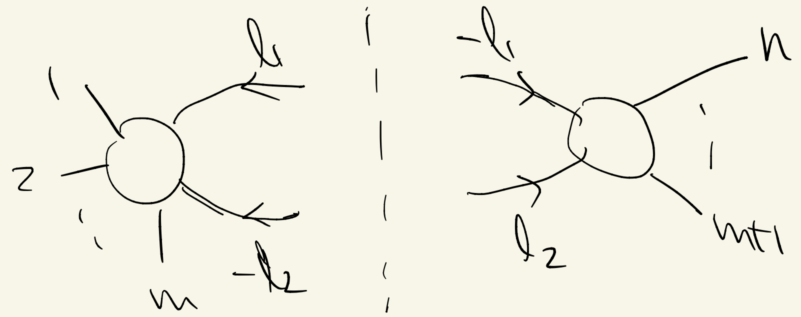
dot w/ eg p_1: RHS $a_2 p_1 \cdot p_2 + a_3 p_1 \cdot p_3$

LHS: num $l \cdot p_1 = -\frac{1}{2}\left[(l-p_1)^2 - l^2\right]$ $\to$ scalar integrals

actually $\exists$ also "rational terms" on RHS: R_n
(no loop integ.); cannot be determined in this way

regular cut: 2 prop's going on-shell ("double cut")

in $s_{1..m}$ channel: $\mathrm{Disc}_{(1..m)} \mathcal{M} =$



$-l_2 = l_1 + (k_1 + \ldots + k_m)$

and will use:

$$\mathrm{Disc}_{(1..m)} \mathcal{M} = \sum_i c_4^i \, \mathrm{Disc}_{(1..m)} I_4^i + \sum_j c_3^j \, \mathrm{Disc}_{(1..m)} I_3^j + \sum_k c_2^k \, \mathrm{Disc}_{(1..m)} I_2^k$$

Disc of boxes, triangles, bubbles known.

now look at LHS: $\frac{i}{l_1^2 + i\varepsilon} \to 2\pi\delta(l_1^2) \qquad \frac{i}{l_2^2 + i\varepsilon} \to 2\pi\delta(l_2^2)$

$$\mathrm{Disc}_{(1..m)} \mathcal{M} = \sum_{h_1, h_2} \int \frac{d^4 l_1}{(2\pi)^2} \delta(l_1^2)\delta(l_2^2) A^{\text{tree}}(l_1^{h_1}, 1, 2, \ldots, m, -l_2^{-h_2}) A^{\text{tree}}(l_2^{h_2}, (m+1), \ldots n, -l_1^{-h_1})$$

$\mathrm{Disc}_{(1..m)} M =$

$$\mathrm{Disc}_{(1..m)} \sum_{h_1,h_2} \int \frac{d^4 l_1}{(2\pi)^2} \frac{i}{l_1^2} \frac{i}{l_2^2} \, A^{tree}(l_1^{h_1}, 1, 2 .., m, -l_2^{-h_2}) \, A^{tree}(l_2^{h_2}, (m+1), .. n, -l_1^{-h_1})$$

in this integrand have the product of 2 on-shell tree amplitudes – can be simplified using all the tools we saw.

---

note: need $\quad l_1^2 = 0 \qquad l_2^2 = (l_1 + 1 + .. + m)^2 = 0$

$$\Rightarrow 2 l_1 \cdot (1 + .. + m) + (1 + .. + m)^2 = 0$$

$\downarrow$

$l_{1x}\ l_{1y}, l_{1z}$ free $\quad]\ $ 2 free par's remain.

---

but here can go on: consider also

triple cuts: 3 prop's go on-shell

quadruple cuts: 4 " "

no more: no more soln's

triple cut: integrand:

$$\sum_{h_1,h_2,h_3} A^{tree}(l_1^{h_1}, 1, \ldots m, -l_2^{h_2}) \cdot A^{tree}(l_2^{h_2}, (m+1) \ldots k, -l_3^{h_3})$$

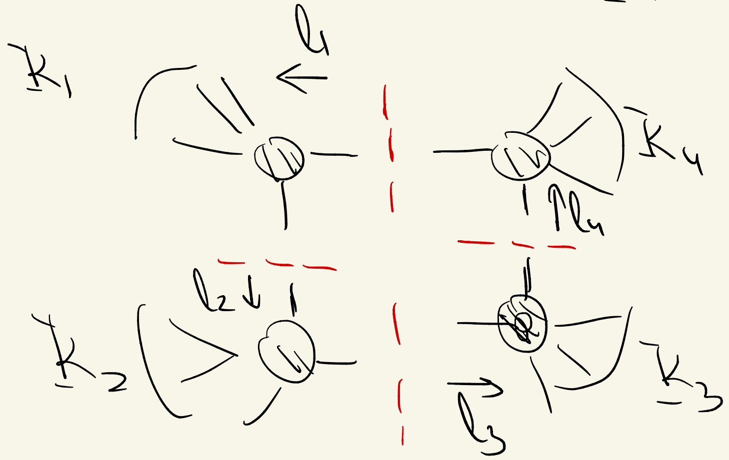
$$\cdot A^{tree}(l_3^{h_3}, k+1, \ldots n, -l_1^{h_1})$$

RHS: box, triangle only

quadruple cut:

now we partition $k_1 \ldots k_n$ into 4 groups

with the sums $\underline{K}_1 \ldots \underline{K}_4$



$l_2 = l_1 + \underline{K}_1 \quad l_3 = l_1 + \underline{K}_1 + \underline{K}_2 \quad l_4 = l_1 + \underline{K}_2 + \underline{K}_3$

4 unknowns l_1^μ ; 4 eqns: $l_i^2 = 0$

$l_1^2 = 0$: $l_1^0 = \pm |\vec{l}_1|$; 3 other equ's linear

$\Rightarrow$ 2 discrete soln's

for each soln +/- have:

LHS integrand: $A_1^{tree}(l^+)\, A_2^{tree}(l^+)\, A_3^{tree}(l^+)\, A_4^{tree}(l^+)\ \frac{1}{l_1^2}\cdots\frac{1}{l_4^2}$

on RHS: just one box contributes (the one w/ same partition)

$c_4^{(i)}\ \frac{1}{l_1^2}\cdots\frac{1}{l_4^2} \qquad l_1 = l_1^{\pm}$

$$\Rightarrow c_4^{(i)} = \frac{1}{2}\left[A_1^{tree}(l^+)\, A_2^{tree}(l^+)\, A_3^{tree}(l^+)\, A_4^{tree}(l^+) + (+\leftrightarrow -)\right]$$

find box part w/out doing any loop integral
from on-shell tree-level ampl. only!

+ typically: many contrib's $=0$ because of tree-level
selection rules (can't sew helicities)

eg $A_n(+\cdots+)$ $n\geq 5$: no box/triang/bubble.

to determine bubble coeff: 1st box (quad cut)
then triang (triang ")
" bubble (double cuts)