ICTP-SAIFR Lectures on Light Feebly Interacting Particles (Personal Notes)

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September 20, 2022

Important Disclaimer: These are not finished lecture notes, but just personal notes. The text is far from polished (or fully comprehensible), but it should help you following the lectures. Please do not share them and please let me know if you find any typo or mistake.

1 Motivation

Why is it interesting to think about light and weakly interacting particles? First of all, light and weakly coupled with respect to what? Two possible answers: historical (draw coupling vs mass plane) and say light with respect to the energy frontier 100 GeV, $1 - 0.1$ couplings. Physical answer: light and weakly coupled with respect to the best answers that we have to some of the biggest problems in particle physics. The light and feeble frontier is interesting because it answers big questions differently, without introducing any theoretical strain compared to more traditional answers.

1. Dark Matter
2. A marginal operator that we do not understand
$$\theta G\widetilde{G} = \frac{\theta}{2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[G_{\mu\nu}G_{\rho\sigma}] = \frac{\theta}{2}\epsilon^{\mu\nu\rho\sigma}\sum_{a=1}^{8} G^a_{\mu\nu}G^a_{\rho\sigma} \tag{1.1}$$
$[\theta] = E^0$ can receive $\mathcal{O}(1)$ contributions from all energy scales. The answer might be at high energy Λ, manifesting itself as small $\mathcal{O}(E/\Lambda)^n$ effects at low energy (Appelquist-Carazzone).
3. Hierarchy problem (see appendix and Gilad's lectures)
4. Top down motivation from string theory (mainly dark photons and axions)
5. Experiment (add compilation Figure or draw sketch). Message: there is a large parameter space that we have not explored. Note that smaller mass = stronger constraints is not always true. See next Section.

The first three are the three big questions in particle physics (ignoring the cosmological constant).

1.1 Are lighter Particles Easier to Detect?

By easier we mean that we can constrain smaller couplings. Naively one would immediately say yes, since

$$\sigma \sim \frac{g^2}{m^2} \,. \tag{1.2}$$

However g is hardly every truly independent of m. For example a scalar of mass m in a theory valid up to a scale Λ naturally can have couplings only $g \lesssim 4\pi m/\Lambda$. This already shows that the question is not as simple as it seemed. Let's start with spin zero scalars.

1.1.1 Spin zero CP-even Scalars

For CP-even bosons the general answer is yes because lighter particles can give a coherent effect on longer distances and generate longer range forces. Let's start with the forces. Born approximation to the scattering amplitude in NR quantum mechanics

$$\langle p|iT|p'\rangle = -i\tilde{V}(\vec{p}-\vec{p}')(2\pi)\delta(E_p - E_{p'}) \tag{1.3}$$

Yukawa interaction $\phi q\bar{q}$

$$\tilde{V}(\mathbf{q}) = -\frac{g^2}{|\mathbf{q}|^2 + m_\phi^2} \,. \tag{1.4}$$

In position space

$$V(\mathbf{x}) = -\int \frac{d^3q}{(2\pi)^3} \frac{g^2}{|\mathbf{q}|^2 + m_\phi^2} e^{i\mathbf{q}\cdot\mathbf{x}} \,. \tag{1.5}$$

Close contour above in the complex plane, catch $q = im_\phi$ pole

$$V(\mathbf{x}) \sim e^{-m_\phi r} \,. \tag{1.6}$$

Dimensional analysis

$$V(\mathbf{x}) \sim \frac{1}{r} e^{-m_\phi r} \,. \tag{1.7}$$

For the actual integral see Peskin.

Let's now look at coherent enahncements. A particle of momentum $q = mv$ is localized over a distance

$$\Delta x \simeq \frac{\hbar}{q} \equiv \lambda_{\rm dB} \,. \tag{1.8}$$

When you compute the amplitude for its interaction with a material you have to sum over all the particles within a sphere of radius $\lambda_{\rm dB}$ so you will have

$$\mathcal{M} = N_p \mathcal{M}_p \,, \tag{1.9}$$

where $\mathcal{M}_p$ is the single particle amplitude and $N_p \simeq \rho\lambda_{\rm dB}^3$ the particles in the sphere. This gives a huge enhancement in the cross section

$$\sigma \sim N_p^2 \sigma_p \,. \tag{1.10}$$

The interactions sum coherently, giving a N^2 enhancement. If $\lambda_{\rm dB}$ is smaller than the inter-particle spacing in the material, our particle can still cross a macroscopic slab, seeing on its path N_p particles (sketch), but in this case we have to sum probabilities, so

$$\sigma_I \sim N_p \sigma_p \,. \tag{1.11}$$

This can be a huge difference for a solid where $N_p \simeq 10^{23}$ (Avogadro's number).

Since we measure quantities with some finite absolute precision the variation of an observable

$$\delta O = g\langle O\rangle \lesssim \delta O_{\rm exp} \tag{1.12}$$

due to a small coupling g is easier to see if the background value $\langle O\rangle$ is big. Summing $\langle O\rangle$ coherently over large distances gives an advantage. For instance $\phi\bar{q}q$ couples to mass $\langle \bar{q}q\rangle \simeq \Lambda_{\rm QCD}^3$ (in a nucleus). If we're looking for a force $\langle O\rangle$ might be a distance and it's enhanced by the first argument. If we're looking for other effects, the second argument might be the relevant one.

However this is not always true (important exceptions):

1. dark photons (see appendix)

2. Pseudo-scalars. If CP is not broken they can only couple to CP-odd bilinears

1.1.2 Spin zero CP-odd scalars

First of all we should ask why a CP-odd scalar might be light and the natural answer is Goldstone's theorem. So it makes sense to focus on pseudo-scalars with derivative couplings to the SM. For example we can have interactions like

$$\frac{\partial_\mu a}{f_a}\bar{\psi}\gamma^\mu\gamma^5\psi \,. \tag{1.13}$$

or integrating by parts

$$g_a a\bar{\psi}\gamma^5\psi \,. \tag{1.14}$$

At low energy

$$\bar{u}^r(p')\gamma^5 u^s(p) \to m\xi^{r\dagger}(\vec{p}-\vec{p}')\cdot\vec{\sigma}\xi^2 \,. \tag{1.15}$$

a is coupling to spin, so there's no collective effects unless we can polarize the material! Exercise Derive the NR limit of $\bar{\psi}\gamma^\mu\gamma^5\psi$.

Nonetheless a pseudo-scalar can still couple coherently to large EM fields

$$aF\widetilde{F} = 4a\mathbf{E}\cdot\mathbf{B} \,. \tag{1.16}$$

Note that this is still a derivative coupling (i.e. $F\widetilde{F}$ is a total derivative). The bottom line is that there is no general intuition and one should check case by case.

1.1.3 Dark Photons

We will consider a sub-component of dark matter to be millicharged, with effective charge $q_{\text{eff}} \ll 1$. If electric charge is quantised, as expected if the electromagnetic $U(1)_{\text{em}}$ originates in a grand unified theory, the simplest realisation of millicharged particles is to invoke kinetic mixing between $U(1)_{\text{em}}$ and a dark sector $U(1)'$. The Lagrangian of the theory is then

$$\mathcal{L} = \mathcal{L}_{\text{mCP,kin.}} - \frac{1}{4}(F'_{\mu\nu}F'^{\mu\nu} + F_{\mu\nu}F^{\mu\nu} - 2\epsilon F'_{\mu\nu}F^{\mu\nu}) + \frac{m_{A'}^2}{2}A'_\mu A'^\mu - eA_\mu J^\mu - e'A'_\mu J^\mu_{\text{mCP}} \ . \quad (1.17)$$

The kinetic mixing parameter is ϵ, while the usual electromagnetic coupling is e, and the mCP gauge coupling is e'. For now we do not specify the mCP spin, and leave their kinetic term $\mathcal{L}_{\text{mCP,kin.}}$ and current J^μ_{mCP} implicit. To leading order in ϵ, the transformation

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \epsilon A'_\mu, \\ A'_\mu &\rightarrow A'_\mu \, , \end{aligned} \quad (1.18)$$

brings the gauge bosons kinetic terms in canonical form. The dark photon A' now couples not only to the mCP current J^μ_{mCP}, but also with a strength suppressed by ϵ to the SM current J^μ,

$$\mathcal{L} = \mathcal{L}_{\text{mCP,kin.}} - \frac{1}{4}(F'_{\mu\nu}F'^{\mu\nu} + F_{\mu\nu}F^{\mu\nu}) + \frac{m_{A'}^2}{2}A'_\mu A'^\mu - e(A_\mu + \epsilon A'_\mu)J^\mu - e'A'_\mu J^\mu_{\text{mCP}} \ . \quad (1.19)$$

As a result, any SM current used to source a visible electromagnetic field is also sourcing an ϵ-suppressed dark field, which can couple to the mCPs. When the dark photon mass is very small, this leads to the "effectively millicharged" limit, where the range of the dark photon is so long that for experimental purposes, one can treat the mCPs as coupling directly to the visible photon. This leads to a natural definition of the millicharge or effective charge of the dark sector particles

$$q_{\text{eff}} \equiv \epsilon \frac{e'}{e} \ . \quad (1.20)$$

The exactly massless limit deserves special attention. In this limit, after the shift in Eq. (1.18) the quadratic part of the Lagrangian is simply

$$\mathcal{L} \supset \mathcal{L}_{\text{mCP,kin.}} - \frac{1}{4}(F'_{\mu\nu}F'^{\mu\nu} + F_{\mu\nu}F^{\mu\nu}) \, , \quad (1.21)$$

and any orthogonal transformation that mixes A and A' leaves it invariant. In particular we can perform the $\mathcal{O}(\epsilon)$ rotation

$$\begin{aligned} A_\mu &\rightarrow A_\mu - \epsilon A'_\mu, \\ A'_\mu &\rightarrow A'_\mu + \epsilon A_\mu \, , \end{aligned} \quad (1.22)$$

and obtain

$$\mathcal{L} = \mathcal{L}_{\text{mCP,kin.}} - \frac{1}{4}(F'_{\mu\nu}F'^{\mu\nu} + F_{\mu\nu}F^{\mu\nu}) - eA_\mu J^\mu - e'(A'_\mu + \epsilon A_\mu)J^\mu_{\text{mCP}} \ . \quad (1.23)$$

Therefore in the massless dark photon limit, we get true mCPs. The visible photon A_μ couples directly to the mCP current $J^\mu_{\rm mCP}$ with strength $q_{\rm eff} = \epsilon e'/e$ relative to its coupling strength to J^μ. We could have seen this also by directly performing the rotation $A_\mu \to A_\mu$, $A'_\mu \to A'_\mu + \epsilon A_\mu$ in the original Lagrangian. However, we went through the trouble of adding one more step, because it is useful to think about the problem in terms of the (A, A') plane and the $SO(3)$ symmetry of the kinetic terms.

When $e' = m_{A'} = 0$ the coupling to J^μ identifies a preferred direction in the (A, A') plane, but the rest of the Lagrangian is $SO(3)$ symmetric so there is no physical meaning to this specific direction. We can rotate it at will without changing the rest of the Lagrangian. An observer will call "photon" anything that couples to J^μ regardless of its composition in terms of A and A'. This makes the dark photon in practice unobservable.

In this language it is easy to see that turning on either $m_{A'}$ or e' makes the dark photon observable. Now we have at least two vectors in the (A, A') plane. One is still given by the field coupling to J^μ, the other either by the massive field or the field coupled to $J^\mu_{\rm mCP}$. A rotation in this plane does not change the scalar product between the two vectors, so regardless of the basis that we choose we can ask physical questions: How much does the massive photon overlap with the photon coupled to SM charges? Or how much does the photon coupled to SM charges couple to dark currents?

So if either $e' \neq 0$ or $m_{A'} \neq 0$ we can perform measurements that reveal the existence of a second vector in the (A, A') plane and thus of the plane itself. In other words the dark photon is observable. On the contrary, if $e' = m_{A'} = 0$ we have access to a single vector and we will never know if it is embedded in a plane or not.

This discussion explains all the standard results expected for massless and massive dark photons. However it is useful to come back to the massless limit and be more explicit. The massless limit of a theory with millicharges has an important difference with respect to traditional dark photon searches that do not postulate the existence of these particles. In the absence of millicharges ($e' = 0$) we can repeat the steps above Eq. (1.23) and get a completely decoupled dark photon,

$$\mathcal{L} = \mathcal{L}_{\rm mCP,kin.} - \frac{1}{4}(F'_{\mu\nu}F'^{\mu\nu} + F_{\mu\nu}F^{\mu\nu}) - eA_\mu J^\mu \ . \tag{1.24}$$

as expected from our general geometric argument. The standard intuition of a decoupled dark photon in the $m_{A'} = 0$ limit holds only in absence of charged dark sector states. Parametrically this means that in absence of millicharges all physical signals go to zero with $m_{A'}$. In our theories that do contain mCPs we will have signals that are not proportional to $m_{A'}$ as in the standard case.

The difference is also geometric (in the physical space of the experiment). A dark photon without a charged dark sector can not produce only transverse signals, because in the massless limit it is decoupled from the SM. It was shown that its longitudinal signals are dominant, when $m_{A'}$ is small compared to the characteristic frequencies of the experiment []. On the contrary our dark photon coupled to millicharges can give an experimental signal also when it is massless and purely transverse.

We give particular emphasis to this limit because we will see that standard searches always outperform searches for the millicharged signal when $m_{A'}$ is parametrically important, so in practice the massless or effectively massless dark photon limit will be the most relevant for our setups.

2 Dark Matter

What do we know?

1. DM has gravitational interactions

2. It has a lifetime comparable to that of the Universe. At least $\mathcal{O}(t_0 - t_{\rm EQ}) \simeq 13.9 \times 10^9$ years. It could be much longer if DM decays in certain energy ranges that can be detected by satellites now in orbit.

3. Removing DM completely is a disaster for the CMB (draw CMB peaks)

4. DM is dark: $\sigma(\chi\gamma \to \chi\gamma) \lesssim 10^{-33}$ cm$^2(m/\text{GeV})$. For reference $\sigma(e^-\gamma \to e^-\gamma) \simeq 0.5 \times 10^{-24}$ cm^2 $\simeq 10^{12}$ $\sigma(\chi\gamma \to \chi\gamma)$ (last equality valid for $m \simeq m_e$, electron cross section computed at $E_\gamma \simeq E_e \simeq m_e$).

5. DM does not form disks of the same thickness as those of baryons. We can have only approximately $3\% \div 5\%$ of it in structures of this type.

6. 10^{-21} eV $\lesssim m \lesssim 10^{48}$ GeV. Lower bound: dwarf galaxies

$$\lambda_{\rm DM} = 1/(m v_{\rm DM}) \simeq \text{kpc}(10^{-22}\text{ eV}/m). \tag{2.1}$$

   Upper bound from lensing (plus other constraints at higher masses: disruption of binaries, friction in halos, CMB [gravitational waves from mergers induce distortions], LSS structure do not form earlier than observed, https://arxiv.org/pdf/2110.02821.pdf).

7. Self-interactions $\sigma/m \lesssim \text{cm}^2/g \simeq 10/\text{GeV}^3$. $\alpha_{\rm SI} \gtrsim 10^{-100}$ (gravity).

8. DM is cold or warm $\lambda_{\rm FS} \lesssim$ Mpc

9. Pauli exclusion principle + density of dwarf galaxies (escape velocity) gives $m \gtrsim$ keV for fermions

$$v_F(r) = \left(\frac{2\pi^2\rho(r)}{m^4}\right)^{1/3} < v_{\rm esc} \tag{2.2}$$

   write above Eq "Quantum Mechanics" and "Newtonian Gravity" as in the colloquium.

10. Model dependent bounds on SM couplings $\mathcal{O}(10^{-66}) \lesssim \alpha_{\rm SM} \lesssim 4\pi$. Lower bound from gravity. Upper bound can be geometric.

Conclude with usual discussion on huge parameter space. We need theory motivation! A priori this is not telling us that DM has to be light or weakly coupled, but while searching for the most theoretically motivated parameter space, we'll find that light, weakly coupled particles are very good candidates.

3 Thermal History of the Universe

We know that

1. On scales > 100 Mpc the Universe is 1) Homogeneous (translational invariant) 2) isotropic (rotational invariant). Below this scale there are galaxies.

2. It is expanding $|v_A - v_B| = H(t)d_{AB}$, $H(t_0) \simeq 70\text{km}/(\text{Mpc} \times \text{s})$.

3. When it was 10^3 smaller than now and about 13 Gy younger than now $\Delta\rho/\rho \simeq 10^{-5}$.

You need something that starts very homogeneous and isotropic, but has the possibility to evolve in time to become more anisotropic and accommodate Hubble expansion. FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{(2)}\right], \quad d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta d\phi^2 \,. \tag{3.1}$$

Hubble expansion $|v_A - v_B| = d_{AB}\dot{a} = d_{AB}H$, $H = \dot{a}/a$. The metric is describing in polar coordinates

$$x = rc_\theta s_\phi, \quad y = rs_\theta s_\phi, \quad z = rc_\phi \,. \tag{3.2}$$

a 3D manifold that is translationally and rotationally invariant. If we were in 4D we would write this manifold as

$$x^2 + y^2 + z^2 + u^2 = \vec{x}\cdot\vec{x} = a^2 \,. \tag{3.3}$$

The length element is

$$dl^2 = dx^2 + dy^2 + dz^2 + du^2 = dr^2 + r^2 d\Omega^2_{(2)} + du^2 \,, \tag{3.4}$$

but

$$du^2 = (du)^2 = (d\sqrt{a^2 - x^2 - y^2 - z^2})^2 = \frac{(xdx + ydy + zdz)^2}{a^2 - r^2} = \frac{(rdr)^2}{a^2 - r^2} \tag{3.5}$$

Then in polar coordinates

$$dl^2 = dr^2 + r^2 d\Omega^2_{(2)} + \frac{(rdr)^2}{a^2 - r^2} = r^2 d\Omega^2_{(2)} + \frac{a^2 dr^2}{a^2 - r^2} = r^2 d\Omega^2_{(2)} + \frac{dr^2}{1 - r^2/a^2} \tag{3.6}$$

Then $r \to r \times |a|^2$ (large coord. freedom of GR) gives us the FRW metric with $k = +1, -1, 0$.

How do we compute $a(t)$? From Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}\left(\frac{R}{2} - \Lambda\right) = 8\pi G_N T_{\mu\nu} \,. \tag{3.7}$$

How does $T_{\mu\nu}$ look like in our Universe?

For one particle we have

$$T_{\mu\nu} = \frac{p_\mu p_\nu}{p_0}\delta^3(\vec{x} - \vec{x}_p(t)) \tag{3.8}$$

Excercise Derive the above $T_{\mu\nu}$ from the single particle action

$$S = -m\int d\tau = -m\int \gamma dt\,. \tag{3.9}$$

End Excercise N particles

$$T_{\mu\nu} = \sum_k \frac{p^k_\mu p^k_\nu}{p^k_0}\delta^3(\vec{x} - \vec{x}^k_p(t)) \tag{3.10}$$

In the limit of a large number of particles per unit volume we can take the continuum limit

$$T_{\mu\nu} \simeq \int d^3p\, n(|\vec{p}|)\frac{p_\mu p_\nu}{p_0}\,. \tag{3.11}$$

where $n(|\vec{p}|)$ is the number of particles with magnitude of the three-momentum $|\vec{p}|$ per unit volume. If the Universe is homogeneous and isotropic, n depends only on the magnitude of the three-momentum. Obviously $n(|\vec{p}|)$ is an even function of each of the components of $\vec{p}$, so all off-diagonal terms vanish

$$T_{ij} = \int d^3p\, n(|\vec{p}|)\frac{p_i p_j}{p_0}\,. \tag{3.12}$$

because where integrating the even function $n(|\vec{p}|)/p_0$ times the odd function p_i over a symmetric interval. How do we interpret the diagonal elements? T_{00} is obviously the energy density

$$T_{00} \;=\; \int d^3p\, n(|\vec{p}|)p_0 \sim \frac{E}{V} \equiv \rho\,. \tag{3.13}$$

To understand T_{ii} imagine a cube of size L and a particle travelling in the i direction that bounces off one of its walls elastically. Then the momentum exchange is $\Delta p_i = 2mv_i$ and the force exerted on the wall

$$\Delta F_i = \frac{\Delta p_i}{\Delta t} = \frac{mv_i v_i}{L} \tag{3.14}$$

Note that the particle travelled a distance $2L$ between to successive bounces. The pressure is

$$p = \frac{\Delta F_i}{L^2} = \frac{p_i v_i}{L^3} = \frac{p_i m\gamma v_i}{L^3 p_0} = \frac{p_i^2}{L^3 p_0} \tag{3.15}$$

This is precisely what our T_{ii} looks like for a single particle. We are in a homogeneous and isotropic Universe so all three directions must be equivalent. So finally for a perfect fluid $T_{\mu\nu} =$

diag(ρ, p, p, p). Note that in general ρ and p are related. For instance for a NR particle $p \simeq 0$, $\rho \simeq m_{\rm tot}/V$. For a relativistic particle $p = (1/3)\vec{p}\cdot\vec{p}/E = (1/3)\rho$.

For a perfect fluid Einstein's equations become

$$\begin{aligned} H^2 &= \frac{8\pi G_N \rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \\ \dot{H} + H^2 &= \frac{\ddot{a}}{a} = -\frac{8\pi G_N}{6}(\rho + 3p) + \frac{\Lambda}{3} \,. \end{aligned} \tag{3.16}$$

What are ρ and p? We know experimentally that the Universe was in thermal equilibrium with $T \simeq$ MeV and almost only SM particles when it was $\simeq 10^6$ times smaller than today. We know this by measuring light elements' abundances He4, Li, D, ... in areas poor of stars and comparing it with the thermal eq. calculation in the SM. We can take this as a starting point and see how quantities evolve from then (going both in the past or in the future according to what we care about). First of all we have then to understand what ρ and p look like in equilibrium.

p_α = probability of state α. Entropy

$$S = -\sum_\alpha p_\alpha \log p_\alpha \,, \tag{3.17}$$

The maximum is

$$S_{\rm max} = \log\Gamma \quad \Gamma = N_s \tag{3.18}$$

where N_s is the total number of states, meaning that all states are equiprobable $p_\alpha = 1/N_s$ for all α. If we define ΔN_ϵ to be the number of particles in the energy interval $[\epsilon, \epsilon + \Delta\epsilon]$ and Δg_ϵ as the. number of microstates in the one particle phase space. This means

$$\Delta g_\epsilon = g\int_\epsilon^{\epsilon+\Delta\epsilon} \frac{d^3p d^3x}{(2\pi\hbar)^3} \,, \tag{3.19}$$

this is because of the uncertainty principle $\Delta p\Delta x \geq h$ so states in a phase space cell $< (2\pi\hbar)^3$ are indistinguishable. Here g are the internal degrees of freedom. Then the total number of microstates for **bose** particles corresponds to how many ways you have of putting ΔN_ϵ particles in Δg_ϵ cells, i.e.

$$\Delta G_\epsilon = \frac{(\Delta N_\epsilon + \Delta g_\epsilon - 1)!}{\Delta N_\epsilon!(\Delta g_\epsilon - 1)!} \tag{3.20}$$

The total number of states is then

$$\Gamma = \prod_\epsilon \Delta G_\epsilon \tag{3.21}$$

Explanation: The ΔN_ϵ particles between $[\epsilon, \epsilon + \Delta\epsilon]$ can be in any one of their ΔG_ϵ states. The same is true for the $\Delta N_{\epsilon'}$ particles between $[\epsilon', \epsilon' + \Delta\epsilon]$ and so on. So the total number of states for the whole system is the product in the above equation. Then

$$S_{\rm max} = \sum_\epsilon \log \Delta G_\epsilon \,. \tag{3.22}$$

For bose particles we can assume $\Delta N_\epsilon, \Delta g_\epsilon \gg 1$ and expand

$$\log N! = \sum_{n=1}^{N} \log N \simeq \int_1^N dx \log x + \frac{\log N}{2} \simeq (N+1/2)\log N - N \tag{3.23}$$

Finally

$$S_{\max} = \sum_\epsilon \left[(n_\epsilon+1)\log(n_\epsilon+1) - n_\epsilon \log n_\epsilon\right]\Delta g_\epsilon, \quad n_\epsilon \equiv \frac{\Delta N_\epsilon}{\Delta g_\epsilon}\,. \tag{3.24}$$

n_ϵ are called occupation numbers, they characterize the average number of particles per microstate of a single particle. Let us maximize S keeping energy and number of particles fixed

$$\begin{aligned} E &= \sum_\epsilon \epsilon \Delta N_\epsilon = \sum_\epsilon \epsilon n_\epsilon \Delta g_\epsilon \\ N &= \sum_\epsilon \Delta N_\epsilon = \sum_\epsilon n_\epsilon \Delta g_\epsilon \end{aligned} \tag{3.25}$$

We can use Lagrange multipliers

$$S + \lambda_1 E + \lambda_2 N \tag{3.26}$$

Taking derivatives (wrt n_ϵ) we get

$$n_\epsilon = \frac{1}{e^{-\lambda_1\epsilon-\lambda_2}-1} = \frac{1}{e^{\frac{\epsilon-\mu}{T}}-1} \tag{3.27}$$

One can then measure n_ϵ and find that the two Lagrangian parameters are temperature and chemical potential. Note that to obtain this equation we have minimized

$$-S - \lambda_1 E - \lambda_2 N \tag{3.28}$$

So it's useful to think of this quantity as if it was a potential determining the equilibrium state of the system. Indeed

$$\frac{F}{T} = S + \frac{E}{T} + \frac{\mu N}{T} \tag{3.29}$$

is the free energy (a thermodynamic potential) we are saying that $\mu = 0$ means that we can change N for free (we're not affecting the potential). Similarly we are saying that when $T \to \infty$ there is no penalty in changing the energy of the system, which is a bit crazy, but quite intutitive. If we take into account that two Fermi particles can't occupy the same microstate we get

$$S_{\max} = \sum_\epsilon \left[(n_\epsilon-1)\log(n_\epsilon-1) - n_\epsilon \log n_\epsilon\right]\Delta g_\epsilon \tag{3.30}$$

and

$$n_\epsilon = \frac{1}{e^{-\lambda_1 \epsilon - \lambda_2} + 1} = \frac{1}{e^{\frac{\epsilon-\mu}{T}} + 1}\,. \tag{3.31}$$

The beauty of this derivation is that used only the thermodynamic principle of maximum entropy and the counting of states. It's valid in any geometry!

In QFT N is not conserved. For instance $e^+e^- \to \gamma\gamma$. However if the reactions

$$A + B \leftrightarrow C + D \tag{3.32}$$

are in equilibrium then

$$\mu_A + \mu_B = \mu_C + \mu_D \tag{3.33}$$

So we have to work out case by case the chemical potentials. Note that for relativistic particles (for instance SM photons) $\mu = 0$, physically this means that you can always produce for free a zero energy photon, so there's no constraint on the total number of photons. So when the previous process is in equilibrium $\mu_{e^+} = -\mu_{e^-}$ (related to charge conservation).

This can be derived explicitly from the equation of state of a relativistic ideal gas

$$pV = E/3 \tag{3.34}$$

The relation above comes from our relativistic stress energy tensor. Plus the fact that we'll derive below that $E = T^4 \times \text{const}$. Exercise Do this derivation using thermodynamic potentials or look up the solution.

We set off to compute ρ and p so let's do it

$$\rho = \frac{E}{V} = \frac{\sum_\epsilon \epsilon n_\epsilon \Delta g_\epsilon}{V} \simeq \frac{g}{(2\pi)^3 V}\int d^3p d^3x \frac{1}{e^{\frac{\epsilon-\mu}{T}} \pm 1} = \frac{g}{(2\pi)^3}\int d^3p \frac{1}{e^{\frac{\epsilon-\mu}{T}} \pm 1} = \frac{g}{2\pi^2}\int_m^\infty \frac{\sqrt{\epsilon^2 - m^2}\epsilon^2 d\epsilon}{e^{\frac{\epsilon-\mu}{T}} \pm 1} \tag{3.35}$$

The number of particles per unit volume is then

$$n \equiv \frac{N}{V} = \frac{g}{2\pi^2}\int_m^\infty \frac{\sqrt{\epsilon^2 - m^2}\epsilon d\epsilon}{e^{\frac{\epsilon-\mu}{T}} \pm 1} \tag{3.36}$$

One can show (see Mukhanov or Kolb and Turner) that

$$p = \frac{\rho}{3} - \frac{g^2 mn}{6\pi^2}\,. \tag{3.37}$$

Finally

$$s \equiv \frac{S}{V} = \frac{\epsilon + p - \mu n}{T}\,. \tag{3.38}$$

Useful expansions (numbers valid for bosons

$$T \gg m, \mu \to n \simeq \frac{\zeta(3)}{\pi^2} T^3, \rho \simeq \frac{g\pi^2}{30} T^4, p \simeq \rho/3, s \simeq \frac{2\pi^2}{45} g T^3 \,. \tag{3.39}$$

for fermions

$$\rho = (7/8)\rho_b, n = (3/4) n_b, s = (7/8) s_b \,. \tag{3.40}$$

Other expansion

$$T \ll m, \quad \frac{m-\mu}{T} \gg 1 \,, n = g \left(\frac{mT}{2\pi^2}\right)^{2/3} e^{-\frac{m-\mu}{T}} \,, \quad \rho = mn \quad s = \frac{m-\mu}{T} n \,.$$

If $\mu \lesssim m$ they all scale as $\sim e^{-m/T}$. This means that ρ_{tot} is dominated by the species that are lighter than T

$$\rho_{\text{tot}} = \sum_{i \in \text{all particles}} \rho_i \simeq \sum_{i \in \text{all particles with } m<T} \frac{g_i \pi^2}{30} T^4 + \mathcal{O}(e^{-m/T}) \tag{3.41}$$

In our Universe $k \simeq 0$ and Λ is comparable to T only today. So for most of the history of the Universe

$$H^2 = \frac{8\pi G_N}{3} \rho = \frac{8\pi^3}{90} g_*(T) \frac{T^4}{M_{\text{Pl}}^2} \tag{3.42}$$

$$g_*(T) = \sum_{i \in \text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i \in \text{fermions}} g_i \left(\frac{T_i}{T}\right)^4 \tag{3.43}$$

Another very important quantity is entropy density because it allows us to keep track of the expansion of the universe

$$l(t_1) = a(t_1) l \to l(t_2) = \frac{a(t_2)}{a(t_1)} l(t_1) \equiv (1+z) l(t_1) \tag{3.44}$$

so we care about quantities like

$$\frac{a(t_2)}{a(t_1)} \tag{3.45}$$

because for instance if the total number of DM particles is conserved

$$n_1 = n_2 \frac{V_2}{V_1} = n_2 \left(\frac{a(t_2)}{a(t_1)}\right)^3 \,. \tag{3.46}$$

However we know that mostly the universe is in equilibrium so entropy is constant

$$S_1 = S_2 \to s_1 a(t_1)^3 = s_2 a(t_2)^3 \tag{3.47}$$

so finally

$$\frac{a(t_2)}{a(t_1)} = \left(\frac{s_1}{s_2}\right)^{1/3} \simeq \frac{T_1}{T_2}\,. \tag{3.48}$$

If you want to be more precise

$$s = \frac{2\pi^2}{45} g_{*S}(T) T^3\,, \qquad \sum_{i\in\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{i\in\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3\,. \tag{3.49}$$

Add sketch of g_* and g_{*S}.

3.1 Boltzmann Equation

We have assumed equilibrium, but a Universe in equilibrium is extremely boring.

Exercise Derive the continuity equation from thermodynamics

$$dE = -pdV \tag{3.50}$$

Solution

$$\begin{aligned} dE = d(\rho V) = V_0 a(t)^3 d\rho + 3a^2 V_0 \rho da \\ \rightarrow \frac{dE}{dt} = V_0 a(t)^3 \frac{d\rho}{dt} + 3a^3 V_0 \rho H \rightarrow -p\frac{dV}{dt} = -p3a^3 V_0 p H \end{aligned} \tag{3.51}$$

so finally

$$\begin{aligned} \frac{d\rho}{dt} = -3H(\rho+p), \quad w \equiv p/\rho \\ d(\log\rho)/dt = -3(1+w)d(\log a)/dt \\ \rho \sim a^{-3(1+w)} \end{aligned} \tag{3.52}$$

End of Solution

For $w \geq -1$, ρ decreases with $a(t)$. If we plug back into the first Friedmann equation

$$\dot{a}^2 = \frac{8\pi G_N \rho_0 a^{-3(1+w)+1}}{3} \tag{3.53}$$

and

$$a(t) \sim t^{\frac{2}{3(1+w)}} \tag{3.54}$$

$a(t)$ always increases for $w \geq -1$. We thus would end up with an empty univerese because

$$T \sim 1/a(t) \tag{3.55}$$

and eventually we are left with only photons because of the exponential suppression of particle number densities seen above.

How do we describe departures from equilibrium? In equilibrium

$$n = g \int \frac{d^3p}{(2\pi)^3} f(|\vec{p}|, t) \tag{3.56}$$

but in general one might have

$$n(x) = g \int d^3p d^3x f(\vec{p}, \vec{x}, t) \tag{3.57}$$

However homogeneity means $n(x) = n, \forall x$ and isotropy that $f(\vec{p}) = f(|\vec{p}|)$. So we have to consider just the simpler case

$$n = g \int \frac{d^3p}{(2\pi)^3} f(|\vec{p}|, t) \tag{3.58}$$

We want to compute the Liouville operator (Exercise)

$$\frac{df}{d\tau} = \frac{\partial f}{\partial t} + \frac{d\epsilon}{dt}\frac{\partial f}{\partial \epsilon} = \frac{\partial f}{\partial t} - H\frac{|\vec{p}|^2}{\epsilon}\frac{\partial f}{\partial \epsilon} \equiv L[f]\,. \tag{3.59}$$

Solution The derivation is as follows. In a gravitational field (we write m_0 to show explicitly the absence of other γ factors)

$$\frac{dp^\sigma}{d\tau} = -\Gamma^\sigma_{\mu\nu}\frac{p^\mu p^\nu}{m_0}\,. \tag{3.60}$$

For the FRW metric the relevant symbols are

$$\Gamma^0_{ij} = \frac{\dot{a}}{a} g_{ij} \tag{3.61}$$

so (the physical magnitude of the 3-momentum is $g_{ij}p^i p^j$)

$$\frac{d\epsilon}{d\tau} = -\frac{H|\vec{p}|^2}{m_0} \tag{3.62}$$

Finally

$$u^0 = m_0\frac{dt}{d\tau} \to \frac{d\epsilon}{d\tau} = \frac{d\epsilon}{dt}\frac{dt}{d\tau} = \frac{d\epsilon}{dt}u^0 \to \frac{d\epsilon}{dt} = -\frac{H|\vec{p}|^2}{m_0 u^0} = -\frac{H|\vec{p}|^2}{\epsilon} \tag{3.63}$$

End of Solution

Boltzmann equation (without proof)

$$C[f] = L[f] \tag{3.64}$$

$C[f]$ = collision operator. With the simplifying assumptions that we will make it is convenient to compute the integrated BE

Exercise Show that

$$g_1 \int \frac{d^3p}{(2\pi)^3} L[f_1] = \dot{n}_1 + 3Hn_1 \tag{3.65}$$

Solution (in the first equality we use $d\epsilon/d|\vec{p}| = |\vec{p}|/\epsilon$, then we integrate by parts)

$$\begin{aligned}
g \int \frac{d^3p}{(2\pi)^3} L[f] &= \dot{n} - Hg \int \frac{d^3p}{(2\pi)^3} \frac{|\vec{p}|^2}{\epsilon} \frac{\partial f}{\partial \epsilon} = \dot{n} - Hg \int \frac{d^3p}{(2\pi)^3} |\vec{p}| \frac{\partial f}{\partial |\vec{p}|} \\
&= \dot{n} - Hg \int \frac{d^3p}{(2\pi)^3} \left(\frac{\partial (|\vec{p}| f)}{\partial |\vec{p}|} - f \right) = \dot{n} + Hn - gH \int \frac{d^3p}{(2\pi)^3} \frac{\partial (|\vec{p}| f)}{\partial |\vec{p}|} \\
&= \dot{n} + Hn - gH \int \frac{d\Omega |\vec{p}|^2 d|\vec{p}|}{(2\pi)^3} \frac{\partial (|\vec{p}| f)}{\partial |\vec{p}|} \\
&= \dot{n} + Hn - gH \int \frac{d\Omega d|\vec{p}|}{(2\pi)^3} \left(\frac{\partial (|\vec{p}|^3 f)}{\partial |\vec{p}|} - 2|\vec{p}|^2 f \right) = \dot{n} + 3Hn - \frac{gH}{2\pi^2} \left. |\vec{p}|^3 f \right|_0^\infty = \dot{n} + 3Hn
\end{aligned} \tag{3.66}$$

End of Solution For a 2 to 2 process with particles 1 and 2 in the initial state the collision operator reads

$$\begin{aligned}
g \int \frac{d^3p}{(2\pi)^3} C[f] &= -\sum_{\text{spins}} \int \frac{d^3p_1}{2\epsilon_1 (2\pi)^3} \frac{d^3p_2}{2\epsilon_2 (2\pi)^3} \frac{d^3p_3}{2\epsilon_3 (2\pi)^3} \frac{d^3p_4}{2\epsilon_4 (2\pi)^3} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \times \\
&\times \left[|\mathcal{M}_{12\to34}|^2 f_1 f_2 (1 \pm f_3)(1 \pm f_4) - |\mathcal{M}_{34\to12}|^2 f_3 f_4 (1 \pm f_1)(1 \pm f_2) \right]
\end{aligned} \tag{3.67}$$

First Assumption: Let us imagine that we start with particles 3 and 4 in equilibrium (they could be SM particles). Both chemical $\mu = 0$ and kinetic $T = T_\gamma$. If $E \gtrsim T$, $f^{\rm eq} = 1/(e^{E/T} \pm 1) \simeq e^{-E/T} \ll 1$. Furthermore if the 2 to 2 process is in equilibrium

$$f_1^{eq} f_2^{eq} = f_3^{eq} f_4^{eq} \tag{3.68}$$

so

$$\begin{aligned}
g \int \frac{d^3p}{(2\pi)^3} C[f] &= -\sum_{\text{spins}} \int \frac{d^3p_1}{2\epsilon_1 (2\pi)^3} \frac{d^3p_2}{2\epsilon_2 (2\pi)^3} \frac{d^3p_3}{2\epsilon_3 (2\pi)^3} \frac{d^3p_4}{2\epsilon_4 (2\pi)^3} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \times \\
&\times \left[|\mathcal{M}_{12\to34}|^2 f_1 f_2 - |\mathcal{M}_{34\to12}|^2 f_1^{eq} f_2^{eq} \right]
\end{aligned} \tag{3.69}$$

unitarity implies

$$\begin{aligned}
&\sum_{\text{spins}} \int \frac{d^3p_3}{2\epsilon_3 (2\pi)^3} \frac{d^3p_4}{2\epsilon_4 (2\pi)^3} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) |\mathcal{M}_{12\to34}|^2 = \\
&\sum_{\text{spins}} \int \frac{d^3p_3}{2\epsilon_3 (2\pi)^3} \frac{d^3p_4}{2\epsilon_4 (2\pi)^3} (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) |\mathcal{M}_{34\to12}|^2 = 4F g_1 g_2 \sigma_{12\to34}\,.
\end{aligned} \tag{3.70}$$

So we have

$$g \int \frac{d^3p}{(2\pi)^3} C[f] \quad = \quad -4g_1g_2 \int \frac{d^3p_1}{2\epsilon_1(2\pi)^3} \frac{d^3p_2}{2\epsilon_2(2\pi)^3} \left[F\sigma_{12\to 34}\right]\left[f_1f_2 - f_1^{eq}f_2^{eq}\right] \tag{3.71}$$

If we define the thermally averaged cross section

$$\begin{aligned} v_{\text{Mol}} &\equiv \frac{F}{\epsilon_2\epsilon_1} \\ \langle\sigma v\rangle &\equiv \frac{g_1g_2 \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\sigma v_{\text{Mol}} f_1^{eq} f_2^{eq}}{n_1^{eq}n_2^{eq}} \end{aligned} \tag{3.72}$$

Second Assumption: kinetic equilibrium. If this is the case $f_1 = kf_1^{eq}$, with k constant. Then

$$\begin{aligned} &- 4g_1g_2 \int \frac{d^3p_1}{2\epsilon_1(2\pi)^3}\frac{d^3p_2}{2\epsilon_2(2\pi)^3} F\sigma_{12\to 34} f_1f_2 = -4g_1g_2k^2 \int \frac{d^3p_1}{2\epsilon_1(2\pi)^3}\frac{d^3p_2}{2\epsilon_2(2\pi)^3} F\sigma_{12\to 34} f_1^{eq}f_2^{eq} \\ &= \langle\sigma v\rangle k^2 n_1^{eq}n_2^{eq} = \langle\sigma v\rangle n_1n_2 \end{aligned} \tag{3.73}$$

We finally have

$$g \int \frac{d^3p}{(2\pi)^3} C[f] \quad = \quad -\langle\sigma v\rangle(n_1n_2 - n_1^{eq}n_2^{eq}) = \dot{n}_1 + 3Hn_1 \tag{3.74}$$

3.2 WIMP Miracle

Consider $\chi\chi \to$ SMSM. The theory is invariant under $\chi \to -\chi$ so this is the leading diagram. The BE for this process is

$$\dot{n}_\chi + 3Hn_\chi = -\langle\sigma v\rangle(n_\chi^2 - (n_\chi^{eq})^2) \tag{3.75}$$

If $3Hn_\chi \gg -\langle\sigma v\rangle n_\chi^2$ we can solve

$$\dot{n}_\chi + 3Hn_\chi \simeq 0 \to \frac{\dot{n}_\chi}{n_\chi} = 3\frac{\dot{a}}{a} \tag{3.76}$$

So

$$n_\chi(t) = n_\chi^* \left(\frac{a_*}{a(t)}\right)^3 \sim \frac{1}{a^3}\,. \tag{3.77}$$

If $3Hn_\chi \ll -\langle\sigma v\rangle n_\chi^2$ interactions are fast, we are in equilibrium with

$$2\mu_\chi = 2\mu_{\text{SM}} = 2\mu_\gamma = 0 \tag{3.78}$$

Therefore

$$n_\chi(t) = \frac{1}{e^{-\frac{\epsilon}{T(t)}} \pm 1}\,. \tag{3.79}$$

Draw picture of $Y_\chi \equiv n_\chi/s$. Therefore everything happens at freezout

$$
\begin{aligned}
&3H_f \simeq -\langle\sigma v\rangle_f n_{\chi,f} \\
&T_f \lesssim m_\chi \to \langle\sigma v\rangle_f \simeq \sigma_0 + \sigma_2 v_\chi^2 + ... \\
&n_{\chi,f} \simeq \frac{T_f^2}{M_{\rm Pl}\sigma_0}
\end{aligned}
\tag{3.80}
$$

Today

$$
\rho_\chi = m_\chi n_\chi = m_\chi n_{\chi,f}\left(\frac{a_f}{a_0}\right)^3 = m_\chi n_{\chi,f}\frac{s_0}{s_f} \simeq m_\chi \frac{T_f^2}{M_{\rm Pl}\sigma_0}\left(\frac{T_0}{T_f}\right)^3 \simeq \frac{T_0^3}{M_{\rm Pl}\sigma_0}
\tag{3.81}
$$

How does this compare to DM?

$$
\rho_{\rm DM}(t_{\rm EQ}) \simeq \rho_\gamma(t_{\rm EQ}) \simeq ({\rm eV})^4
\tag{3.82}
$$

Therefore

$$
\rho_{\rm DM}(t_0) \simeq {\rm eV}T_0^3\,.
\tag{3.83}
$$

So finally today

$$
\frac{\rho_\chi}{\rho_{\rm DM}} \simeq \frac{1}{M_{\rm Pl}{\rm eV}\sigma_0} \to \sigma_0 \simeq \frac{1}{(10\ {\rm TeV})^2}
\tag{3.84}
$$

The miracle is that

$$
\frac{1}{(10\ {\rm TeV})^2} \simeq \frac{\alpha_W^2}{m_W^2}
\tag{3.85}
$$

A 100 GeV particle with weak interactions to the SM could be dark matter!

3.3 A few selected variations on the WIMP

To get the WIMP miracle at some point we used unitarity on the collision operator $C[f]$

$$
\begin{aligned}
\sum_s \int d\Pi_F |\mathcal{M}_{F\to I}|^2 &= \sum_s \int d\Pi_F |\mathcal{M}_{I\to F}|^2 \\
d\Pi_F &= \delta^4(...)\prod_{i\in F}\frac{d^3p_i}{2\epsilon_i(2\pi)^3}\,.
\end{aligned}
\tag{3.86}
$$

However we don't have to. Consider again $\chi\chi \to SM\ SM$ for simplicity. Instead of using unitarity we can do the following

$$
\begin{aligned}
f_\chi^2 \sum_s \int d\Pi_{SM}|\mathcal{M}_{\chi\to SM}|^2 = 4Fg_\chi^2\sigma_{\chi\to SM}f_\chi^2 \\
f_{SM}^2 \sum_s \int d\Pi_\chi|\mathcal{M}_{SM\to\chi}|^2 = 4Fg_{SM}^2\sigma_{SM\to\chi}f_{SM}^2
\end{aligned}
\tag{3.87}
$$

Then the BE becomes

$$\dot{n}_\chi + 3Hn_\chi = -\langle\sigma_{\chi\to SM}v\rangle n_\chi^2 + \langle\sigma_{SM\to\chi}v\rangle n_{SM}^2 \tag{3.88}$$

Other than more intuitive this equation is telling us what happens if we relax some of the hidden assumptions in the WIMP calculation. Let's say that $m_{SM} > m_\chi$. Then $SMSM \to \chi\chi$ is allowed also at zero temperature and we can use the usual expansion of the cross section

$$\langle\sigma_{SM\to\chi}v\rangle \simeq \sigma_0 + \sigma_2 v^2 + \dots . \tag{3.89}$$

However we have no idea of the value of $\langle\sigma_{\chi\to SM}v\rangle$. In equilibrium

$$-\langle\sigma_{\chi\to SM}v\rangle(n_\chi^{eq})^2 + \langle\sigma_{SM\to\chi}v\rangle(n_{SM}^{eq})^2 = 0 . \tag{3.90}$$

This means

$$\langle\sigma_{\chi\to SM}v\rangle = \langle\sigma_{SM\to\chi}v\rangle\frac{(n_{SM}^{eq})^2}{(n_\chi^{eq})^2} \sim \sigma_0 e^{-2\frac{m_{SM}-m_\chi}{T}} \tag{3.91}$$

So now

$$\frac{\rho_\chi}{\rho_{\rm DM}} \simeq \left(\frac{\rho_\chi}{\rho_{\rm DM}}\right) e^{2\frac{m_{SM}-m_\chi}{T_f}} . \tag{3.92}$$

You need σ_0 exponentially larger than that of a WIMP!

$$\sigma_0 \simeq \frac{\alpha_\chi^2}{m_\chi^2} . \tag{3.93}$$

So either much lighter or much more strongly coupled DM! This is known as forbidden dark matter (Griest and Seckel '89, RTD and Rudermann '15).

Exercise Show the relation between T_f of forbidden DM and that of a WIMP. Let's define for convenience $x \equiv m/T$, where m is the DM mass and we dropped the subscript f of freeze-out for convenience. For a WIMP at freeze-out

$$\begin{aligned} n_\chi &= g_\chi m^3 \left(\frac{1}{2\pi^2 x}\right)^{3/2} e^{-x} = \frac{H(x)}{\sigma_0} = \frac{2\pi\sqrt{2\pi}}{\sqrt{90}}\sqrt{g_*(m/x)}\frac{m^2}{x^2 M_{\rm Pl}\sigma_0} \\ x_{\rm WIMP} &\simeq \log\frac{\pi^{3/2}\sqrt{90}g_\chi m M_{\rm Pl}\sigma_0}{\sqrt{x}\sqrt{g_*(m/x)}} \end{aligned} \tag{3.94}$$

For forbidden the calculation is identical with $\sigma_0 \to \sigma_0 e^{-2\Delta x}$, where $\Delta = (m_{SM} - m)/m$, so since $\log(1/\sqrt{x}\sqrt{g_*(m/x)})$ gives only a small additive correction (verify) we can conclude that approximately

$$x_{\rm F} \simeq \frac{x_{\rm WIMP}}{1+2\Delta} \tag{3.95}$$

End of Exercise

Other hidden assumptions

1. Kinematics [Forbidden]
   Griest and Seckel '89, RTD and Ruderman '15

2. Dynamics [SIMP]
   Hochberg, Kuflik, Volansky, Wacker, '14

3. Thermal History [Cannibal]
   Carlson, Machacheck, Hall '92
   Pappadopulo, Ruderman, Trevisan '16

4. Single χ hypothesis [Coannihilation]
   Griest and Seckel '89, RTD, Mondino, Ruderman, Wang '15

5. $\mathcal{O}(\alpha_W)$ couplings [Freeze-In, WIMPless]
   Hall, Jedamzik, March-Russell, West, '09
   Feng, Kumar, '09

6. Kinetic Equilibrium [Coscattering]
   RTD, Pappadopulo, Ruderman, '16

Freeze-In Tiny coupling between DM and SM no kinetic and no chemical equilirbium. Assumption: initially $n_\chi \simeq 0$.

$$\dot{n}_\chi + 3Hn_\chi \simeq \langle \sigma_{SM\to\chi} v\rangle n_{SM}^2 \tag{3.96}$$

Interactions stop being relevant when

$$n_{\chi,f} \simeq \frac{\langle \sigma_{SM\to\chi} v\rangle_f n_{SM,f}^2}{H_f} \simeq \sigma_0 M_{\rm Pl} T_f^4 \tag{3.97}$$

so finally

$$\frac{\rho_\chi}{\rho_{\rm DM}} \simeq \frac{m_\chi \sigma_0 M_{\rm Pl} T_f}{\rm eV} \simeq \frac{\alpha_\chi^2 M_{\rm Pl}}{\rm eV} \to \alpha_\chi \simeq 10^{-14}\,. \tag{3.98}$$

We have seen two examples where DM can be either much lighter or much more weakly coupled than a WIMP and there are many more in the references listed above.

4 Ultralight Scalar Dark Matter

We saw that we can get pretty small couplings and masses from the thermal models that we discussed. What is the limit? Experimentally as we know we can go all the way down to $m \simeq 10^{-21}$ eV (for bosons) and in principle to gravitational couplings. Are there models that can do it?

In principle you could say: let me just do the same calculation of the WIMP, but in a dark sector with tiny couplings, then I'm gonna get a tiny mass as well. If the dark sector is sufficiently cold its contribution to Hubble can be hidden from our probes of the expansion of the Universe (BBN and CMB). In reality it is not quite so simple. If dark matter has any thermal origin it will have some velocity distribution. After decoupling from radiation it will move randomly in the Universe (free-streaming). On average this will reduce the primordial over- and under-densities that give birth to galaxies, today we should see less structure on scales over which DM could propagate.

A DM particle will propagate over a distance

$$\lambda_{FS} = a_0 \left[\int_{t_{keq}}^{t_{NR}} \frac{c}{a(t)} dt + \int_{t_{NR}}^{t_{EQ}} \frac{v(t)}{a(t)} dt \right] . \tag{4.1}$$

The best case is $t_{keq} \gg t_{NR}$ so the DM free-streams only for a short time while being non-relativistic. Then

$$\lambda_{FS}^{\text{best}} = a_0 a(t_{NR}) c \int_{t_{keq}}^{t_{EQ}} \frac{1}{a(t)^2} dt \simeq 0.1 \text{ Mpc} \sqrt{\frac{g_*(\text{keV})}{g_*(m_{DM})} \frac{\text{keV}}{m_{DM}}} \log \frac{T_{\text{keq}}}{\text{eV}} . \tag{4.2}$$

So no galaxies if $m_{DM} \lesssim$ keV.

What did we assume? That DM has some initial velocity $\sim c$. What if there was a way to produce DM (non-thermally) with zero velocity? It turns out that there is!

Consider

$$S_\phi = \int d^4x \sqrt{-g} \left[\frac{g^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi - \frac{m_\phi^2}{2} \phi^2 \right] , \tag{4.3}$$

let's say that we want a ϕ with $m_\phi \lesssim$ eV to be DM. We know that

$$\begin{aligned} \rho_{\text{DM}} \simeq \text{eV} \times T_0^3 \simeq \text{eV} \times (0.1 \text{ meV})^3 \\ n_{\text{DM}} = \frac{\rho_{\text{DM}}}{m_\phi} \\ p_{\text{DM}} = m_\phi \sqrt{\langle v_{\text{DM}}^2 \rangle} \end{aligned} \tag{4.4}$$

So in the smallest indistinguishable phase space cell $(2\pi\hbar)^3$ we have potentially lots of particles!

$$\frac{(2\pi\hbar)^3}{p_{\text{DM}}^3} n_{\text{DM}} = \frac{(2\pi\hbar)^3}{p_{\text{DM}}^3 V} N_{\text{DM}} \simeq \frac{\text{eV}^4}{m_\phi^4} \tag{4.5}$$

This means that for $m_\phi \ll$ eV we can use a classical description!

Since we are always in an approximately homogenous Universe we can write the equation of motion for $\phi(\vec{x}, t) = \phi(t)$ verify as an exercise that the result is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = \ddot{\phi} + 3H\dot{\phi} + m_\phi^2 \phi = 0 \tag{4.6}$$

When $H \gg m_\phi$ then the scalar is effectively stuck. In a Hubble time $H\dot{\phi} \simeq H^2\phi \gg m_\phi^2 \phi$ so

$$\ddot{\phi} + 3H\dot{\phi} \simeq 0 \to \dot{\phi} \simeq 0 \to \phi \simeq \phi_0 . \tag{4.7}$$

As the universe cools down and expands, H decreases ($H \sim T^2 \sim 1/a^2$) eventually $H \simeq m_\phi$ and we can solve the full damped harmonic oscillator equation

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2 \phi = 0 \to \phi(t) = \frac{1}{a(t)^{3/2}} \left[C_1 \cos(m_\phi t) + C_2 \sin(m_\phi t)\right] . \tag{4.8}$$

The energy density is

$$\rho_\phi = T_{00} = \frac{\dot{\phi}^2}{2} + \frac{m_\phi^2 \phi^2}{2} . \tag{4.9}$$

If we average over multiple oscillations we get

$$\langle \dot{\phi}^2 \rangle = \langle \phi^2 \rangle \tag{4.10}$$

and

$$\langle \rho_\phi \rangle = m_\phi^2 \langle \phi^2 \rangle \tag{4.11}$$

from now on I will mean this to be the energy density of ϕ. So can ϕ be DM? It depends on initial conditions, let's take $\phi(t_0) = \phi_0$, at times t_0 sufficiently early on that ϕ is stuck $\dot{\phi}(t_0) = 0$. Then

$$\langle \rho_\phi \rangle = \frac{m_\phi^2 \phi_0^2}{2a(t)^3} . \tag{4.12}$$

We notice that the energy density redshifts like matter! Numerically we need

$$\frac{\langle \rho_\phi \rangle}{\rho_{DM}} \simeq \frac{m_\phi^2 \phi_0^2}{\mathrm{eV} \times T_{\mathrm{osc}}^3} \simeq 1 \tag{4.13}$$

where $H(T_{\mathrm{osc}}) \simeq m_\phi$. There's no problem with free-streaming because $\partial_x \phi \simeq 0$. Nightmare scenario! No coupling to the SM. Simple mechanism to get the relic density.