Scaling functions in disordered elastic materials (Jim Sethna)

P. W. Anderson, 1978, Les Houches school on Ill Condensed Matter

New paradigm

"How do disordered systems *differ* from regular systems?"

- Today: Avalanches and emergent scale invariance (Dahmen, Myers, Perković, Kuntz, …)
- Tomorrow: Scaling and avalanches (Hayden, Raju, Shekhawat, Chen, Zapperi, Rosso, Wyart)

Old paradigm

"How may they be reduced to them?"

- Thursday: Materials properties near rigidity transitions (Liarte, Thornton; Liarte, Mao, Lubensky)
- Friday: Mean-field theory for jamming and rigidity percolation (Thornton, Liarte)

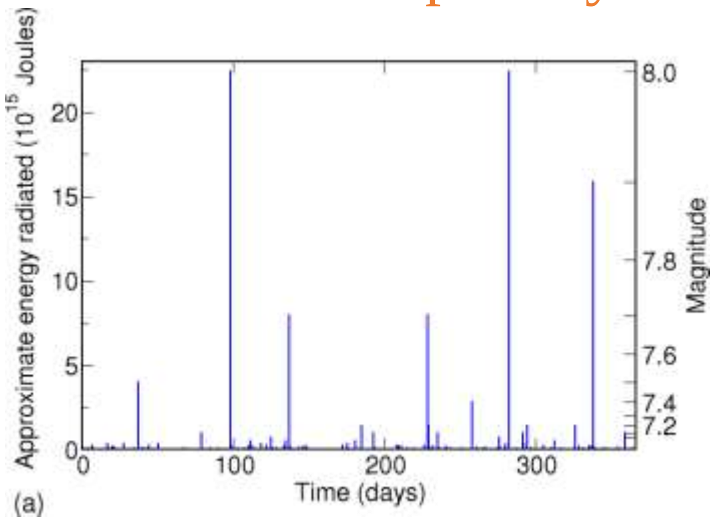
Everywhere, I will focus on *universal scaling functions*…

How do disordered systems ***agree with***/***differ from*** gases, liquids and crystals?

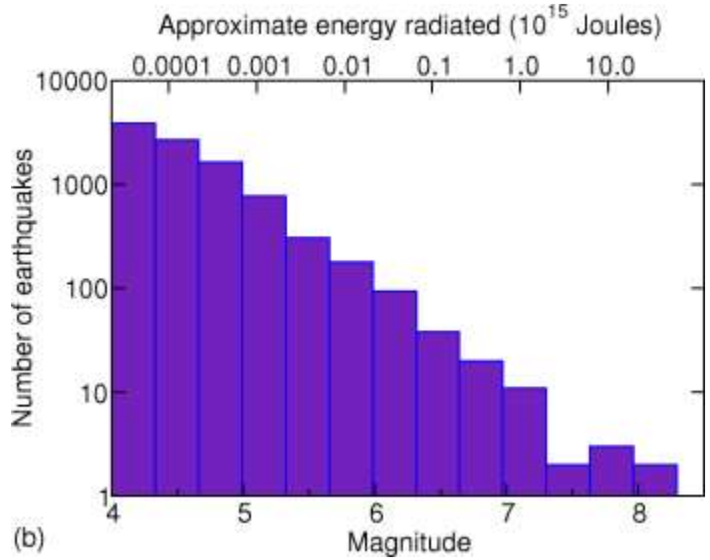
Systems	Fluid/Floppy state	Elastic/Rigid state
Granular materials (sands and powders)	Flows like liquid or gas when dilute, sheared, or agitated (shaken powders, beer bubbles and wet foams, glass melts); viscosities	Supports shear when dense and weakly disturbed (handful of dry suds, glasses); sound waves, elastic constants
Colloidal suspensions (milk, cornstarch)	Complex, multiscale, jerky flow as particles move and avoid one another when density increases (non-Newtonian fluids, diverging viscosities)	Complex "non-affine" deformation under small shears; avalanches on all scales, crackling noise
Foams, emulsions (heads of beer)		
Glasses (window glass, hard candy / doce duro)		

Earthquakes

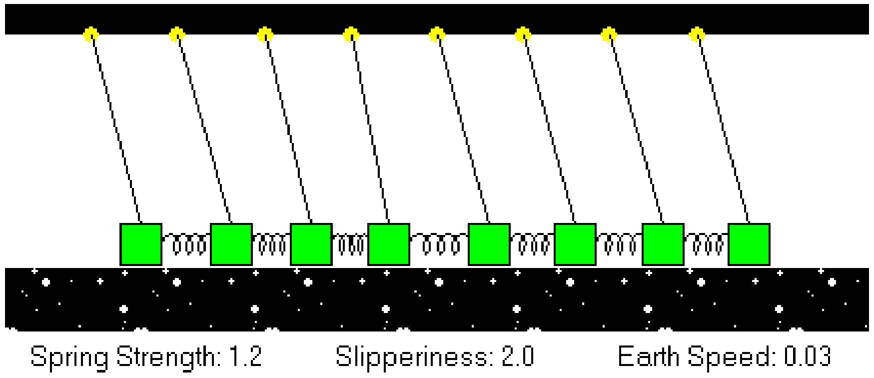
Spatially Extended Events of All Sizes



Earthquakes of Many Sizes: 1995



- Earthquakes of All Sizes
- Gutenberg-Richter Law:
  Probability ~ Size$^{-\text{Power}}$
- Simple Block-Spring Model
  - No disorder
  - Slow driving rate (cm/year)

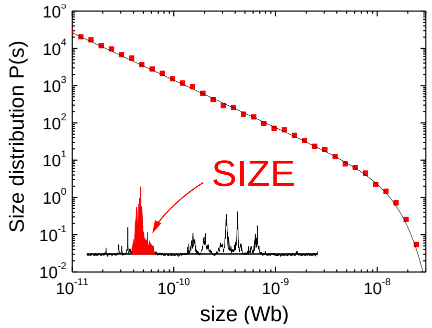


Burridge-Knopoff (Carlson & Langer)

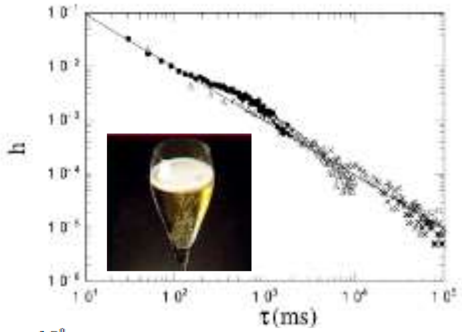
http://simscience.org/crackling/Advanced/Earthquakes/EarthquakeSimulation.html

Crackling noise

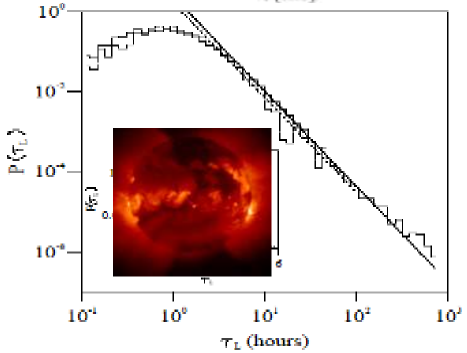
Magnets



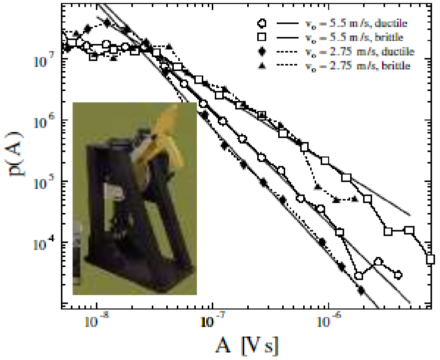
Foams



Solar Flares

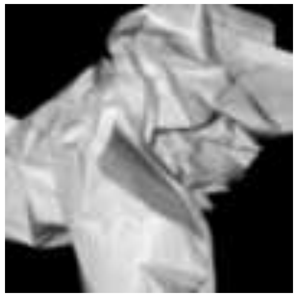


Fracture



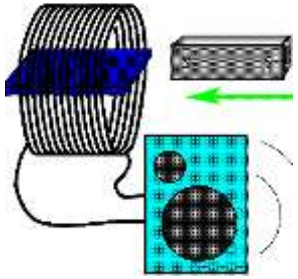
Discrete crackles span enormous range of sizes. Should be comprehensible; *scaling theory*.

Analogy with hydrodynamics:
Molecules don’t matter for Navier-Stokes fluid flow
Microscopics won’t matter for crackling



Paper Crumpling

Rice Krispies



Magnets

Tearing Paper

Kuntz

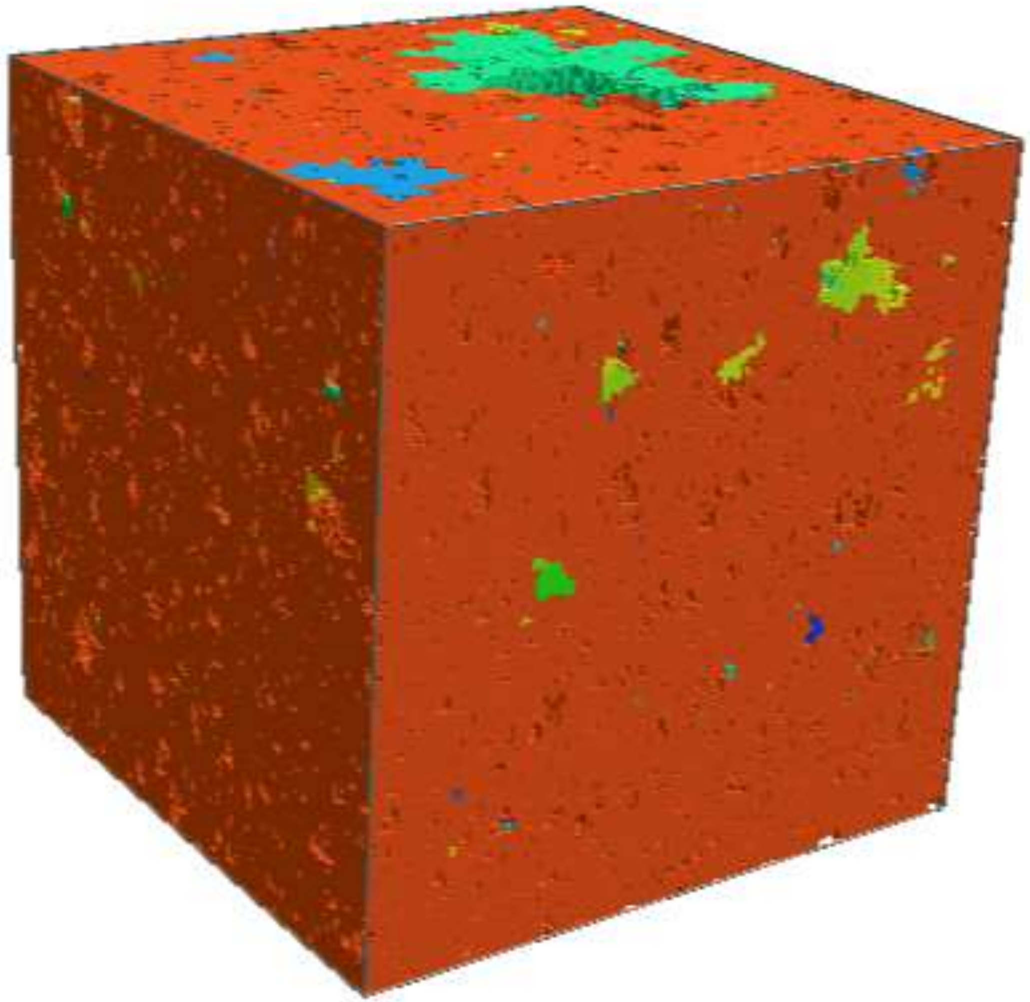
Avalanches at criticality (RFIM)

Ising spins each with random field, variance R and external field H. Avalanche starts when net field

H + neighbors

changes sign. System tuned to critical disorder R_c

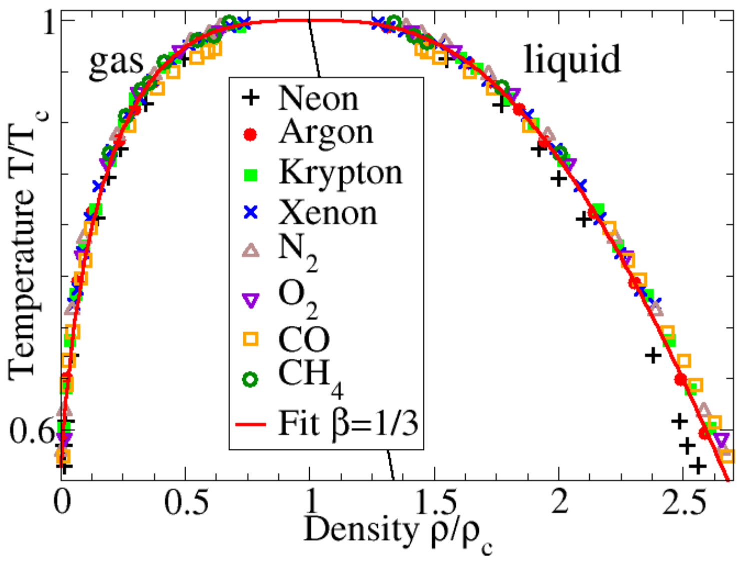
- Avalanches of all scales
- Early small avalanches, growing in size
- “Infinite” (red) avalanche, large jump in magnetization near H_c
- Small final avalanches fill in gaps
- Emergent scale invariance at R_c , H_c
- ‘Self-organized’ when integrated over field H



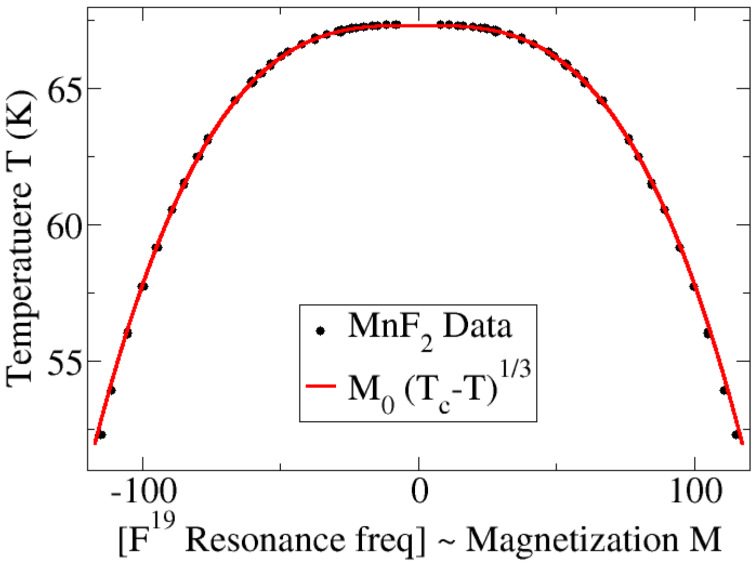
Dahmen, Perković, Pelkie

Universality: Shared Critical Behavior

Ising Model and Liquid-Gas Critical Point



Same critical exponent β=0.326419!



Liquid-Gas Critical Point

$$\rho - \rho_c \sim (T_c\text{-}T)^{\beta}$$

$$\rho^{Ar}(T) = A\,\rho^{CO}(BT)$$

Ising Critical Point

$$M(T) \sim (T_c\text{-}T)^{\beta}$$

$$\rho^{Ar}(T) = A(M(BT),T)$$

Universality: Same Behavior up to Change in Coordinates

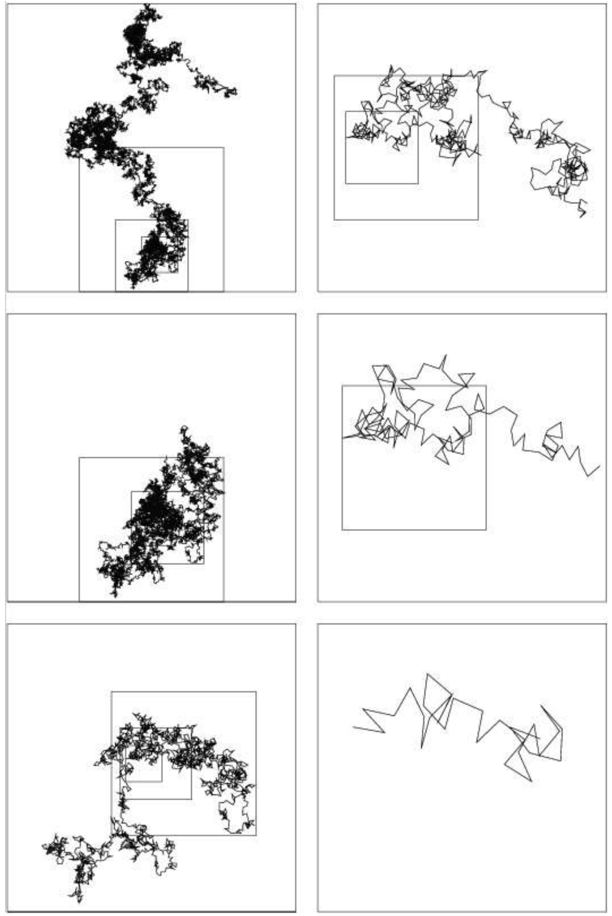
$$A(M,T) = a_0 + a_1\,(M\text{-}M_c) + a_2(T\text{-}T_c) + (\textit{other singular terms})$$

Nonanalytic behavior at critical point (not parabolic top)

All power-law singularities (χ, c_v, ξ) are shared by magnets, liquid/gas

Self-Similarity in Space

Self-Universality on Different Scales



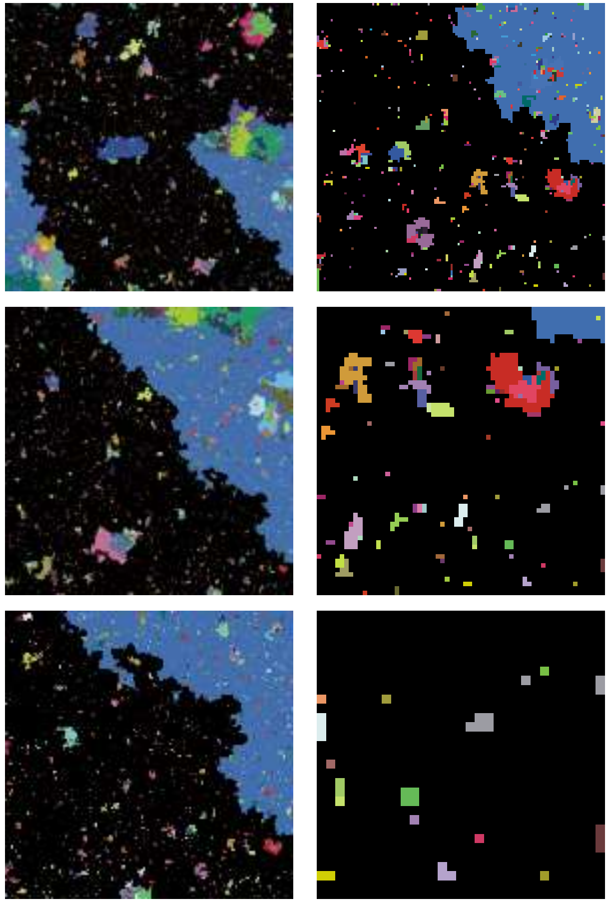
Random Walks

Magnify portion by factor of two: statistically similar on all scales (until 'lattice cutoff').

Random walks – generic scale invariance.

Hysteresis model: Emergent scale invariance at critical point R_c.

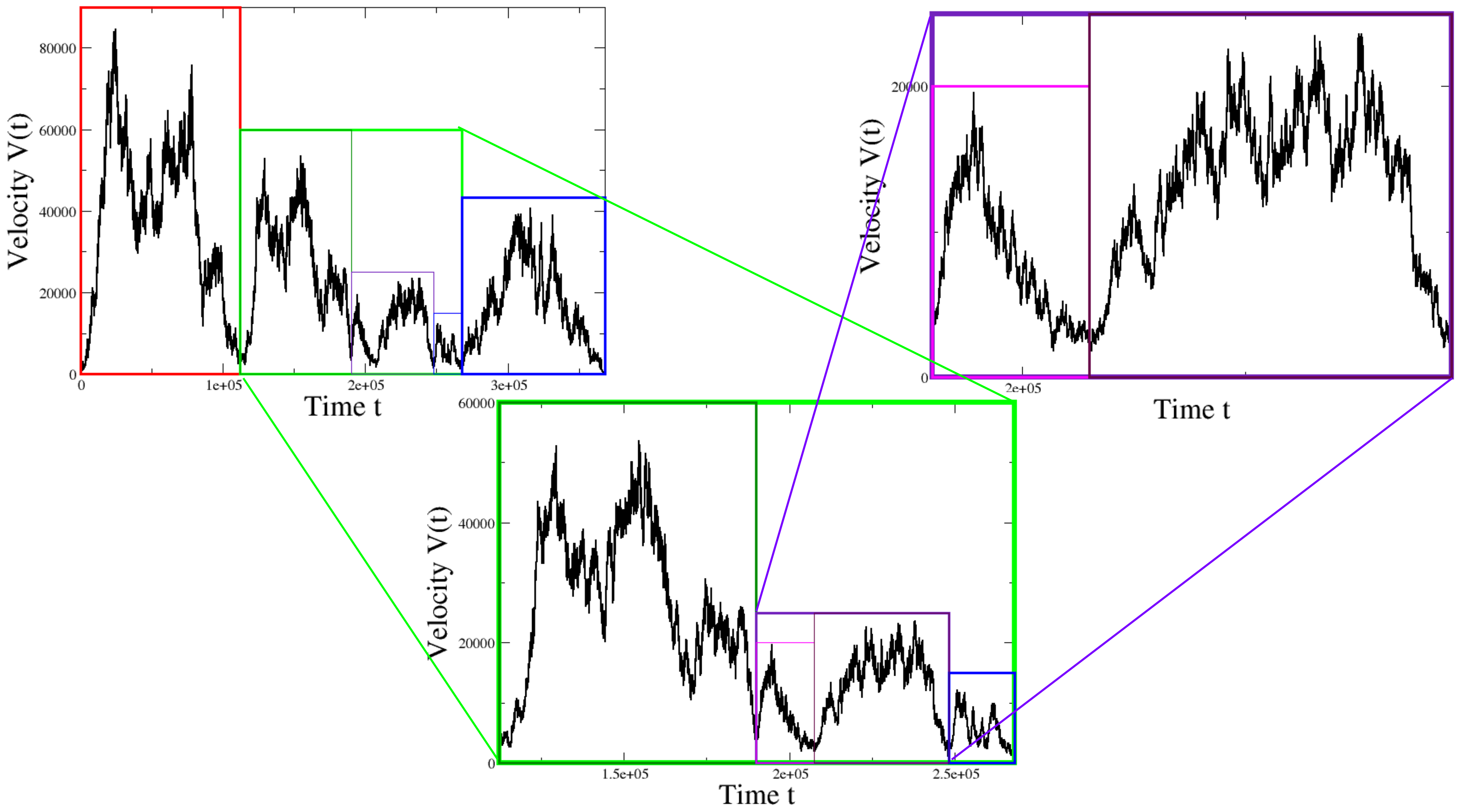
Self-similarity → Power Laws



Hysteresis Model at R_c

Self-similar in Time

Big avalanches made of little ones

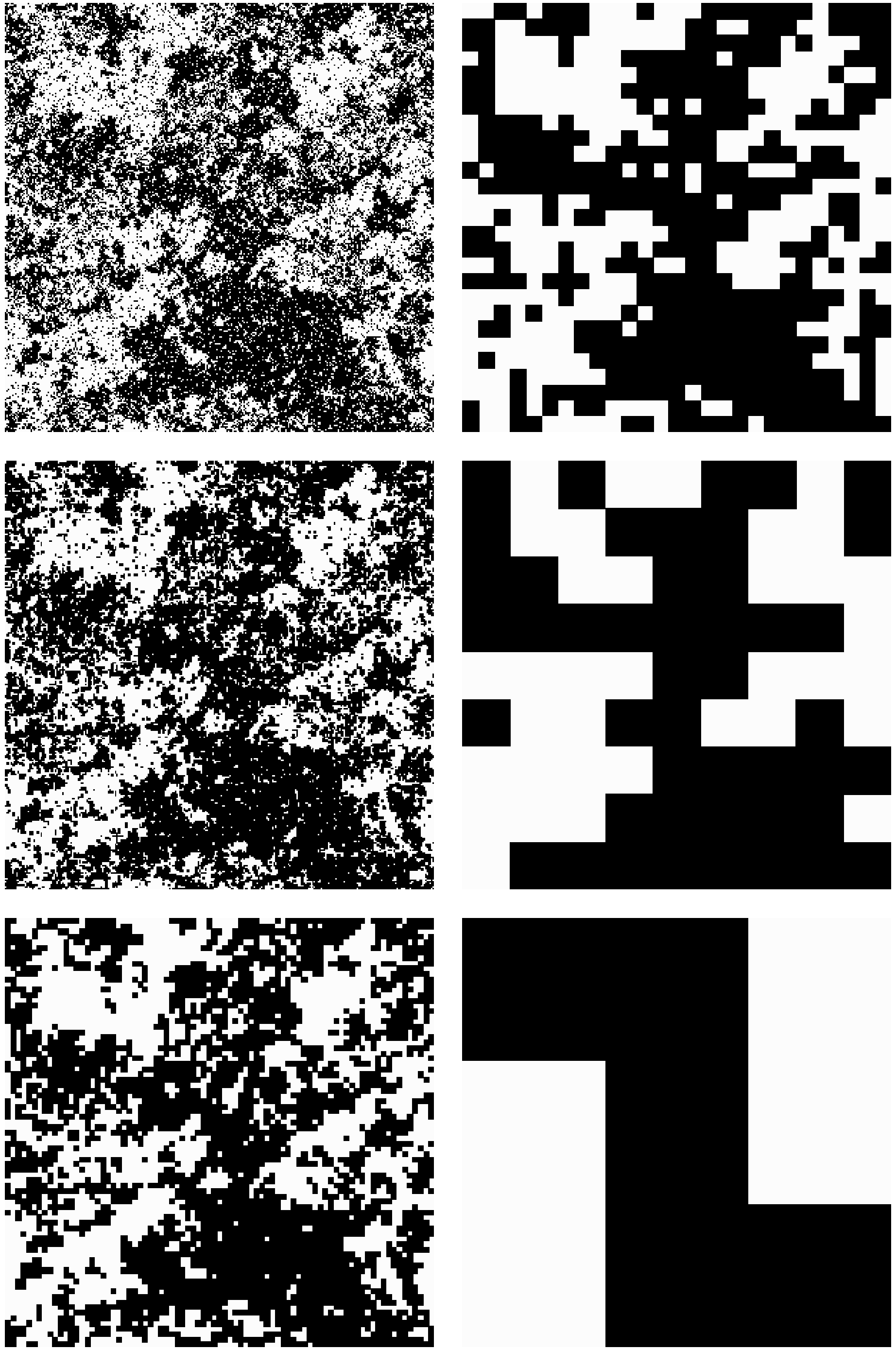


Avalanche almost halts many times: pieces look like whole

Coarse Graining

Remove microscopic details

- 'Continuum limit' – average over details in small regions, get effective laws for coarser system
- Example: majority-rule block-spin transformation (3x3 blocks)
- Renormalization group: find effective block-spin free energy: new interactions from old by tracing over microscopic variables

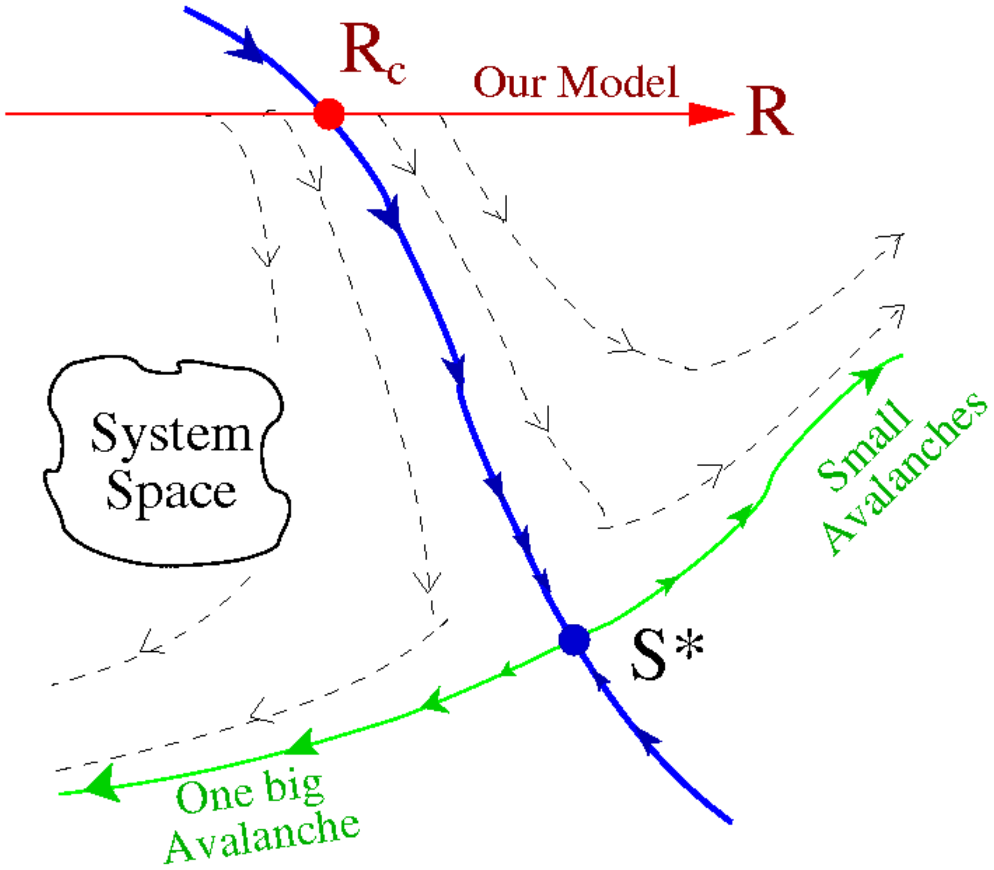


The Renormalization Group

Coarse Graining in the Space of All Systems

Ken Wilson's
amazing abstraction
Space of all possible systems
(experiment or theory)

Coarse laws give
new point in
system space.
Many coarsenings?
Stops changing at S*



Self-Similarity:
S* similar to itself
under coarsening

Universality:
Many different
systems go to same S*

Dimensionality of stable and unstable manifolds

Theory describes experiment if both coarsen to same S.*

Dahmen, Myers

Universality

Universal Power Laws: The Fractal Dimension

Avalanche fractal dimension: length L, mass/volume/size L^{d_f}, universal d_f:

$$S/S_0 \sim (L/L_0)^{d_f}. \tag{1}$$

Derive from RG: coarse grain a small factor $(1 + d\ell)$:

$$L' = L/(1 + d\ell) \sim L - L d\ell \quad dL/d\ell = (L' - L)/d\ell = -L.$$

avalanches rescale by a different factor $1/(1 + a d\ell)$

$$dS/d\ell = -aS \quad dS/dL = (dS/d\ell)/(dL/d\ell) = aS/L.$$

Check that this is satisfied by Eq. 1:

$$\begin{aligned} dS/dL = d(S_0(L/L_0)^{d_f}/dL &= S_0 L_0^{-d_f}(d_f L^{d_f - 1}) \\ &= d_f S_0 (L/L_0)^{d_f}/L = d_f S/L, \end{aligned}$$

so $d_f = a$ is the universal critical exponent.

Universal Scaling Function

Consider the scaling relation $z(x, y)$ between z, x, and y (say, M, $T - T_c$, and H). Check that the RG predicts the universal scaling form near $x = 0$:

$$z(x, y)/z_0 = (x/x_0)^{\beta} \mathcal{Z}((y/y_0)/(x/x_0)^{\beta\delta}) = x^{\beta} \mathcal{Z}(y/x^{\beta\delta}). \tag{2}$$

(1) Check that $Y = y/x^{\beta\delta}$ is a constant under the RG.

If $dx/d\ell = ax$, $dy/d\ell = by$, and $dz/d\ell = cz$, then

$$\begin{aligned} dY/d\ell = d(yx^{-b/a})/d\ell &= (dy/d\ell)x^{-b/a} + y(-b/a)x^{-b/a-1}(dx/d\ell) \\ &= (by)x^{-b/a} - y(b/\cancel{a})x^{-b/a\cancel{-1}}(\cancel{a}\cancel{x}) = 0. \end{aligned}$$

Y is invariant with $\beta\delta = b/a$.

(2) Check the scaling form, Eq. 2:

$$\begin{aligned} cz = dz/d\ell &= \beta x^{\beta-1}(dx/d\ell)\mathcal{Z}(Y) + x^{\beta}\mathcal{Z}'(Y)\cancel{dY/d\ell} \\ &= \beta x^{\beta-1}(ax)\mathcal{Z}(Y) = a\beta x^{\beta}\mathcal{Z}(Y) = a\beta z. \end{aligned}$$

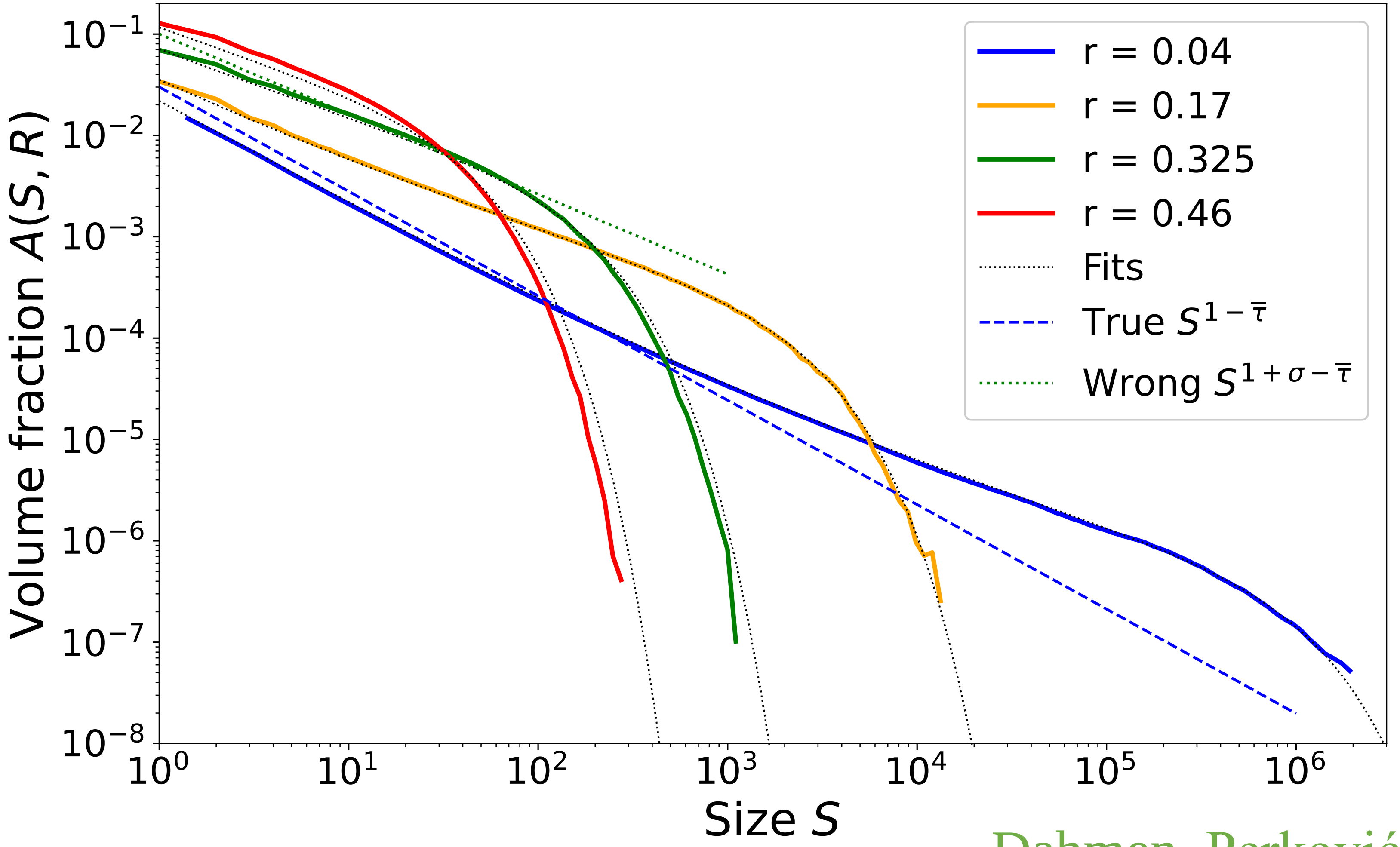
Thus $c = a\beta$, so $\beta = c/a$, $\delta = b/c$, and $\mathcal{Z}(Y)$ is universal.

Universal power laws: Avalanche Size Distribution

Probability $A(S,R)$ that a site is in an avalanche of size S for disorder R, for various

$$r = (R-R_c)/R_c$$

- Straight lines are power laws.
- Power law $A \sim S^{1-\bar{\tau}}$ at R_c.
- Four decades of scaling to get correct power law
- Dotted lines are fit to universal scaling function



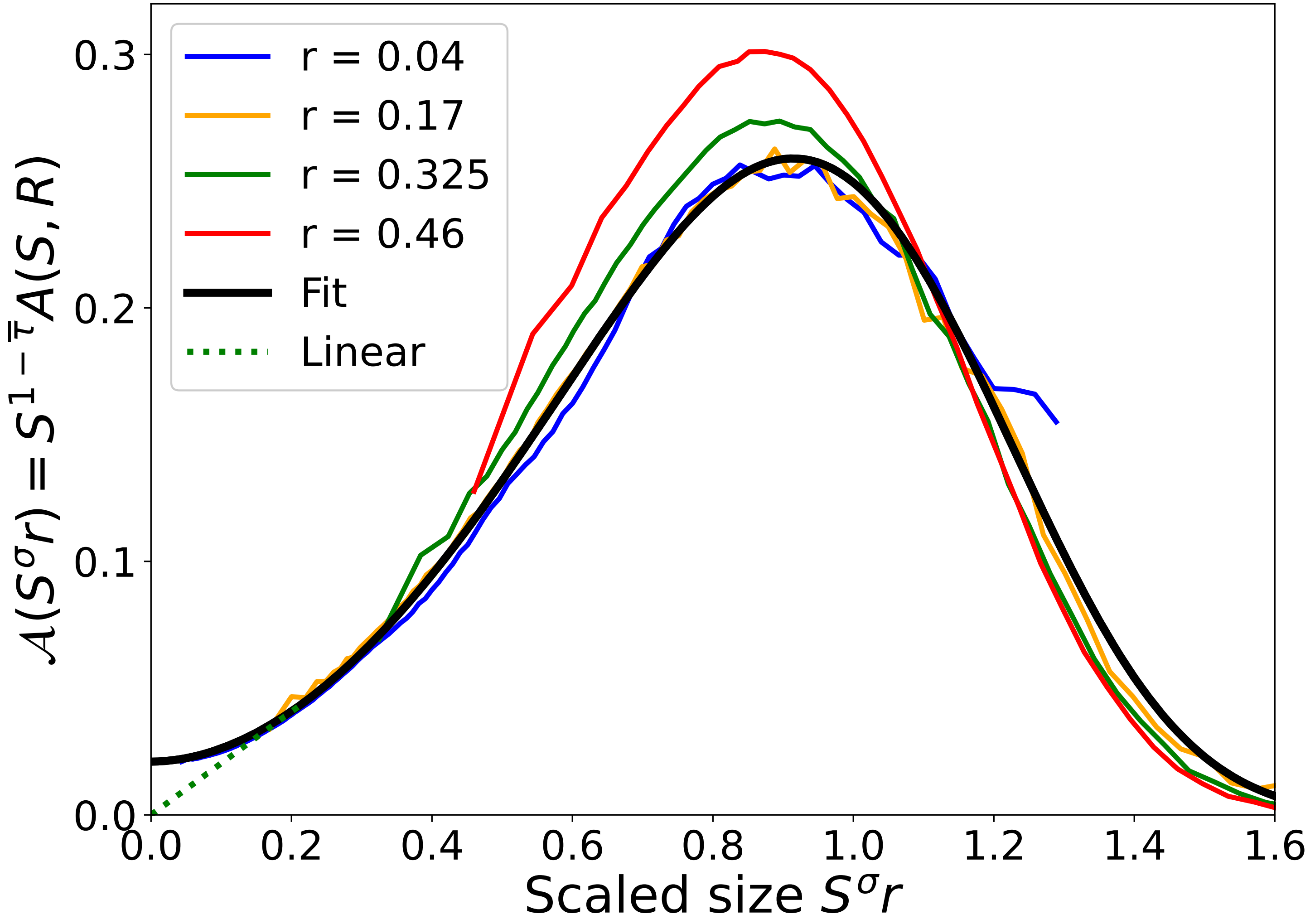
Dahmen, Perković

Universal scaling function: Avalanche Size Distribution

Scaling collapse of the avalanche size distribution *A(S,R)*. The dark line is a fit to the universal scaling function

$$\mathcal{A}(S^\sigma r) = S^{\bar{\tau}-1} A(S,R)$$

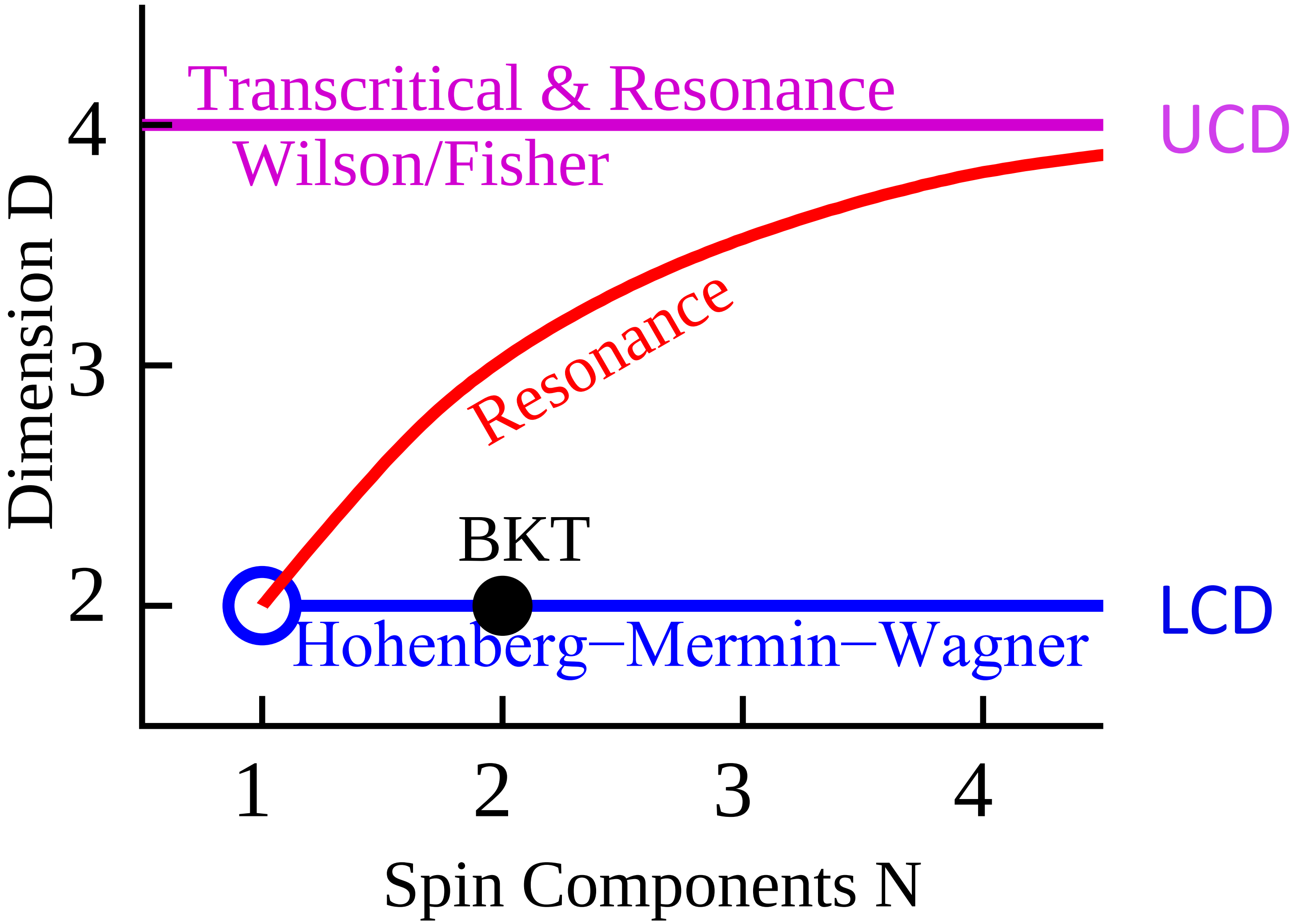
Note that the function nearly vanishes at zero. This is why systems far from the critical point showed a 'wrong' power law (green dotted line).



Dahmen, Perković

Upper and lower critical dimensions: Thermodynamic critical points and mean-field theory

The renormalization group was invented to study phase transitions as a function of temperature in pure systems. Below the lower critical dimension (LCD), the transitions were at zero temperature. Above the upper critical dimension (UCD), the properties near the transition are described by *mean field theory.*



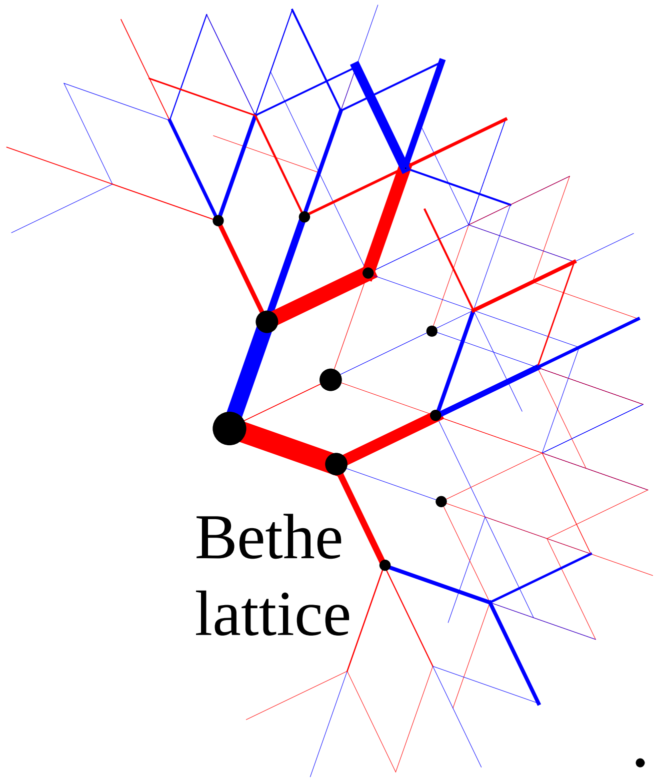
Mean field theories

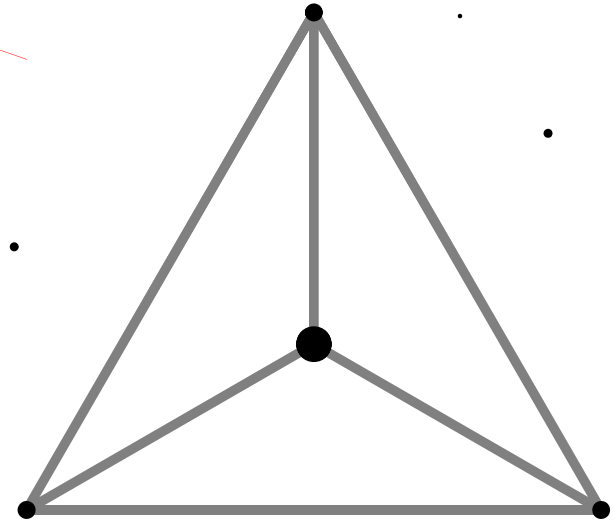
There are many kinds of mean-field theories.

- Curie-Weiss type (bounding free energy)
- Bethe lattice (branching tree)
- Infinite-range models (hypertetrahedron)
- Infinite dimensions: Replica theory
- EMT, CPA, Mode Coupling Theory

They give the correct critical behavior above the upper critical dimension, which can be

- Higher (six for random-field Ising models)
- Or lower, for long-range interactions (d=3 for elastic interactions in earthquakes and fracture)
- D=2 appears to be both LCD and UCD for jamming, and for slip bands formed during yielding.





Mean field theory: why universal scaling fcts?

Mean field theories can be solved for the entire behavior. Why bother with extracting a universal scaling function that is only valid near the transition?

- Mean-field theories do have universal scaling functions

$$z(x,y) = x^{\beta} \mathcal{Z}(y/x^{\beta\delta}) + \textit{higher order}$$

near the critical point, with $\mathcal{Z}$ a universal scaling function (even when there is no RG).

- Mean-field theories make predictions both near and far from the critical point, but not all the behavior is predicted correctly.
- Only the universal exponents and universal scaling functions apply quantitatively for finite dimensions, or for experiments.

Extracting the universal predictions is thus an important task, even for a solvable model.

Mean field theory for the Ising model

(0) Derive mean-field theory. Curie-Weiss, z neighbors, self-consistent equation: spin feels field h + mz from neighbors.

$$m = \frac{e^{(h+mz)/T} - e^{-(h+mz)/T}}{e^{(h+mz)/T} + e^{-(h+mz)/T}}$$ (self-consistent theory)

(1) Change to $t = T_c - T$ with $T_c = z$.
(2) Substitute $m/m_0 = t^\beta \mathcal{M}$, $h/h_0 = t^{\beta\delta} H$, $\beta = ½$, $\delta = 3$.
(3) Assume t going to zero. Derive an equation for $m(t,h) = t^\beta \mathcal{M}(H) = t^\beta \mathcal{M}(h/t^{\beta\delta})$:

$$\boxed{\mathcal{M} = H - \mathcal{M}^3/3}$$ (Universal scaling function relation)

The Curie-Weiss theory gives a reasonable behavior, but it just approximates, say, the 5D Ising behavior. The universal scaling part specifies what the theory guarantees to be true.