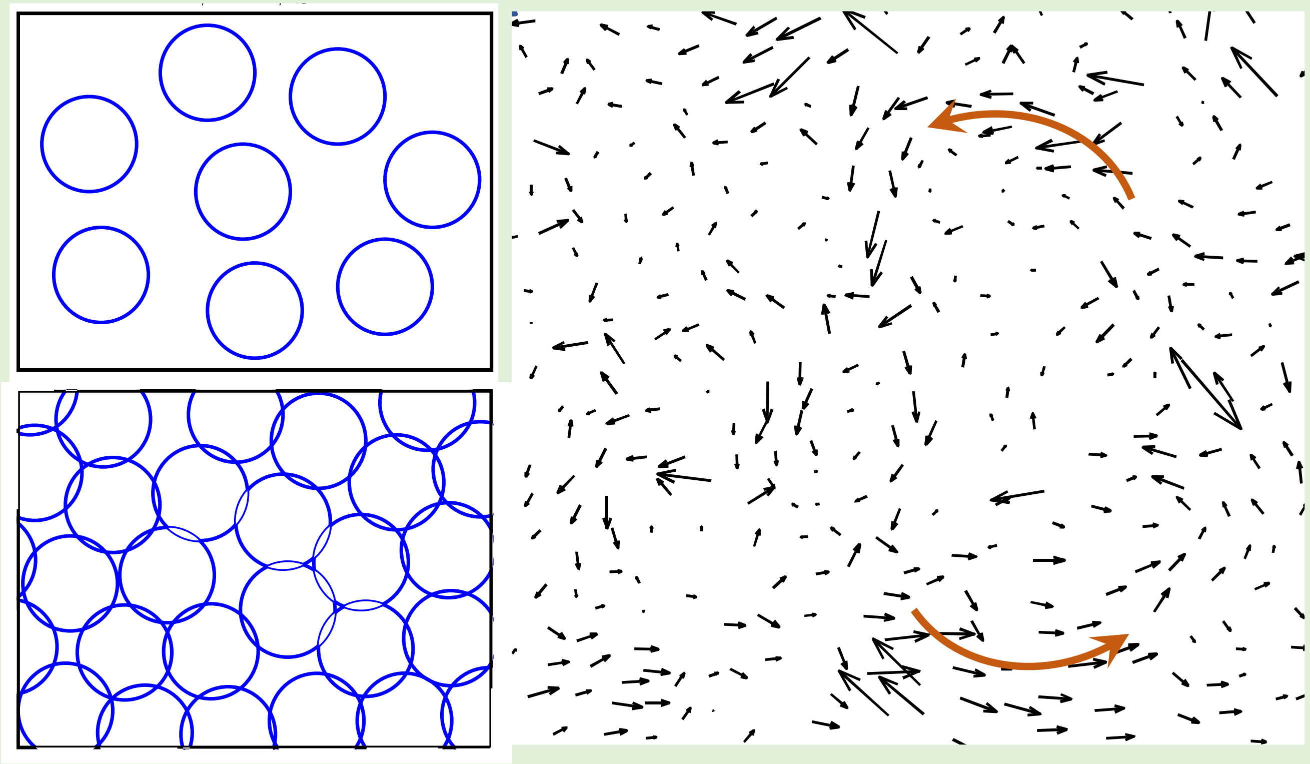
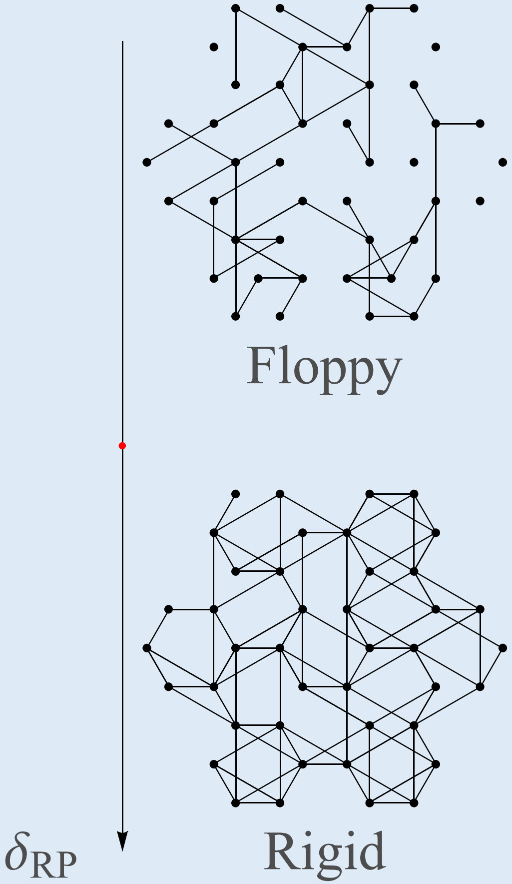
Scaling functions at rigidity transitions



Jamming, theory and experiment (Pouliquen)
Bulk modulus jumps, shear modulus grows from zero
Expt: Complex, multiscale, jerky flow near onset
Uniform continuum behavior on long enough scales

What about glass?



Liarte, Mao, Stenull, Lubensky 2019

Rigidity percolation
Bulk and shear modulus grow from zero

Linear response

Correlations, fluctuations, dissipation

Expand about equilibrium

- How things wiggle: Correlations *C*(**x**-**x**', *t*-*t*'), *S*(**q**, *ω*)
- How things evolve: Greens function *G*(**q**, *ω*)
- How things move when kicked: Susceptibility *χ*(**q**, *ω*)
  - Moduli and yielding from χ' = Re[χ]
  - Dissipation, viscosity from χ'' = Im[χ]

Deep relationships

- Onsager regression hypothesis:

$$\widehat{C}(\mathbf{k},t) = \widehat{G}(\mathbf{k},t)\widehat{C}(\mathbf{k},0)$$
$$C(\mathbf{x},t) = \int \mathrm{d}\mathbf{x}'\, G(\mathbf{x}-\mathbf{x}',t)C(\mathbf{x}',0)$$

- Susceptibility

$$\widetilde{\chi}(\mathbf{k},\omega) = \chi'(\mathbf{k},\omega) + \mathrm{i}\chi''(\mathbf{k},\omega)$$

- Power dissipated:

$$p(\omega) = (\omega|f_\omega|^2/2)\chi''(\omega)$$

- Fluctuation-response theorem, static susceptibility:

$$\widetilde{\chi}_0(\mathbf{k}) = \beta\widehat{C}(\mathbf{k},0)$$

- Fluctuation-response theorem, uniform susceptibility:

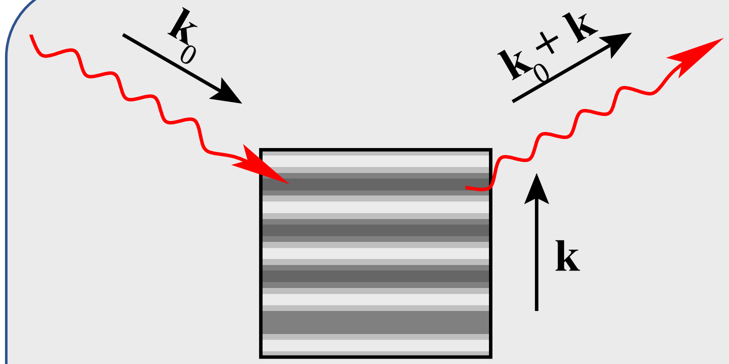
$$\left\langle \langle s\rangle^2_{\text{space}}\right\rangle = k_B T\widetilde{\chi}_0(\mathbf{0})/V$$
$$\text{Ising model } \chi \propto \langle M^2\rangle - \langle M\rangle^2$$
$$\approx \text{specific heat } c_v = (1/Nk_BT^2)(\langle E^2\rangle - \langle E\rangle^2)$$

- Fluctuation-dissipation theorem

$$\chi(\mathbf{x},t) = -\beta\partial C(\mathbf{x},t)/\partial t \quad (t>0)$$
$$\chi''(\omega) = (\beta\omega/2)\widetilde{C}(\omega)$$

- Kramers–Kronig relations (from causality):

$$\chi'(\omega) = (2/\pi)\int_0^\infty \chi''(\omega')(\omega'/(\omega'^2-\omega^2))\,\mathrm{d}\omega'$$
$$\chi''(\omega) = -(2\omega/\pi)\int_0^\infty \chi'(\omega')/(\omega'^2-\omega^2)\,\mathrm{d}\omega'$$



Materials Experiments

- Structure from elastic scattering: X-rays, neutron, acoustic, optical
- Dissipation, time dependence from inelastic scattering
- Pulse-probe experiments…

What is an elastic material?

Deforming **x**' = **x** + **u**(**x**)

Continuum elastic potential energy

- Gradient expansion: small q
- Translations $\mathbf{u} \to \mathbf{u} + \mathbf{u_0}$: depends on $\nabla \mathbf{u} = \partial_i u_j$
- Rotations $\mathbf{u} \to R \cdot \mathbf{u}$, depends on

$$\epsilon_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$$

- Dynamical matrix: $(1/2) \sum D_{ijkl} \epsilon_{ij} \epsilon_{kl}$
- Homogeneous, isotropic

$$D = (\lambda + 2\mu)\epsilon_{ii}^2 + \mu \epsilon_{ij} \epsilon_{ij}$$

$$D_{\mathbf{q}}^{ij} = (\lambda + 2\mu) q^2 \widehat{q}_i \widehat{q}_j + \mu q^2 (\delta_{ij} - \widehat{q}_i \widehat{q}_j)$$

- Bulk $B = \lambda + 2\mu/D$, shear $G = \mu$

Dynamics

- Described by Greens function $\mathcal{G}_{\mathbf{q}}$
- Long time scales: low frequency ω
- If energy conserved,

$$\mathcal{G}_{\mathbf{q}} = (D_{\mathbf{q}} - \mathbb{1}\omega^2)^{-1}$$

- If overdamped (colloids in water), drag γ,

$$\mathcal{G}_{\mathbf{q}} = (D_{\mathbf{q}} - i\gamma\omega\mathbb{1})^{-1}$$

- If damped springs (momentum conserved): Kelvin damping, Maxwell damping …

What will our theory predict?

Example: Longitudinal susceptibility near jamming

Y	y	$\mathcal{Y}$
B	$\beta_{\rm B} \equiv \gamma - 2\nu$	$\mathcal{B} = (\partial \mathcal{L}^{-1}/\partial Q)/(2\,Q)$
ζ	$-\gamma_{\rm B} \equiv \gamma - (2+z)\nu$	$\mathcal{Z} = (1/\Omega)\,\mathrm{Im}\,[\mathcal{B}]$
S	$(2+z)\nu - \gamma$	$\mathcal{S} = (1/\Omega)\,\mathrm{Im}[\mathcal{P}]$

$$\frac{\chi_L}{\chi_0} \approx |\delta_{\rm RP}|^{-\gamma}\mathcal{L}\left(\frac{q/q_0}{|\delta_{\rm RP}|^{\nu}}, \frac{\omega/\omega_0}{|\delta_{\rm RP}|^{z\nu}}, \frac{\delta_{\bf J}/\delta_0}{|\delta_{\rm RP}|^{\phi}}\right)$$

where (for no damping) $\nu = 1$, $z = 1$, $\phi = 1$, and

$$\mathcal{L}(Q, \Omega, W) = \left[\frac{Q^2}{1 + W/(\sqrt{1-\Omega^2} \pm 1)} - \Omega^2\right]^{-1}$$

The bulk modulus B, the viscosity ζ, the density response Π, and the correlation function S, are given in the table. For example, the correlation function

$$S/S_0 = |\delta_{\rm RP}|^{(2+z)\nu-\gamma}\mathcal{S}(Q, \Omega, W).$$

Basically all linear response quantities are related

Emergent continuum rigidity

It seems possible that any uniform isotropic elastic medium must be describable using frequency-dependent elastic moduli $\lambda(\omega)$ and $\mu(\omega)$ at long wavelengths and low frequencies. Symmetries, gradient expansions, and low frequency approximations would tell us that the emergent theory should have the same form for the dynamical matrix

$$(\lambda + 2\mu)\epsilon_{ii}^2 + \mu\epsilon_{ij}^2 \equiv (\lambda + 2\mu)q^2\hat{q}_i\hat{q}_j + \mu q^2(\delta_{ij} - \hat{q}_i\hat{q}_j).$$

If so, a glasse, gel, foam, or sandpile may have local fluctuations on many time and length scales, but above their correlation length and time scales we can describe them just with $\lambda(\omega)$ and $\mu(\omega)$.

Just because there is a continuum theory doesn't mean that it is a mean-field theory! Our calculation makes sense because jamming is above its upper critical dimension $(D = 2)$: there are fluctuations on all scales, but they are too weak to change the mean-field behavior.

Liarte, Thornton, Chowdhury

Are we describing glasses?

Our particular calculations of $\lambda(\omega)$ and $\mu(\omega)$ do not describe the glass transition.

• Melted glasses resist compression, and our floppy states do not.

• Our states have viscosities $\zeta \sim (p - p_c)^x$, while glasses diverge roughly like $\zeta \sim \exp(A/(T - T_c))$.

• Mode coupling theory for glasses predicts properties similar to ours, but need to be replaced near the transition with Gardiner transitions, ...

But many properties of glasses are nicely described by the jamming transition (like the Boson peak). The attractive interactions between atoms are not big enough to wipe out these features except near the transition.

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