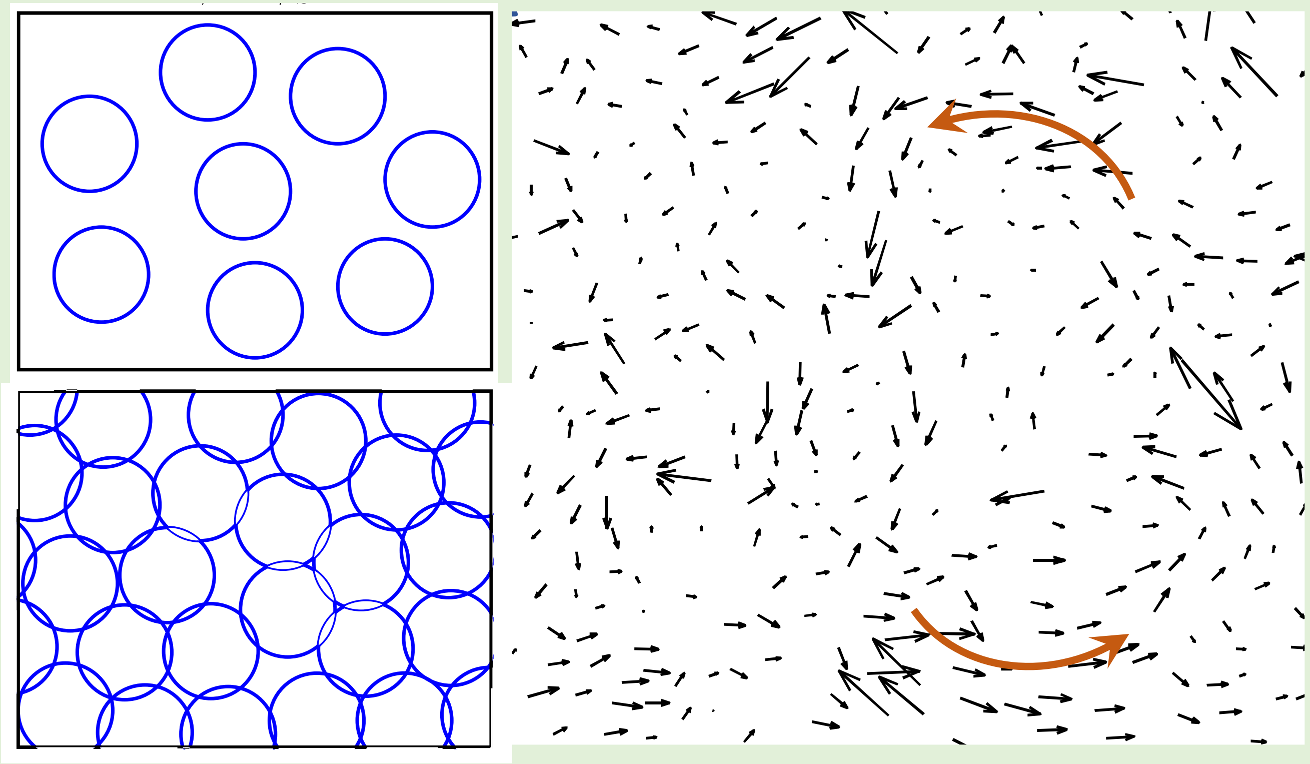
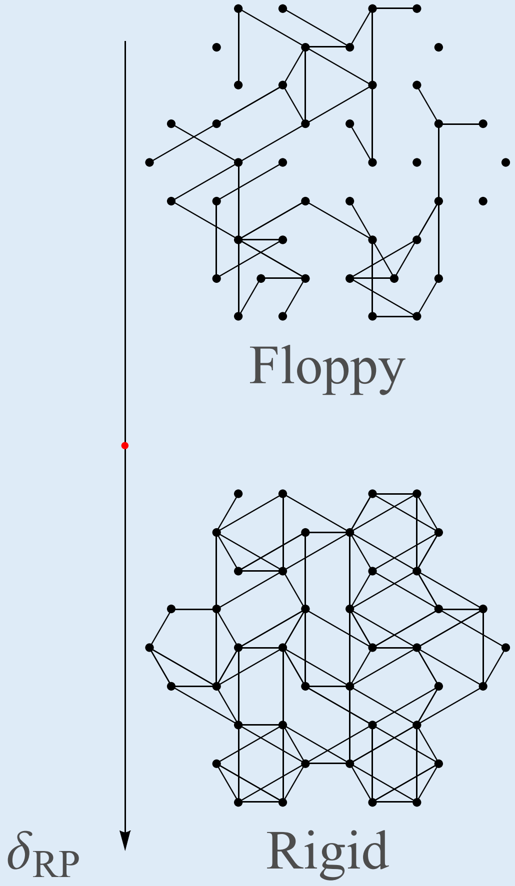
Scaling functions at rigidity transitions



Jamming, theory and experiment (Pouliquen)
Bulk modulus jumps, shear modulus grows from zero
Expt: Complex, multiscale, jerky flow near onset
Uniform continuum behavior on long enough scales

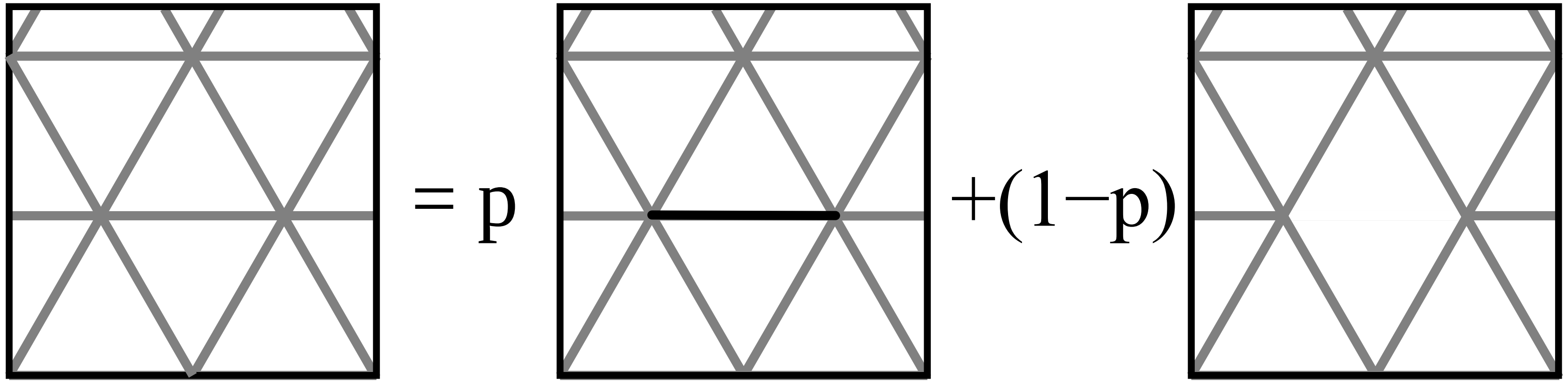
What about glass?



Liarte, Mao, Stenull, Lubensky 2019

Rigidity percolation
Bulk and shear modulus grow from zero

Lattice CPA

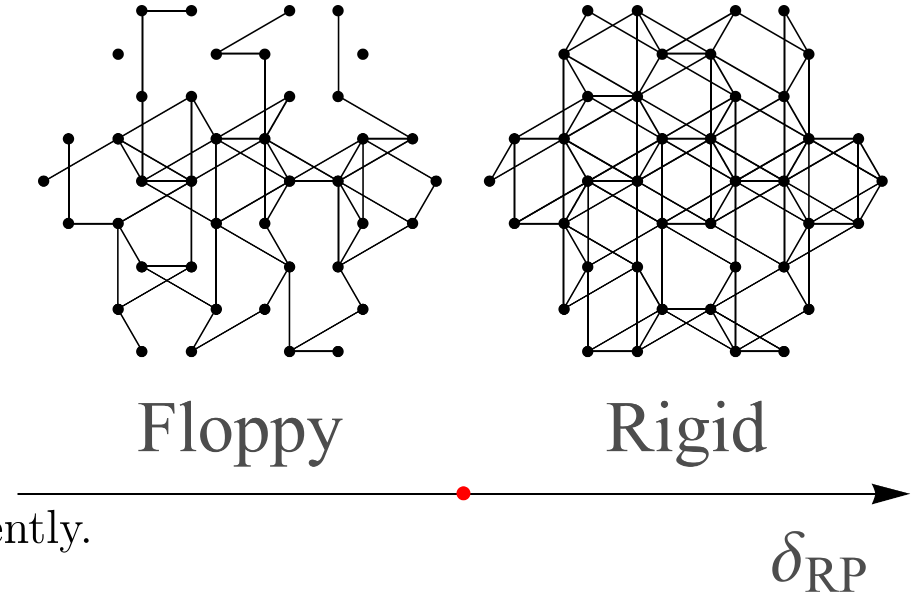


The CPA replaces a lattice with spring K_1 filled with probability p with an effective weaker lattice with complex $K_p(\omega)$.

- Replaces random lattice, spring K_1 w/probability p, with pure lattice w/complex $K_p(\omega)$
- Includes self-consistent effects on all phonons $\mathbf{q}$
- New bond is weaker by $K_p(\omega)/K_1 = (p-h)/(1-h)$, where

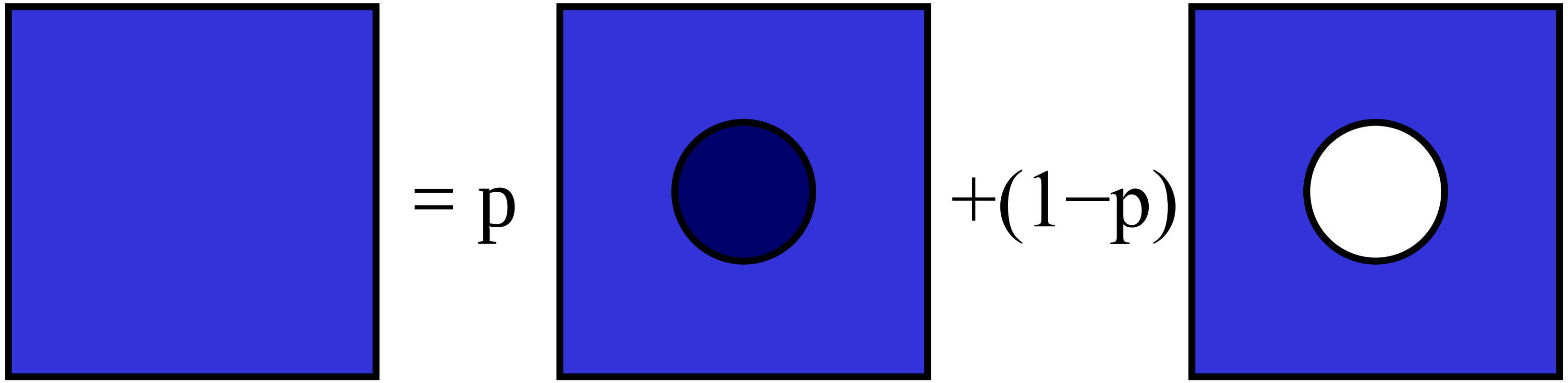
$$h(\omega) \equiv (1/z) \int_{\mathrm{BZ}} \mathrm{d}^d\mathbf{q}\, \mathrm{Tr}\left(D_{\mathbf{q}}(\omega)\, \mathcal{G}_{\mathbf{q}}(\omega)\right).$$

- Here z is the number of bonds per node, BZ is the Brillouin zone
- $D_{\mathbf{q}}$ is the dynamical matrix of the particular lattice, and
- $\mathcal{G}_{\mathbf{q}} = \left(D_{\mathbf{q}} - I\omega^2\right)^{-1}$ is the Greens function.
- $D_{\mathbf{q}}$ and $\mathcal{G}_{\mathbf{q}}$ depend on the weakened $K(\omega)$, so this is solved self-consistently.
- $K(\omega)$ is complex; imaginary part is damping.
- Energy is conserved: damping plane waves from disorder scattering.



Liarte, Lubensky, Mao

Continuum CPA



The lattice calculation is complicated, and deriving the scaling form is difficult due to the lattice anisotropy and shape of the Brillouin zone. Jamming is isotropic. Why not do a CPA by punching circular holes in a continuum elastic sheet? Two kinds of holes: one that zeros λ, and one that zeros μ.

We set a maximum wavevector $|\boldsymbol{q}| < q_D$ that is circularly symmetric. The CPA dynamical matrix is that of an isotropic elastic sheet, which we decompose into $D_{\mathbf{q}}^{\lambda}$ and $D_{\mathbf{q}}^{\mu}$, defining the two integrals h_μ and h_λ, governing the self-consistent equations for the two moduli…

$$h_\alpha \equiv \frac{1}{z_\alpha} \int_0^{q_D} \mathrm{d}^d\mathbf{q}\ \mathrm{Tr}\left(D^\alpha_{\mathbf{q}}\, \mathcal{G}_{\mathbf{q}}\right)$$

The dynamical matrix and the Greens function are given in terms of $\lambda(\omega)$ and $\mu(\omega)$

Thornton, Liarte

Reminder: universal scaling function for mean-field Ising: finding universal parts

(0) Find mean-field theory as self-consistent equation:

$$m = \frac{e^{(h+mz)/T} - e^{-(h+mz)/T}}{e^{(h+mz)/T} + e^{-(h+mz)/T}}$$

(1) Change variables to $t = T_c - T$ with $T_c = z$.

(2) Substitute scaling variables and exponents: $m/m_0 = t^{\beta} \mathcal{M}$, $h/h_0 = t^{\beta\delta} H$, $\beta = ½$, $\delta = 3$.

(3) Keep only the leading order in t. Derive scaling form $m(t,h) = t^{\beta}\mathcal{M}(H) = t^{\beta} \mathcal{M}(h/t^{\beta\delta})$:

$$\mathcal{M} = H - \mathcal{M}^3/3$$

(Universal scaling function relation)

Thornton, Liarte

(0) Self-consistent equations: 3D

The integrals for h_α can be done exactly. (Not yet in Mathematica.) In three dimensions,

$$\frac{\mu}{\mu_1} = \frac{p_\mu - h_\mu}{1 - h_\mu}$$

$$\frac{\lambda}{\lambda_1} = \frac{p_\lambda - h_\lambda}{1 - h_\lambda}$$

$$z_\mu h_\mu = \frac{2\mu}{\lambda + 2\mu}\left(1 + 3\left(\frac{\omega^2}{(\lambda+2\mu)\,q_D^2}\right) + \frac{3}{2}\left(\frac{\omega^2}{(\lambda+2\mu)\,q_D^2}\right)^{3/2}\log\left(\frac{\left(\frac{\omega^2}{(\lambda+2\mu)q_D^2}\right)^{1/2} - 1}{\left(\frac{\omega^2}{(\lambda+2\mu)q_D^2}\right)^{1/2} + 1}\right)\right)$$
$$+ 2\left(1 + 3\left(\frac{\omega^2}{\mu q_D^2}\right) + \frac{3}{2}\left(\frac{\omega^2}{\mu q_D^2}\right)^{3/2}\log\left(\frac{\left(\frac{\omega^2}{\mu q_D^2}\right)^{1/2} - 1}{\left(\frac{\omega^2}{\mu q_D^2}\right)^{1/2} + 1}\right)\right)$$

$$h_\lambda = \frac{\lambda}{\lambda + 2\mu}\left(1 + 3\left(\frac{\omega^2}{(\lambda+2\mu)\,q_D^2}\right) + \frac{3}{2}\left(\frac{\omega^2}{(\lambda+2\mu)\,q_D^2}\right)^{3/2}\log\left(\frac{\left(\frac{\omega^2}{(\lambda+2\mu)q_D^2}\right)^{1/2} - 1}{\left(\frac{\omega^2}{(\lambda+2\mu)q_D^2}\right)^{1/2} + 1}\right)\right)$$

Thornton, Liarte

Universal scaling function for 3D Jamming

Pulling out the universal parts

Just as we did for the Curie-Weiss law for the mean-field Ising model, we
(1) Change coordinates to the distance δ_{J}, δ_{RP} from the critical point,

$$p_\lambda = 1 - \delta_{\mathrm{J}} \qquad p_\mu = \delta_{\mathrm{RP}} + (2 + \delta_{\mathrm{J}})/z_\mu$$

(2) Substitute the scaling variables and exponents

$$\lambda = \lambda_1 \Lambda \quad \mu = |\delta_{\mathrm{RP}}|\, \mu_1 M \quad \omega = |\delta_{\mathrm{RP}}|\, q_D \sqrt{z_\mu \mu_1/24}\, \Omega \quad z_\mu = 24\, Z^2/\,|\delta_{\mathrm{RP}}| \qquad |\delta_{\mathrm{RP}}| = \lambda_1 \delta_{\mathrm{J}}/\mu_1 W$$

(3) Keep only the leading order term in δRP.
This gives us self-consistent equations for the universal scaling functions:

$$M_\pm^2 \mp M_\pm + \Omega^2/4 + Z\Omega^3 \log\left(\left(Z\Omega - \sqrt{M_\pm}\right) \Big/ \left(Z\Omega + \sqrt{M_\pm}\right)\right) /8\sqrt{M_\pm} = 0$$

$$\Lambda_\pm = \left(2M_\pm - 3Z^2\Omega^2\right) / \left(2M_\pm + W - 3Z^2\Omega^2\right)$$

(Here $\pm$ indicates the sign of δ_{RP} – minus for floppy, plus for rigid.)

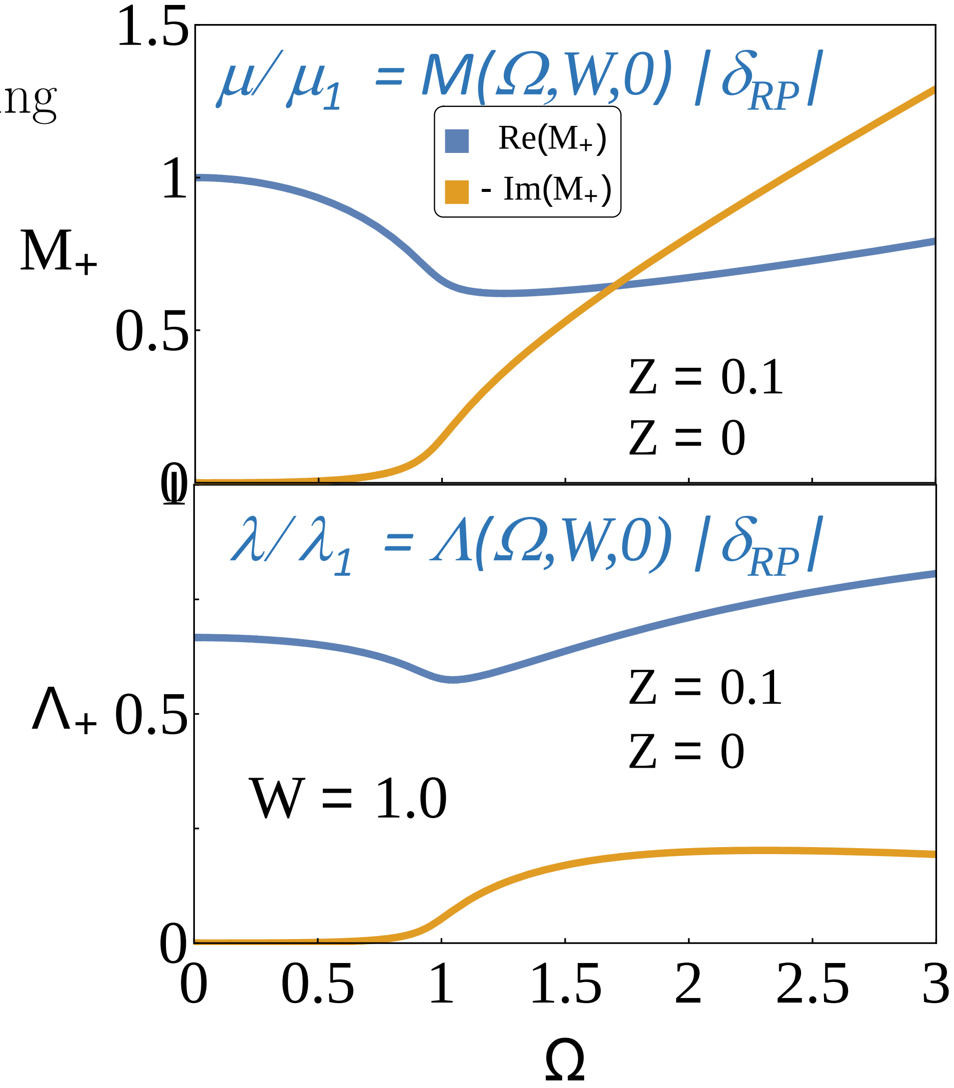
Thornton, Liarte

Leading order scaling: 3D Jamming

Note that $Z = z_\mu |\delta_{\mathrm{RP}}|$ is an irrelevant variable. Ignoring it and setting $Z = 0$, we can solve the equations:

$$M_\pm(\Omega, W, 0) = \left(\pm 1 + \sqrt{1-\Omega^2}\right)/2$$

$$\Lambda_\pm(\Omega, W, 0) = \left(\pm 1 + \sqrt{1-\Omega^2}\right)/\left(W \pm 1 + \sqrt{1-\Omega^2}\right)$$



At left we plot the imaginary part of *M(Ω,0)* and *Λ*(*Ω*, 0), together with *M(Ω,Z)* and *Λ*(*Ω, Z)* with Z = 0.05. Even though Z is irrelevant, it makes an important qualitative change in the prediction – it gives our elastic constants an imaginary part at low frequency. Z is a *dangerous* irrelevant variable.

Thornton, Liarte

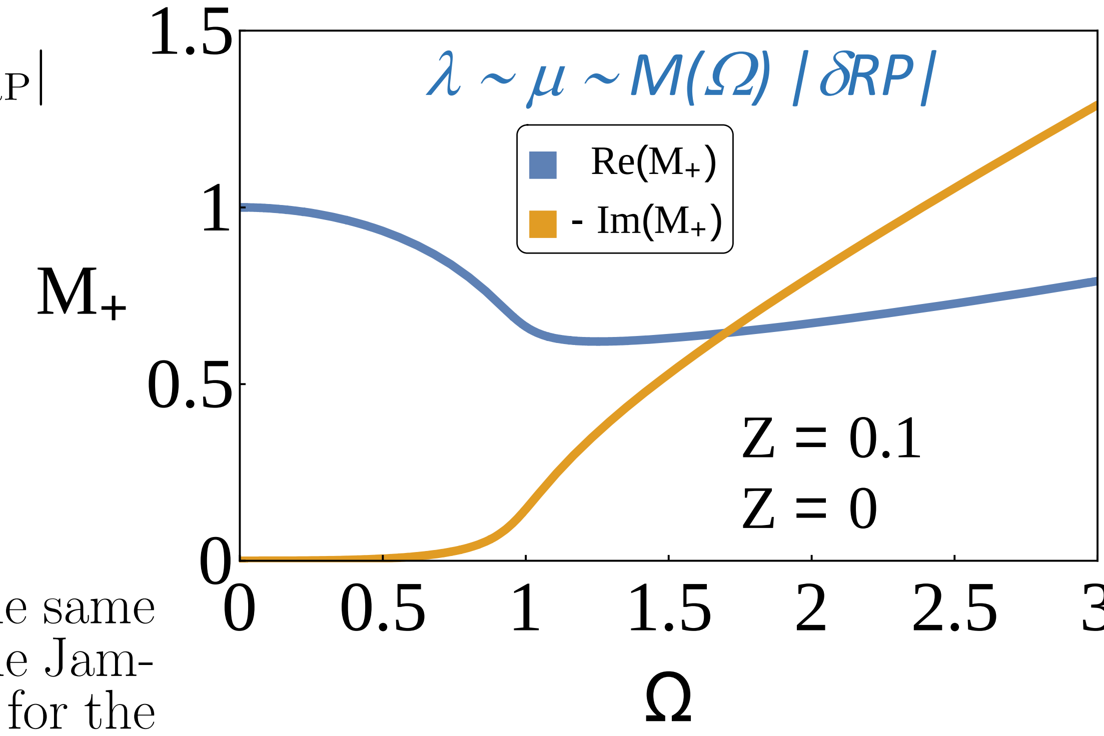
Leading order scaling: 3D RP

Close to rigidity percolation (i.e., as $W \to \infty$), we rescale our rescaled λ to be

$$\overline{\Lambda}_{\pm} = (W/2)\,\Lambda_{\pm} = (\delta_{\mathrm{J}}\cancel{\lambda_1}/2\mu_1\,|\delta_{\mathrm{RP}}|)\,(\lambda/\cancel{\lambda_1}) = \lambda\delta_{\mathrm{J}}/2\mu_1\,|\delta_{\mathrm{RP}}|$$

Then taking $W \to +\infty$ one finds

$$M_{\pm}(\Omega, +\infty, 0) = \left(\pm 1 + \sqrt{1-\Omega^2}\right)\Big/2$$
$$\Lambda_{\pm}\,(\Omega, +\infty, 0) = \left(\pm 1 + \sqrt{1-\Omega^2}\right)\Big/W$$
$$\implies \overline{\Lambda}_{\pm}\,(\Omega, +\infty, 0) = M_{\pm}(\Omega, +\infty, 0)$$



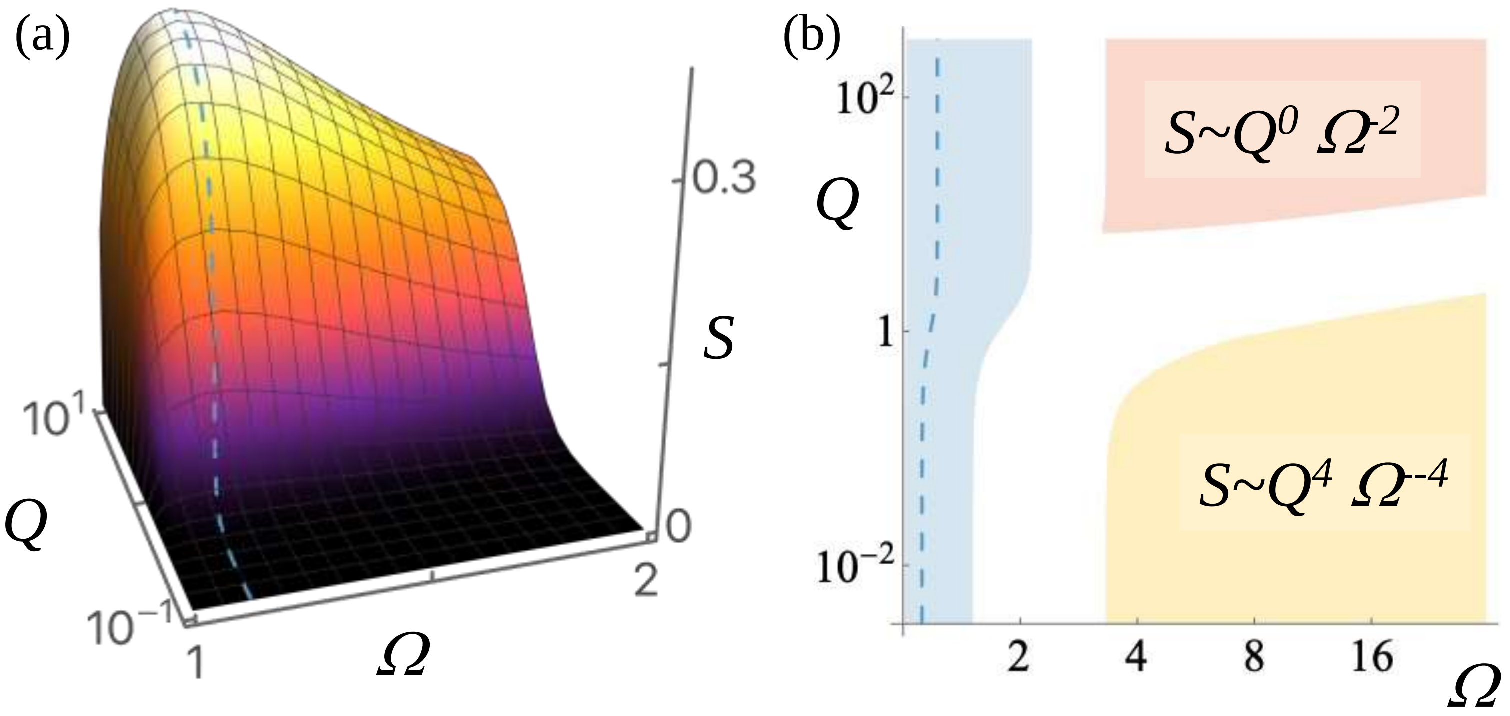
so the two are equal near rigidity percolation, and both have the same dependence as the shear modulus μ near jamming. The unstable Jamming scaling function includes the RP critical behavior, just as for the fracture roughness example.

Thornton, Liarte

Predictions? Example: the Boson Peak

Glasses show a large extra density of states at low frequencies: this is often called the ‘boson peak’. Jamming and RP both show a huge extra density of states at low frequencies, extending down to a frequency ω* which vanishes at jamming.

This is reflected nicely in the CPA calculations, here the boson peak leads to a peak in the correlation function.



(a) Scaling function *S(Q,Ω)* for the correlation function for undamped fluids, (b) power law regimes and boson peak (dashed line)

Thornton, Liarte

2D Jamming

2D Jamming solution at various Z. As Z goes to zero, omega* also goes to zero (very slowly).

Goodrich and Liu: log corrections seen in finite-size scaling for B, mu. Perhaps connection?

Thornton, Liarte