Magnetising the $\mathcal{N}=4$ Super Yang-Mills plasma

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Reference: *JHEP* 06 (2022) 154 2003.00050 [hep-th]

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Summary

1. Magnetised fluid with conformal symmetry
2. AdS/CFT correspondence and the magnetic black brane
3. Thermodynamics of the magnetic black brane
4. Conclusions

1. Magnetised fluid with conformal symmetry

We start with the Gibbs free energy [1] $$G = E - TS = G(T,V,\mathcal{B})$$

where $E = U - VM\mathcal{B}$: magnetic enthalpy , U: internal energy and M: magnetisation density

From conformal symmetry and extensivity $$G = V\,T^D\,g(b)$$

where $b = \mathcal{B}/T^2$, $\mathcal{B}$: magnetic field

Under a scaling transformation $x' = \lambda^{-1}x$ we obtain $$G'(T',V',\mathcal{B}') = \lambda\, G(T,V,\mathcal{B})$$

where $T' = \lambda\, T$, $V' = \lambda^{1-D}V$, $\mathcal{B}' = \lambda^2\mathcal{B}$

From these results we can obtain the equation of state [2] $$E = (D-1)PV - 2\,VM\mathcal{B}$$

where $P = -\partial G/\partial V = -G/V$ is the thermodynamic pressure

[1] R. L. Carlin, Magnetochemistry. Springer-Verlag Berlin Heidelberg, 1986
[2] M. M. Caldarelli, O. J. C. Dias, and D. Klemm, JHEP 03 (2009) 025, arXiv:0812.0801 [hep-th]

When the volume is fixed we define the densities

$$\mathcal{G} = G/V \ , \ \mathcal{F} = F/V \ , \boldsymbol{\rho} = E/V \ , \ \mathcal{U} = U/V \ , \mathcal{S} = S/V$$

For a magnetised conformal fluid in $D = 4$ we obtain $\mathcal{G} = T^4 g(b)$

and the following thermodynamic relations

$$\mathcal{S} = -\frac{\partial \mathcal{G}}{\partial T} = T^3[2bg'(b) - 4g(b)] \ , \quad M = -\frac{\partial \mathcal{G}}{\partial \mathcal{B}} = -T^2 g'(b)$$

$$\mathcal{F} = \mathcal{G} + M\mathcal{B} = T^4[g(b) - bg'(b)] \ , \quad \mathcal{U} = \mathcal{F} + T\mathcal{S} = T^4[bg'(b) - 3(b)]$$

$$\boldsymbol{\rho} = \mathcal{G} + T\mathcal{S} = T^4[2bg'(b) - 3g(b)]$$

From the derivatives of the magnetisation density we obtain

$$\chi = \frac{\partial M}{\partial \mathcal{B}} = -g''(b) \ , \quad \xi = \frac{\partial M}{\partial T} = 2T[bg''(b) - g'(b)]$$

χ: magnetic susceptibility , ξ: pyro-magnetic coefficient

The equation of state takes the form $$\boldsymbol{\rho} = 3P - 2M\mathcal{B}$$

and we also find the conformal identities

$$\mathcal{G} = -P = -\tfrac{1}{4}T\mathcal{S} - \tfrac{1}{2}M\mathcal{B}$$

$$M = \chi\mathcal{B} + \frac{1}{2}\xi\, T$$

The specific heat at fixed V and $\mathcal{B}$ is given by

$$c_{V,B} = T\frac{\partial S}{\partial T} = \frac{\partial \rho}{\partial T} = T^3[-12g(b) + 10\, b\, g'(b) - 4b^2 g''(b)]$$

The stress-energy of the magnetic conformal fluid can be written as [3]

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P\,\eta^{\mu\nu} - \mathcal{M}^{\mu\rho}\mathcal{F}_\rho{}^\nu$$

where $\mathcal{M}^{\mu\rho} = -\partial\mathcal{G}/\partial\mathcal{F}_{\mu\rho}$ is the polarisation tensor

For a magnetic field in the z direction we have $\mathcal{F}_{12} = -\mathcal{F}_{21} = \mathcal{B}$ and $\mathcal{M}^{12} = -\mathcal{M}^{21} = M$

[3] S. A. Hartnoll, P. K. Kovtun, M. Muller and S. Sachdev, Phys. Rev. B 76 (2007) 144502 [arXiv:0706.3215 [cond-mat]

In the fluid rest frame the stress-energy takes the form

$$T^{\mu\nu} = \mathrm{diag}(\rho, P_x, P_x, P_z)$$

where

$$\boldsymbol{\rho} = \mathcal{G} + T\mathcal{S} = \frac{3}{4}T\mathcal{S} - \frac{1}{2}M\mathcal{B}$$

$$P_x = P - MB = \frac{1}{4}T\mathcal{S} - \frac{1}{2}M\mathcal{B}$$

$$P_z = P = \frac{1}{4}T\mathcal{S} + \frac{1}{2}M\mathcal{B}$$

The trace of the stress-energy tensor vanishes, i.e. $T_\mu{}^\mu = -\boldsymbol{\rho} + 2P_x + P_z = 0$

Note that we identify the hydrodynamic pressures P_x and P_z with $-\boldsymbol{\mathcal{G}}$ and $-\boldsymbol{\mathcal{F}}$ respectively

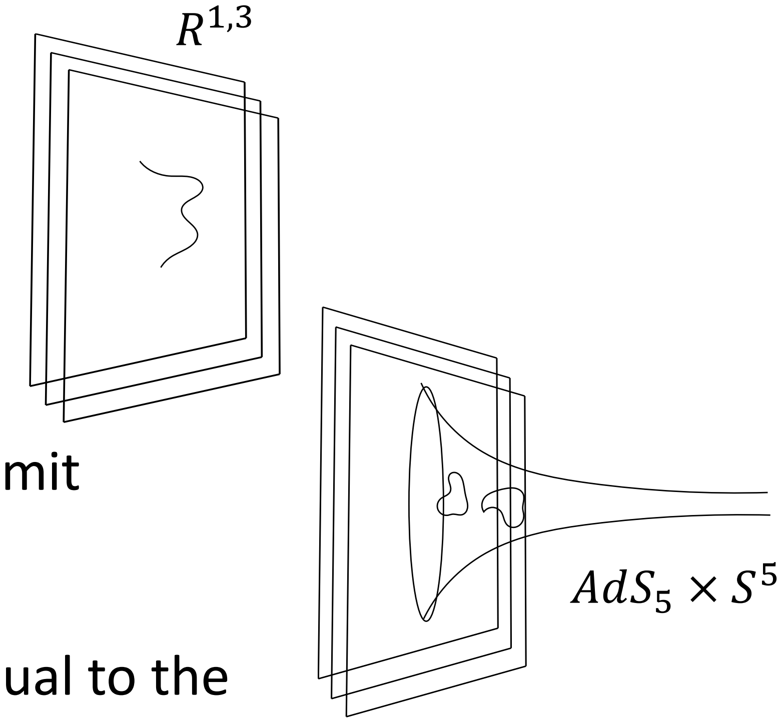
Due to anisotropy, sound propagates in the x and z direction with different speeds

$$c_{s,x}^2 = \left(\frac{\partial P_x}{\partial \rho}\right)_B = \frac{S - \xi B}{C_{V,B}}$$

$$c_{s,z}^2 = \left(\frac{\partial P_z}{\partial \rho}\right)_B = \frac{S}{C_{V,B}}$$

2. The AdS/CFT correspondence and the magnetic black brane

Dp-branes [4] are non-perturbative objects in string theory that allow for dual descriptions in terms of open strings or closed strings



Weakly coupled $SU(N_c)$ $\mathcal{N}=4$ Super Yang-Mills theory in 4d arises from the low energy limit of open strings attached to the stack of N_c D3-branes

Type IIB supergravity in $AdS_5 \times S^5$ arises from the low energy limit of closed strings sourced by the stack of N_c D3-branes

<u>Maldacena conjecture [5]:</u> Type IIB supergravity in $AdS_5 \times S^5$ is dual to the strongly coupled regime of $SU(N_c)$ $\mathcal{N}=4$ Super Yang-Mills theory in 4d

[4] Polchinski, Joseph. 1995. Phys. Rev.Lett., 75, 4724–4727
[5] Maldacena, Juan Martin. 1998. Adv. Theor. Math. Phys., 2, 231–252

AdS/CFT dictionary:

$$g_{YM}^2 = 4\pi g_s$$

$$L^4 = g_{YM}^2 N_C \ell_s^4$$

g_s: open string coupling , ℓ_s: string fundamental length, L: radius of AdS_5 spacetime

Isometry group of AdS_5 maps to the conformal group $SO(2,4)$ of the 4d conformal field theory

Isometry group of S^5 maps to the R-symmetry group of the $\mathcal{N} = 4$ Super Yang-Mills theory

Fields $\phi_{...}$ in $AdS_5 \times S^5$ map to operators $O_{....}$ in the 4d CFT

Correlation functions of the 4d CFT are obtained from the holographic dictionary [6,7]

$$W_{CFT}[\phi_{...}^0] = S_{Sugra}[\phi_{...}]$$

where ϕ^0 arises from the asymptotic expansion of $\phi_{...}$ near the boundary

Finite temperature:

The thermal state of the 4d CFT maps to a 5d black brane that is asymptotically AdS_5 [8]

[6] Witten, Edward. 1998. Adv. Theor. Math. Phys., 2, 253–291.
[7] Gubser, S. S., Klebanov, Igor R., and Polyakov, Alexander M. 1998. Phys. Lett., B428, 105–114
[8] Witten, Edward. 1998. Adv. Theor. Math. Phys., 2, 505–532

The magnetic black brane

Consider Einstein-Maxwell theory in 5d with negative cosmological constant

$$S = \sigma \int d^5x \sqrt{-g}\,[R + 12 - F_{mn}^2]$$

where $$\sigma = 1/(16\pi G_5) = N_c^2/(8\pi^2)$$

The Einstein-Maxwell equations are

$$R_{mn} - \frac{R}{2} g_{mn} - 6\, g_{mn} = 2T_{mn} \quad , \quad \nabla_m F^{mn} = 0$$

where $$T_{mn} = F_{mp} F_n^{\ p} - \frac{1}{4} g_{mn} F_{pq} F^{pq}$$ (5d stress-energy tensor)

Contracting the Einstein equations with g^{mn} we obtain

$$R = -20 - \frac{4}{3} T = -20 + \frac{1}{3} F_{pq} F^{pq}$$

Ansatz for the metric and field strength [9]

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + e^{2V(r)}(dx^2 + dy^2) + e^{2W(r)}dz^2 \quad , \quad F = B\, dx \wedge dy$$

where $B = \mathcal{B}/\sqrt{3}$

Field rescaling

$$U(r) = r_h^2\, \tilde{U}(r) \ , \ \left(e^{V(r)}, e^{W(r)}\right) = r_h\left(e^{\tilde{V}(r)}, e^{\tilde{W}(r)}\right) \ , \ B = r_h^2\, \tilde{B} \ , \ x_\mu = r_h^{-1}\, \tilde{x}_\mu$$

The Einstein-Maxwell equations reduce to

$$\tilde{U}\left(\tilde{V}'' - \tilde{W}''\right) + \left[\tilde{U}' + \tilde{U}\left(2\tilde{V}' + \tilde{W}'\right)\right]\left(\tilde{V}' - \tilde{W}'\right) = -2\tilde{B}^2 e^{-4\tilde{V}}$$
$$2\,\tilde{V}'' + \tilde{W}'' + 2\tilde{V}'^2 + \tilde{W}'^2 = 0$$
$$\frac{1}{2}\tilde{U}'' + \frac{1}{2}\tilde{U}'\left(2\tilde{V}' + \tilde{W}'\right) = 4 + \frac{2}{3}\tilde{B}^2 e^{-4\tilde{V}}$$
$$2\tilde{U}'\tilde{V}' + \tilde{U}'\tilde{W}' + 2\tilde{U}\tilde{V}'^2 + 4\tilde{U}\tilde{V}'\tilde{W}' = 12 - 2\tilde{B}^2 e^{-4\tilde{V}}$$

The first three equations are dynamical (2nd order) and the last equation is a constraint (1st order)

[9] E. D'Hoker and P. Kraus, JHEP 10 (2009) 088, arXiv:0908.3875 [hep-th].

Asymptotic solution near the horizon

$$\widetilde{U}(r) = \widetilde{U}_{h,1}(\tilde{r}-1) + \left(\frac{5}{3}\frac{\tilde{B}^2}{\tilde{v}_{h,0}^4} - 2\right)(\tilde{r}-1)^2 + \cdots$$

$$\frac{e^{\widetilde{V}(r)}}{\tilde{v}_{h,0}} = 1 - \frac{4}{3}\frac{\tilde{B}^2 - 3\tilde{v}_{h,0}^4}{\widetilde{U}_{h,1}\tilde{v}_{h,0}^4}(\tilde{r}-1) + \cdots$$

$$\frac{e^{\widetilde{W}(r)}}{\widetilde{w}_{h,0}} = 1 + \frac{2}{3}\frac{\tilde{B}^2 + 6\tilde{v}_{h,0}^4}{\widetilde{U}_{h,1}\tilde{v}_{h,0}^4}(\tilde{r}-1) + \cdots$$

We can choose $\tilde{v}_{h,0} = 1$ whist the parameters $\widetilde{U}_{h,1}$ and $\widetilde{w}_{h,0}$ are obtained numerically

Asymptotic solution near the boundary

$$\frac{\widetilde{U}(r)}{\tilde{r}^2} = 1 + \widetilde{U}_{\infty,1}\tilde{r}^{-1} + \frac{\widetilde{U}_{\infty,1}^2}{4}\tilde{r}^{-2} - \frac{2}{3}\tilde{B}^2\tilde{r}^{-4}\ln\tilde{r} + \widetilde{U}_{\infty,4}\tilde{r}^{-4} + \cdots$$

$$\frac{e^{\widetilde{V}(\tilde{r})}}{\tilde{r}} = 1 + \frac{\widetilde{U}_{\infty,1}}{2}\tilde{r}^{-1} + \frac{1}{6}\tilde{B}^2\tilde{r}^{-4}\ln\tilde{r} + v_{\infty,4}\tilde{r}^{-4} + \cdots$$

$$\frac{e^{\widetilde{W}(\tilde{r})}}{\tilde{r}} = 1 + \frac{\widetilde{U}_{\infty,1}}{2}\tilde{r}^{-1} - \frac{1}{3}\tilde{B}^2\tilde{r}^{-4}\ln\tilde{r} - 2v_{\infty,4}\tilde{r}^{-4} + \cdots$$

The UV parameters $\widetilde{U}_{\infty,1}$, $\widetilde{U}_{\infty,4}$ and $\tilde{v}_{\infty,4}$ are obtained numerically

Temperature, entropy density and the physical magnetic field

$$T = \frac{U'(r_h)}{4\pi} = \frac{r_h}{4\pi} \widetilde{U}_{h,1}$$

(absence of conical singularity)

$$\mathcal{S} = \frac{S}{V_3} = \frac{A_h}{4G_5 V_3} = 4\pi\sigma\, e^{2V(r_h)} e^{W(r_h)} = 4\pi\sigma\, r_h^3\, \tilde{v}_{h,0}^2\, \widetilde{w}_{h,0}$$

(Bekenstein-Hawking formula)

Using these results we obtain the dimensionless ratio

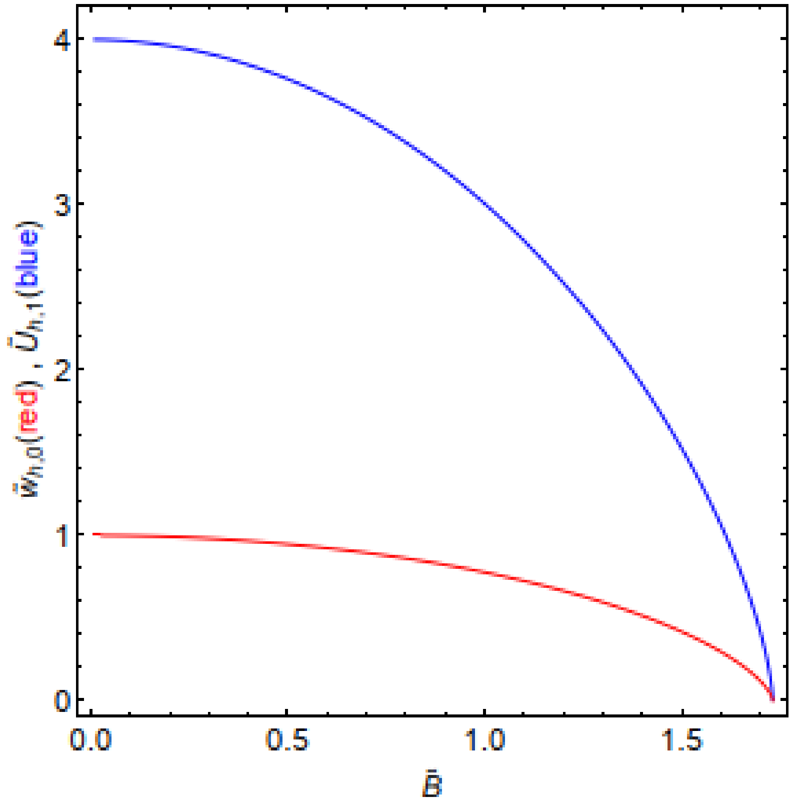
$$\frac{\mathcal{S}}{T^3} = (4\pi)^4 \sigma \frac{\tilde{v}_{h,0}^2 \widetilde{w}_{h,0}}{\widetilde{U}_{h,1}^3}$$

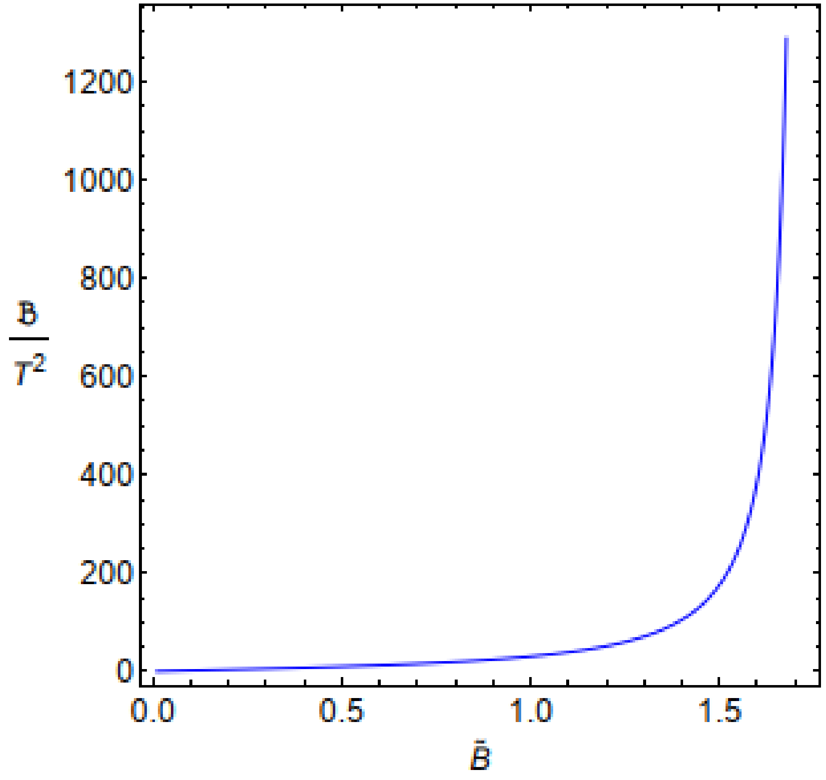
The physical magnetic field can be obtained from the dimensionless ratio

$$b = \frac{\mathcal{B}}{T^2} = 16\sqrt{3}\pi^2 \frac{\tilde{B}}{\widetilde{U}_{h,1}^2}$$

The horizon parameter $\tilde{v}_{h,0}$ was set to 1 whilst the horizon parameters $\widetilde{U}_{h,1}$ and $\widetilde{w}_{h,0}$ are obtained numerically as function of $\tilde{B}$

Some numerical results





Left panel: Horizon parameters $\tilde{U}_{h,1}$ and $\tilde{w}_{h,0}$ as functions of $\tilde{B}$

Right panel: dimensionless ratio $b = \mathcal{B}/T^2$ as a function of $\tilde{B}$

3. Thermodynamics of the magnetic black brane

The Euclidean on-shell action and holographic renormalisation

$$S_{on-shell} = S_M + S_{\partial M}$$

where

$$S_M = -\sigma \int_M d^5x \sqrt{g}\ (R + 12\ - F_{mn}^2)$$

(Einstein-Maxwell term)

$$S_{\partial M} = -2\sigma \int_{\partial M} d^4x \sqrt{\gamma}\ K$$

(Gibbons-Hawking-York boundary term)

Evaluating both terms, the on-shell action becomes a sum of surface terms

$$S_M + S_{\partial M} = -\sigma\, V_3\, \beta\, r_h^4 \left\{ 2\left[\tilde{U}\left(e^{2\tilde{V}+\tilde{W}}\right)'\right]_{\tilde{r}=\tilde{r}_0} + \left[e^{2\tilde{V}+\tilde{W}}\tilde{U}'\right]_{\tilde{r}=1} \right\}$$

Plugging the asymptotic behaviour of the fields $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ we obtain

$$\begin{aligned} &S_M + S_{\partial M} \\ &= -\sigma\, V_3\, \beta\, r_h^4 \left\{ 6\,\tilde{r}_0^4 + 12\,\tilde{U}_{\infty,1}\tilde{r}_0^3 + 9\tilde{U}_{\infty,1}^2\tilde{r}_0^2 + 3\tilde{U}_{\infty,1}^3\tilde{r}_0 - 4\tilde{B}^2 \ln\tilde{r}_0 + \frac{3}{8}\tilde{U}_{\infty,1}^4 + 6\tilde{U}_{\infty,4} + \tilde{U}_{h,1}\tilde{v}_{h,0}\tilde{w}_{h,0} \right\} \end{aligned}$$

Diffeomorphism invariant counter-term

$$S_{ct} = \sigma \int_{\partial M} d^4x \sqrt{\gamma}\,[a_1 + a_2 F_{\mu\nu}F^{\mu\nu} \ln(F_{\mu\nu}F^{\mu\nu}) + a_3\, F_{\mu\nu}F^{\mu\nu}]$$

UV divergences are cancelled choosing $a_1 = 6$, $a_2 = 1/4$ and we obtain

$$S_r = -\sigma V_3 \beta\, r_h^4\, [3\widetilde{U}_{\infty,4} + \widetilde{U}_{h,1}\tilde{v}_{h,0}^2 \widetilde{w}_{h,0} - \tilde{B}^2 \ln \tilde{B} - \left(2a_3 + \frac{1}{2}\ln 2\right)\tilde{B}^2]$$

Gibbs free energy density

$$\mathcal{G} = \frac{TS_r}{V_3} = -\sigma\, r_h^4\, [3\widetilde{U}_{\infty,4} + \widetilde{U}_{h,1}\tilde{v}_{h,0}^2 \widetilde{w}_{h,0} - \tilde{B}^2 \ln \tilde{B} - \left(2a_3 + \frac{1}{2}\ln 2\right)\tilde{B}^2]$$

Magnetic enthalpy density

$$\boldsymbol{\rho} = -\sigma\, r_h^4\, [3\widetilde{U}_{\infty,4} - \tilde{B}^2 \ln \tilde{B} - \left(2a_3 + \frac{1}{2}\ln 2\right)\tilde{B}^2]$$

Scheme-independent quantities: $\mathcal{G}_{\mathrm{r}} = \mathcal{G} - \mathcal{G}_{T=0}$, $\boldsymbol{\rho_r} = \boldsymbol{\rho} - \boldsymbol{\rho}_{T=0}$

Holographic stress-energy tensor

$$< T_r^{\mu\nu} > = < T_{reg}^{\mu\nu} > + < T_{ct}^{\mu\nu} >$$

where

$$< T_{reg}^{\mu\nu} > = \frac{2\, r_0^6}{\sqrt{-\gamma}} \frac{\delta(S_M + S_{\partial M})}{\delta\gamma_{\mu\nu}} = -2\sigma(K^{\mu\nu} - K\gamma^{\mu\nu})$$

$$< T_{ct}^{\mu\nu} > = -\frac{2r_0^6}{\sqrt{-\gamma}} \frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}$$

Result:

$$< T_r^{\mu\nu} > = \mathrm{diag}(\boldsymbol{\rho}_r, P_{x,r}, P_{x,r}, P_{z,r})$$

where

$$\boldsymbol{\rho}_r = -\frac{3N_c^2}{8\pi^2}\, r_h^4 \left\{ \tilde{U}_{\infty,4} - \frac{\tilde{B}^2}{6}\left[2\tilde{U}_{\infty,4}(\sqrt{3}) + \ln\left(\frac{\tilde{B}^2}{3}\right)\right]\right\}$$

$$P_{x,r} = -\frac{N_c^2}{8\pi^2}\, r_h^4 \left\{ \tilde{U}_{\infty,4} - 8\tilde{v}_{\infty,4} - \frac{\tilde{B}^2}{2}\left[2\tilde{U}_{\infty,4}(\sqrt{3}) + \ln\left(\frac{\tilde{B}^2}{3}\right)\right]\right\}$$

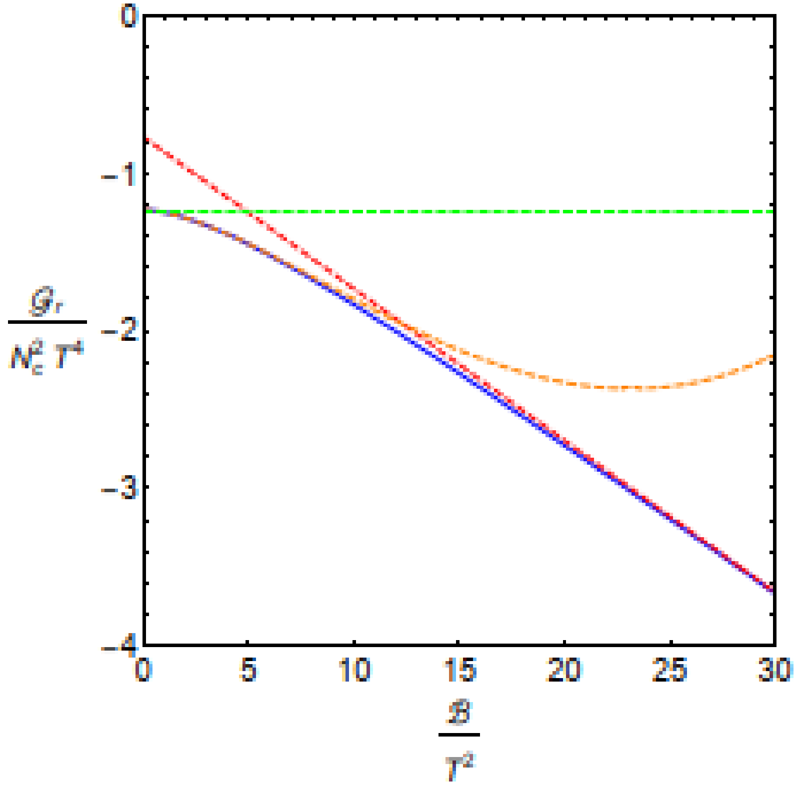
$$P_{z,r} = -\frac{N_c^2}{8\pi^2}\, r_h^4 \left\{ \tilde{U}_{\infty,4} + 16\, \tilde{v}_{\infty,4} + \frac{\tilde{B}^2}{2}\left[2\tilde{U}_{\infty,4}(\sqrt{3}) + \ln\left(\frac{\tilde{B}^2}{3}\right)\right]\right\}$$

The trace of the stress-energy tensor vanishes, as expected for a conformal fluid

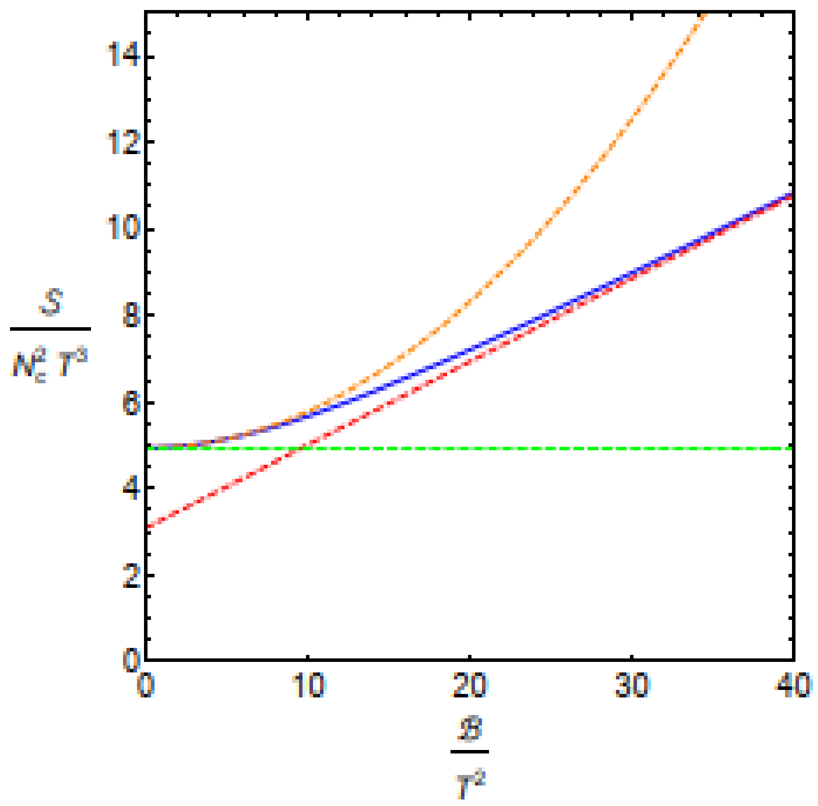
A non-diffeomorphism invariant counter-term can lead to a T independent trace anomaly that disappears when subtracting the $T = 0$ result

Numerical results

Gibbs free energy density



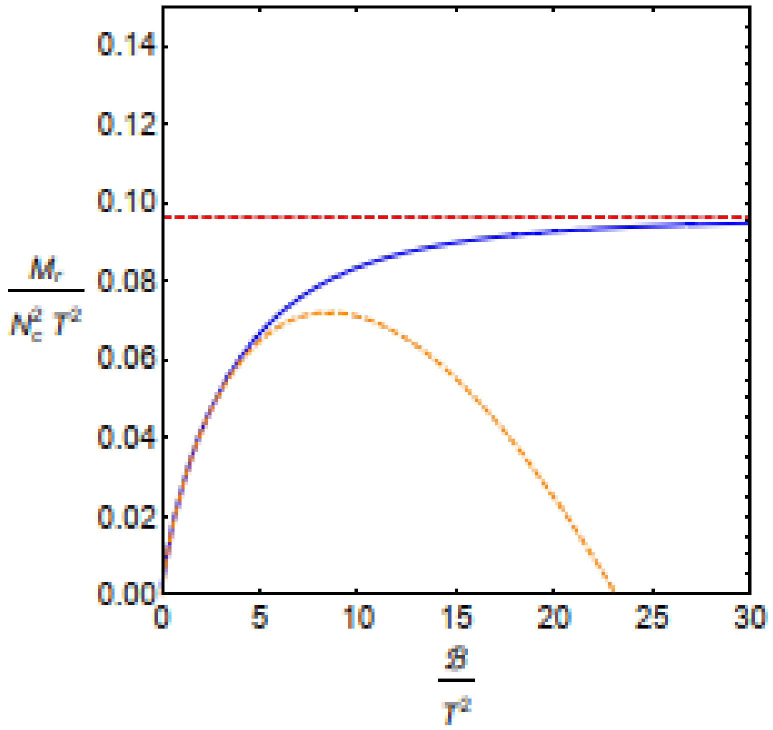
Entropy density



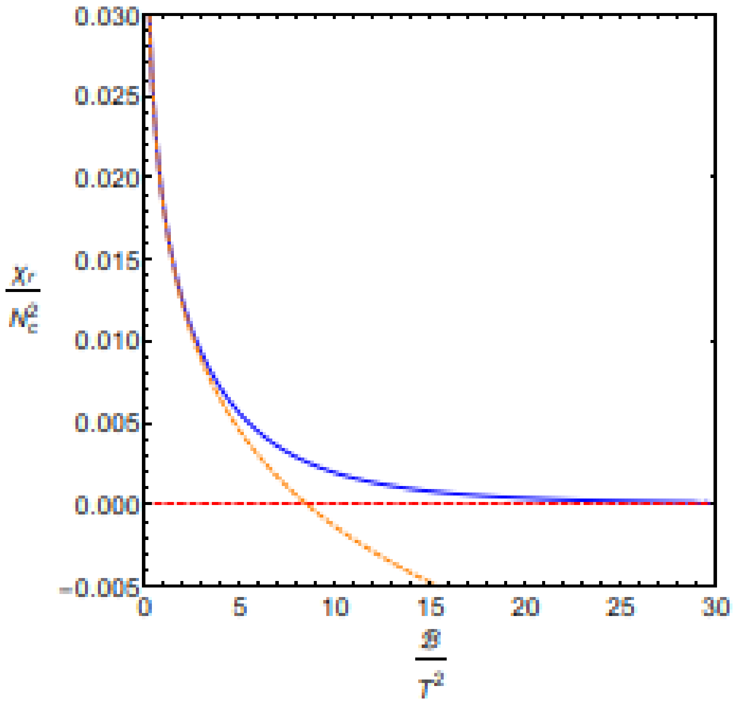
Blue curves represent the full numerical results

Orange and red curves represent analytical results at small b or large b

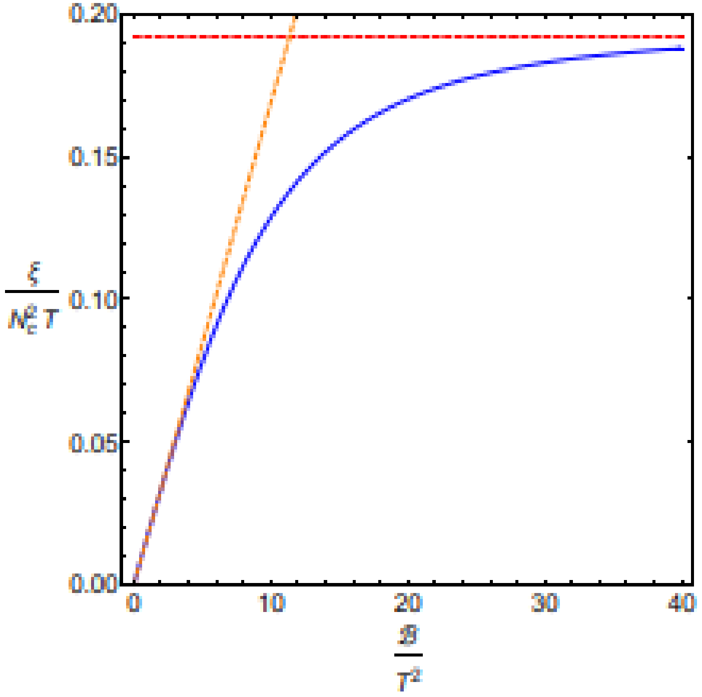
Magnetisation



Susceptibility



Pyro-magnetic coefficient



Blue curves represent the full numerical results

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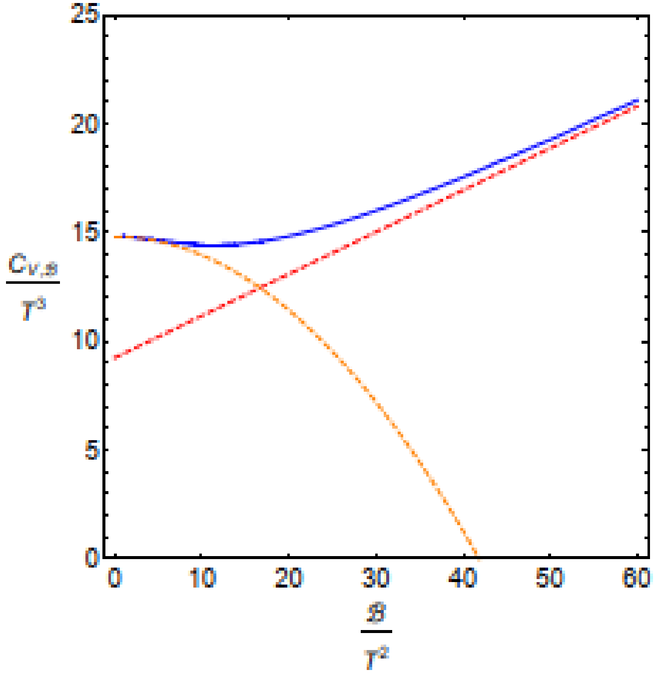
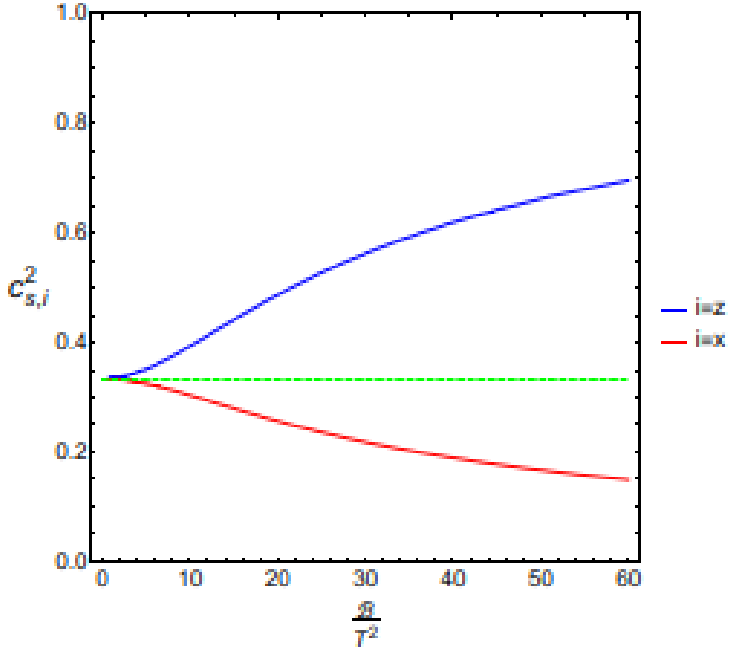
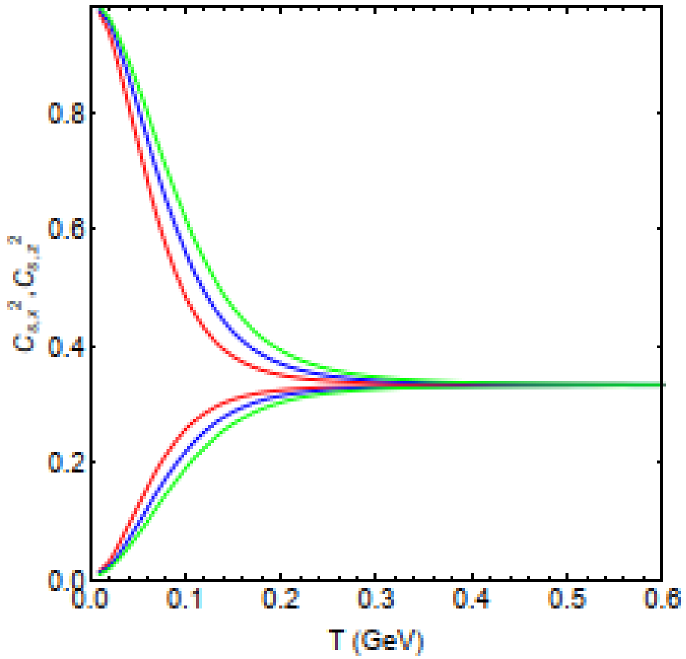
Specific heat



Speeds of sound





Blue curves represent the full numerical results

Orange and red curves represent analytical results at small b or large b

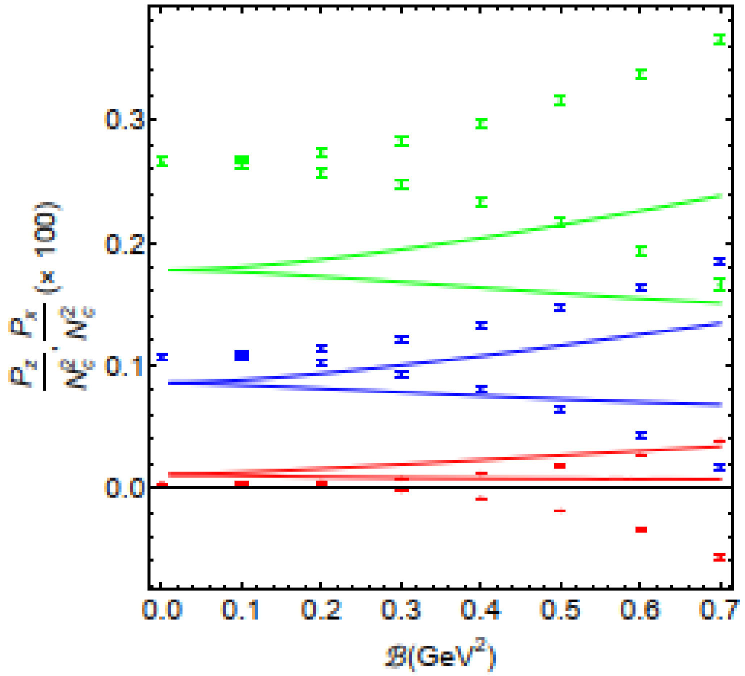
Important conclusion: The behaviour of all the thermodynamic quantities at large b is consistent with $\mathbf{3+1 \rightarrow 1+1}$ **dimensional reduction** in the magnetised conformal plasma

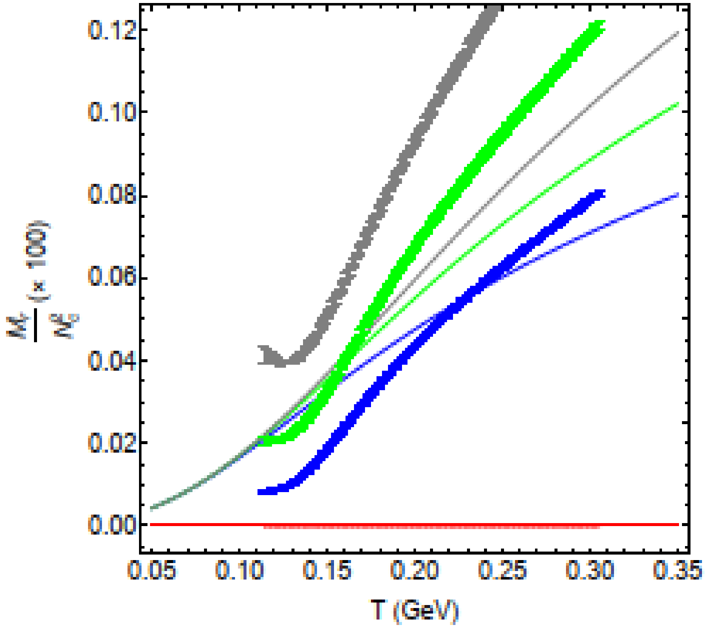
The analytical result at large b is consistent with the $\boldsymbol{BTZ \times R^2}$ where the BTZ black brane [10] is the gravity dual of a 2d CFT at finite temperature

[10]M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849–1851, arXiv:hep-th/9204099

Phenomenological comparison to lattice QCD results

In this case we fix $\sigma = N_c^2/(45\pi^2)$ in order to match the Stefan-Boltzmann result for a free Yang-Mills plasma in the large N_c limit. We compare against the lattice QCD results of [11]





Left panel: Red, blue and green correspond to $T = 0.15\text{GeV}$, $T = 0.25\text{GeV}$ and $T = 0.3\text{GeV}$

Right panel: Red, blue, green and grey correspond to $\mathcal{B} = 0\text{GeV}^2$, $\mathcal{B} = 0.2\text{GeV}^2$, $\mathcal{B} = 0.3\text{GeV}^2$ and $\mathcal{B} = 0.4\text{GeV}^2$

[11] G. S. Bali, F. Bruckmann, G. Endrödi, S. D. Katz, and A. Schäfer, JHEP 08 (2014) 177, arXiv:1406.0269 [hep-lat]

4. Conclusions

- We used the AdS/CFT correspondence to investigate the strongly coupled regime of a $\mathcal{N} = 4$ Super Yang-Mills plasma in the presence of a finite magnetic field
- We obtained a Gibbs free energy density and a stress-energy tensor consistent with a magnetised fluid with conformal symmetry
- The behaviour of the thermodynamic quantities at large magnetic fields is compatible with a $3+1 \rightarrow 1+1$ dimensional reduction of the CFT
- We found that the anisotropy between the pressures in the magnetised conformal fluid increases with the magnetic field in a qualitatively similar way than the quark-gluon plasma
- Possible future directions: incorporate confinement and chiral symmetry breaking in order to describe magnetic catalysis and inverse magnetic catalysis, describe perturbations of the magnetic black brane and compare with magnetohydrodynamics.

THANK YOU!