Finite Energy Sum Rules at finite magnetic fields: advances and perspectives

Cristián Villavicencio

CCE – Universidad del Bio-Bio

Workshop on Electromagnetic Effects in Strongly Interacting Matter

São Paulo - October 2022

Based on

Phys. Rev. D 92, 016006 (2015)

Phys. Rev. D 98, 034015 (2018)

Phys. Rev. D 102, 094007 (2020)

in collaboration with

Marcelo Loewe
Cesáreo Dominguez
Alejandro Ayala
Luis Hernandez
Rtenato Zamora

Outline

- Introduction to finite energy sum rules (FESR)
  - FESR formalism
  - Axial channel (vacuum)
  - Nucleon channel (vacuum)

- FESR at finite magnetic field
  - Propagators
  - condensates
  - Axial channel
  - Nucleon channel
  - Nucleon axial coupling constant

- Summary and outlook

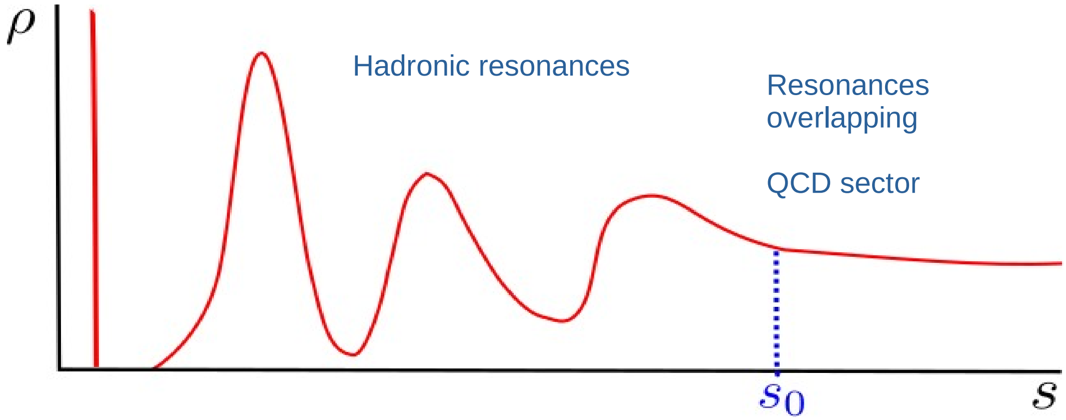
Introduction to FESR

The spectral function

Two current correlator $$\Pi_{\mu\nu}(x-y) = i\langle 0|T J_\mu(x) J_\nu^\dagger(y)|0\rangle$$

Fourier transformation $$\Pi_{\mu\nu}(q) = q_\mu q_\nu \Pi_L(q^2) + (g_{\mu\nu} q^2 - q_\mu q_\nu)\Pi_T(q^2)$$

Spectral function $$\rho(s) = \frac{1}{\pi}\mathrm{Im}\Pi(s+i\epsilon)$$



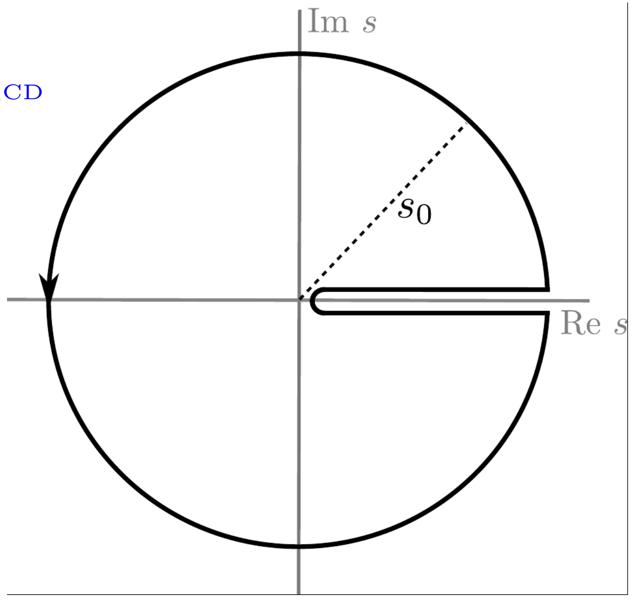
$s_0 \rightarrow$ Hadronic continuum threshold

narrow resonance approximation $$\rho(s) = 2f^2M^2\delta(s-M^2) + \rho^{\mathrm{QCD}}(s)\theta(s-s_0)$$

FESR Quark-hadron duality $\quad \Pi^{\mathrm{Had}} \leftrightarrow \Pi^{\mathrm{QCD}}$

Cauchy's theorem

$$\frac{1}{\pi}\int_0^{s_0} ds\, K(s)\, \mathrm{Im}\Pi^{\mathrm{had}}(s+i\epsilon) = \frac{-1}{2\pi i}\oint_{|s|=s_0} ds\, K(s)\, \Pi^{\mathrm{QCD}}(s) + \mathrm{Res}\, K\Pi^{\mathrm{QCD}}$$



$K(s) = s^N$ analytic kernel

Operator Product Expansion (OPE)

$$\Pi^{\mathrm{QCD}}(s) = \Pi^{\mathrm{pQCD}}(s) = \sum_{n>0} C_{2n}(s,\mu)\frac{\langle 0|:O_{2n}:|0\rangle(\mu)}{s^n} \qquad s \gg m_q^2$$

C_{2n} Wilson coefficients

μ $\overline{\mathrm{MS}}$ subtraction scale

$\langle : O_{2n} : \rangle$ Normal-ordered condensates

Non-normal ordered condensates $\rightarrow$ needed to correct inconsistencies in the chiral limit

$$\langle : \bar{q}_i q_j : \rangle = \langle \bar{q}_i q_j \rangle - S_{ij}(x, x; G, \mu)$$

$$\Pi^{\mathrm{pQCD}}(s) \sim \log(-s)$$

$$\begin{aligned}
\langle O_2 \rangle &= 0 \\
\langle O_4 \rangle &\sim \langle g_s^2 G^a_{\mu\nu} G^{a\mu\nu} \rangle,\ m_q \langle \bar{q} q \rangle \\
\langle O_6 \rangle &\sim \langle (\bar{q} q)^2 \rangle,\ \langle g_s^3 G^a_{\mu\nu} G^b_{\nu\alpha} G^c_{\alpha\mu} f^{abc} \rangle,\ m_q \langle \bar{q} G^a_{\mu\nu} t^a \sigma^{\mu\nu} q \rangle
\end{aligned}$$

FESR features

- cuts the OPE series (vacuum with no radiative corrections)
- Hadronic threshold acts as an order parameter (finite T)
- No need to calculate the full form factor (can integrate contour before loop momentum integration or Feynman parameters integration)

Axial – Axial correlator and derivatives

$$\Pi^A_{\mu\nu}(q^2) = i\int d^4x\, e^{iqx}\,\langle 0|T[A_\mu(x)A^\dagger_\nu(0)]|0\rangle \;=(q_\mu q_\nu - q^2 g_{\mu\nu})\,\Pi_T(q^2) + g_{\mu\nu}\,\Pi_d(q^2)$$

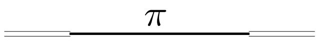
$$\Pi_{5\nu}(q^2) = i\int d^4x\, e^{iqx}\,\langle 0|T\,[i\partial\cdot A(x)\,A^\dagger_\nu(0)]|0\rangle = \; q_\nu\,\Pi_5(q^2)$$

$$\psi_5(q^2) = i\int d^4x\, e^{iqx}\langle 0|T\,[\partial\cdot A(x)\,\partial\cdot A^\dagger_\nu(0)]|0\rangle$$

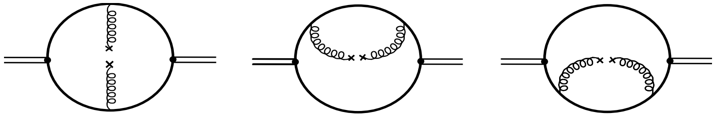
Ward identities ➡

$$q^\mu\Pi^A_{\mu\nu}(q^2) = \Pi_{5\nu}(q^2) + \langle 0|[\bar u\gamma_\nu u - \bar d\gamma_\nu d]|0\rangle$$

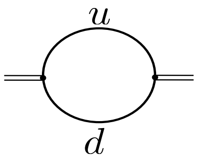
$$q^\nu\Pi_{5\nu}(q^2) = \Psi_5(q^2) + \langle 0|[m_u\bar u u + m_d\bar d d]|0\rangle$$



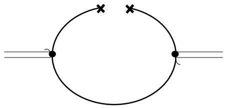
$$\sim \frac{f_\pi^2}{s-m_\pi^2}$$

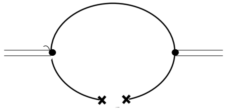


$$\sim \frac{\langle\alpha G^2\rangle}{s^2}$$



$$\sim \log(-s)$$





$$\sim \frac{m_q\langle\bar q q\rangle}{s^2}$$

FESR up to $\langle \boldsymbol{O_4} \rangle$

$$\int ds\, \Pi_0 \rightarrow \qquad 2\, f_\pi^2 = \frac{s_0}{4\,\pi^2}$$

$$\int ds\, s\, \Pi_0 \rightarrow \qquad 2\, f_\pi^2\, m_\pi^2 = \frac{s_0^2}{8\,\pi^2} - 2m_q\, \langle \bar{q}q \rangle - \frac{1}{12\pi}\, \langle \alpha_s\, G^2 \rangle$$

$$\int ds\, \Pi_5 \rightarrow \qquad 2\, f_\pi^2\, m_\pi^2 = -\,4m_q \langle \bar{q}q \rangle + \frac{3}{2\pi^2} {m_q}^2 s_0$$

$$\int ds\, \Psi_5 \rightarrow \qquad 2 f_\pi^2\, m_\pi^4 = \frac{3m_q^2\, s_0^2}{4\,\pi^2} - 4m_q^3\, \langle \bar{q}q \rangle + \frac{m_q^2}{2\pi}\, \langle \alpha_s\, G^2 \rangle$$

$\Rightarrow$ $s_0 \approx 0.67 \ \mathrm{GeV}^2$

Nucleon – nucleon correlator

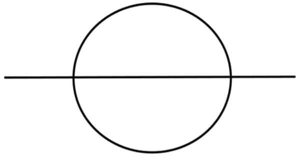
nucleon interpolating function(QCD sector)

$$\eta_N(x) = \varepsilon^{abc}[(u^a)^T(x)C\gamma_\mu u^b(x)]\gamma^\mu\gamma_5 d^c(x),$$

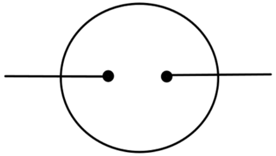
nucleon field (hadronic sector)

$$\eta_N(x) = \lambda_N \Psi_N(x)$$

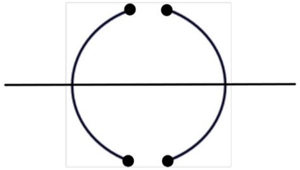
$$\Pi(q) = i\int d^4x e^{iqx}\langle 0|T\eta(x)\bar{\eta}(0)|0\rangle \quad = \not{q}\Pi_1(q^2) + \Pi_2(q^2)$$



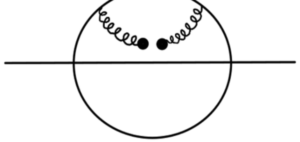
~ 1



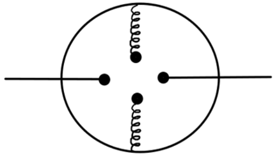
$\sim \langle \bar{q}q \rangle$



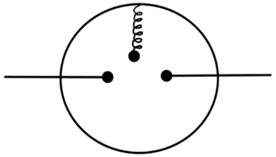
$\sim \langle \bar{q}q\bar{q}q \rangle$



$\sim \langle \alpha_s G^2 \rangle$



$\sim \langle \alpha_s G^2 \bar{q}q \rangle$



$\sim \langle g_s \bar{q}Gq \rangle$

QCD sector

Hadronic sector

$$\Pi_1(s) = -\frac{1}{64\pi^4} s^2 \ln(-s/\nu^2) - \frac{1}{32\pi^3}\langle \alpha_s G^2\rangle \ln(-s/\nu^2) - \frac{2}{3}\frac{\langle \bar{q}q\bar{q}q\rangle}{s} + C_8 \frac{\langle \mathcal{O}_8\rangle}{s^2} + C_{10}\frac{\langle \mathcal{O}_{10}\rangle}{s^3} + \ldots,$$

$$\Pi_1(s) = \frac{-\lambda_N^2}{s - m_N^2}$$

$$\Pi_2(s) = \frac{1}{4\pi^2}\langle \bar{q}q\rangle s \ln(-s/\nu^2) - \frac{1}{12\pi}\frac{\langle \alpha_s G^2 \bar{q}q\rangle}{s} + C_9' \frac{\langle \mathcal{O}_9\rangle}{s^2} + C_{11}' \frac{\langle \mathcal{O}_{11}\rangle}{s^3} + \ldots,$$

$$\Pi_2(s) = \frac{-\lambda_N^2 m_N}{s - m_N^2}$$

FESR (K=1)

$$\lambda_N^2 = \frac{s_0^3}{192\pi^4} + \frac{s_0}{32\pi^3}\langle \alpha_s G^2\rangle + \frac{2}{3}\langle \bar{q}q\bar{q}q\rangle$$

$$\lambda_N^2 m_N = -\frac{s_0^2}{8\pi^2}\langle \bar{q}q\rangle + \frac{1}{12\pi}\langle \alpha_s G^2 \bar{q}q\rangle$$

$\Rightarrow$

$$\lambda_N = 0.017\ \mathrm{GeV}^3$$

$$s_0 = 1.26\ \mathrm{GeV}^2$$

FESR at finite magnetic field

Propagators

$$G(x,y) = e^{iq\phi(x,y)} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \sum_n G^{(n)}(k) B^n \qquad \phi(x,y) = -\frac{1}{2} F_{\mu\nu} x^\mu y^\nu$$

EM fields can be expanded in the same way as Gluon fields $\rightarrow \left(\frac{eB}{s_0}\right)^n$

pions $$D_\pi(p;B) = \frac{i}{p_\parallel^2 - v_\perp \boldsymbol{p}_\perp{}^2 - m_\pi^2} + \dots \qquad v_\perp^2 = \frac{f_\perp}{f_\parallel}$$

fermions

$$S(p) = \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon} - (qB)\frac{i\sigma_{12}(\not{p}_\parallel + m)}{(p^2 - m^2 + i\epsilon)^2} + 2i(qB)^2 \frac{(\not{p}_\parallel + m)\left[p_\perp^2 - \not{p}_\perp(\not{p}_\parallel - m)\right]}{(p^2 - m^2 + i\epsilon)^4}$$
$$- (\kappa B)\frac{i(\not{p}+m)\sigma_{12}(\not{p}+m)}{(p^2 - m^2 + i\epsilon)^2} + \dots$$

$p = p_\parallel + v_\perp p_\perp$ for nucleons

External magnetic field $F_{\mu\nu} = B\epsilon_{0\mu\nu3} \equiv B\boldsymbol{\epsilon}^{\perp}_{\boldsymbol{\mu\nu}}$

Tensorial structures → combination of $p_\mu, \quad g_{\mu\nu}, \quad \epsilon^{\perp}_{\mu\nu}$

$$m_q\langle\bar{q}q\rangle = \langle\bar{q}i\not{D}q\rangle \quad \Rightarrow \quad \langle\bar{q}i\not{D}_{\parallel}q\rangle, \quad \langle\bar{q}i\not{D}_{\perp}q\rangle, \quad \langle\bar{q}i\tilde{\gamma}\cdot D_{\perp}q\rangle$$

$$m_q\langle\bar{q}\sigma_{12}q\rangle = \langle\bar{q}\sigma_{12}i\not{D}q\rangle \quad \Rightarrow \quad \langle\bar{q}\sigma_{12}i\not{D}_{\parallel}q\rangle, \quad \langle\bar{q}\sigma_{12}i\not{D}_{\perp}q\rangle, \quad \langle\bar{q}\sigma_{12}i\tilde{\gamma}\cdot Dq\rangle$$

$$\langle\alpha_s G^2\rangle \quad \Rightarrow \quad \langle\alpha_s(B^2_{\perp} - E^2_{\perp})\rangle, \quad \langle\alpha_s E^2_3\rangle, \quad \langle\alpha_s B^2_3\rangle$$

$$\tilde{\gamma}_\mu = \epsilon^{\perp}_{\mu\nu}\gamma^\nu$$

Axial-Axial correlator and derivatives

Ward identities

$$Q^{\mu}\Pi^{A}_{\mu\nu}(q^2) = \Pi_{5\nu}(q^2) - \Delta_{\nu}(q^2)$$
$$Q^{*\nu}\Pi_{5\nu}(q^2) = \psi_5(q^2) + \Delta_5(q^2),$$

$$Q_{\mu} = q_{\mu} + \frac{ie}{2}F_{\mu\nu}\frac{\partial}{\partial q_{\nu}}$$

$$\Pi^{A}_{\mu\nu}(q) = g^{\parallel}_{\mu\nu}\,\Pi^{\parallel}_{1}(q^2_{\parallel}, q^2_{\perp}) + g^{\perp}_{\mu\nu}\,\Pi^{\perp}_{1}(q^2_{\parallel}, q^2_{\perp}) + i\epsilon^{\perp}_{\mu\nu}\,\tilde{\Pi}_1(q^2_{\parallel}, q^2_{\perp})$$
$$+q^{\parallel}_{\mu}\,q^{\parallel}_{\nu}\,\Pi^{\parallel}_{0}(q^2_{\parallel}, q^2_{\perp}) + \text{ combinations of } \quad q^{\parallel}_{\mu}, \quad q^{\perp}_{\mu}, \quad \epsilon^{\perp}_{\mu\nu}q^{\nu}$$

$$\Pi_{5\mu}(q) = q^{\parallel}_{\mu}\Pi^{\parallel}_{5}(q^2_{\parallel}, q^2_{\perp}) + q^{\perp}_{\mu}\Pi^{\perp}_{5}(q^2_{\parallel}, q^2_{\perp}) + i\epsilon^{\perp}_{\mu\nu}q^{\nu}\tilde{\Pi}_5(q^2_{\parallel}, q^2_{\perp})$$

we set $q_{\perp} = 0$

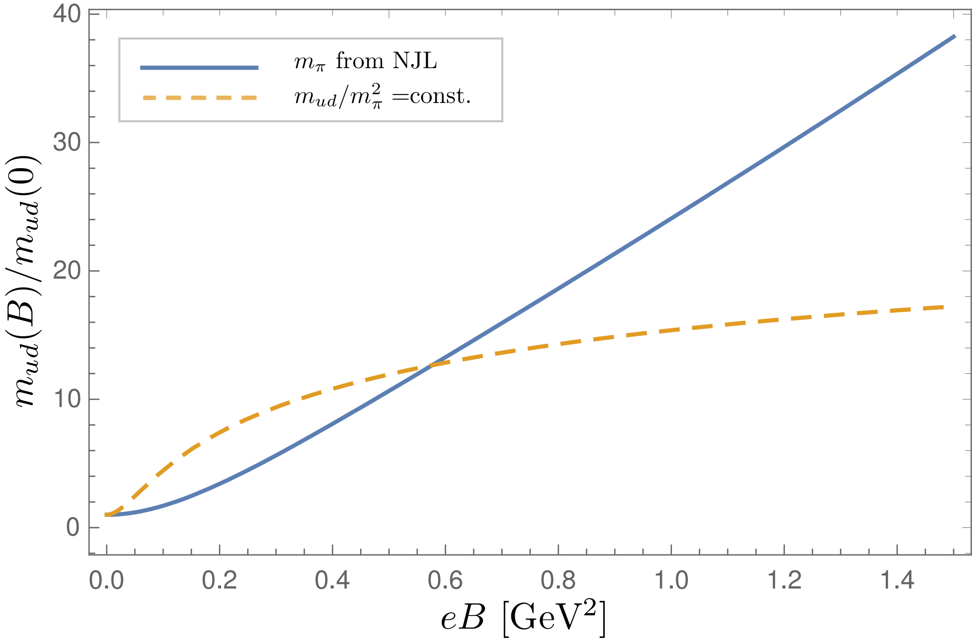
Inputs

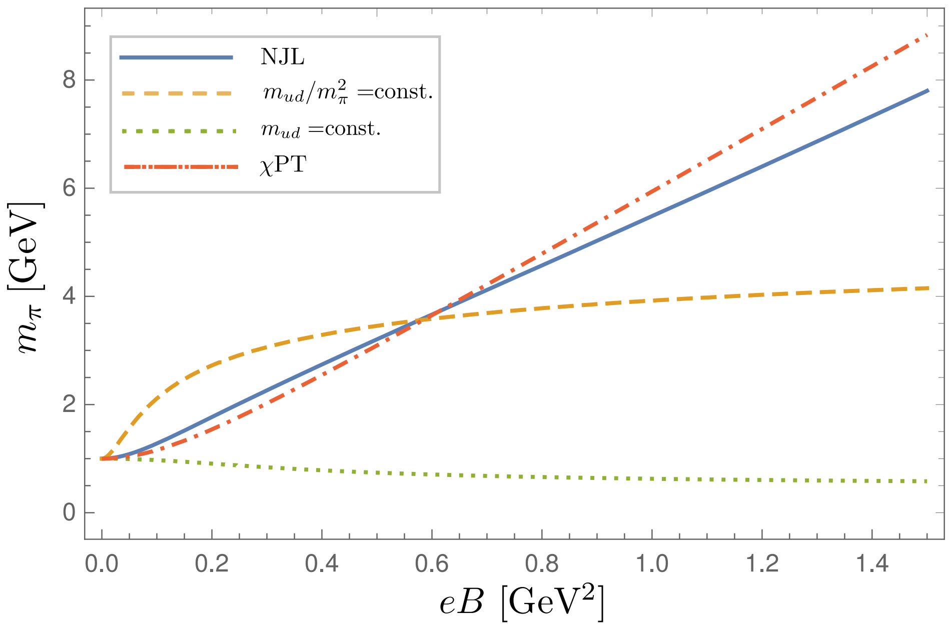
1. Chiral condensate from LQCD or NJL $\langle \bar{u}u + \bar{d}d \rangle (B)$

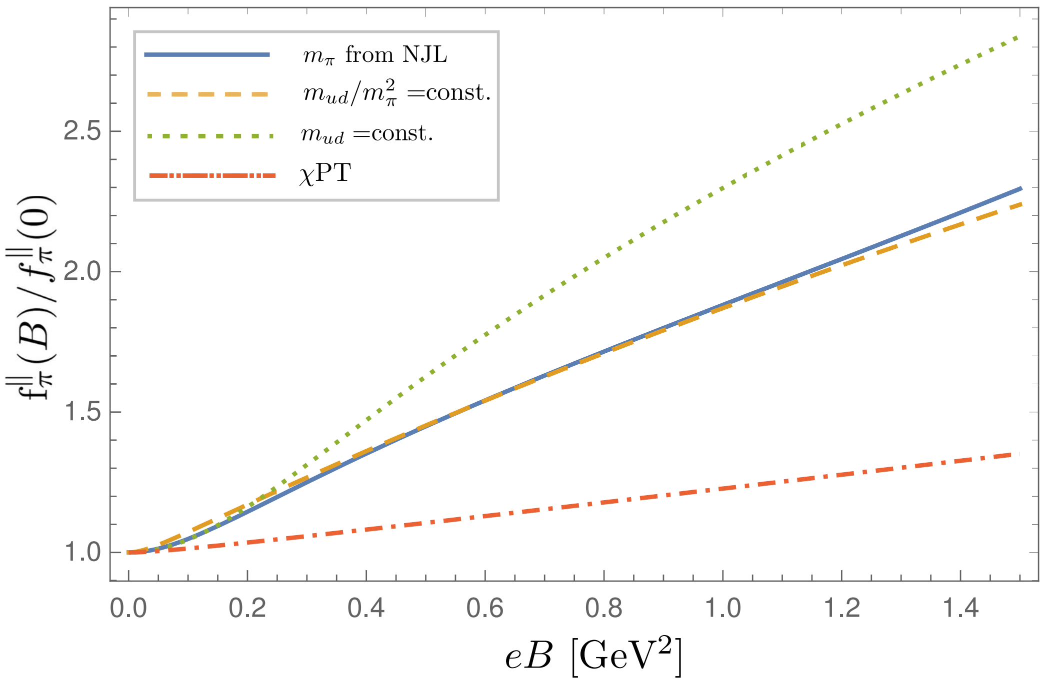
2. A. Pion mass from NJL

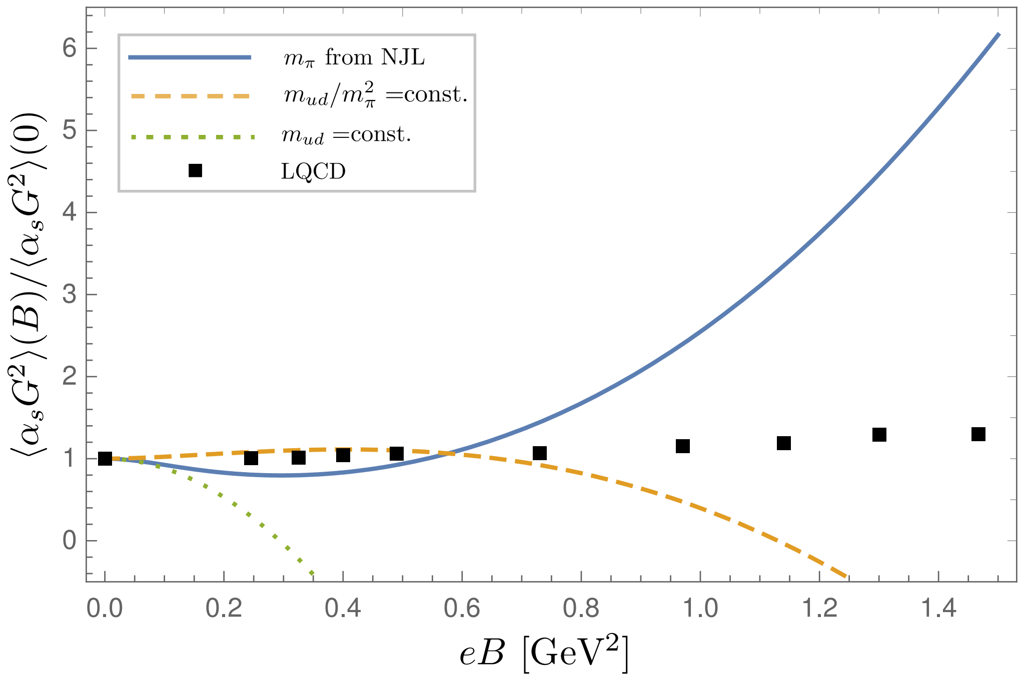
   B. Condition $m_q(B)/m_\pi^2(B) = \text{const.}$

   C. Condition $m_q = \text{const}$









$$\frac{s_0(B)}{s_0(0)} = \frac{f_\pi^{\|}(B)}{f_\pi^{\|}(0)}$$

LQCD D’Elia, Meggiolaro, Mesiti, Negro, Phys. Rev. D 93, 054017 (2016).

Decreasing <GG> Agasian, Shushpanov, Phys. Lett. B 472, 143 (2000).

Nucleon-nucleon correlator

$$\Pi = \boxed{\Pi_S} + i\gamma_5 \cancel{\Pi_P} + \gamma_\mu \Pi_V^\mu + \gamma_\mu \gamma_5 \Pi_A^\mu + \sigma_{\mu\nu} \Pi_T^{\mu\nu}$$

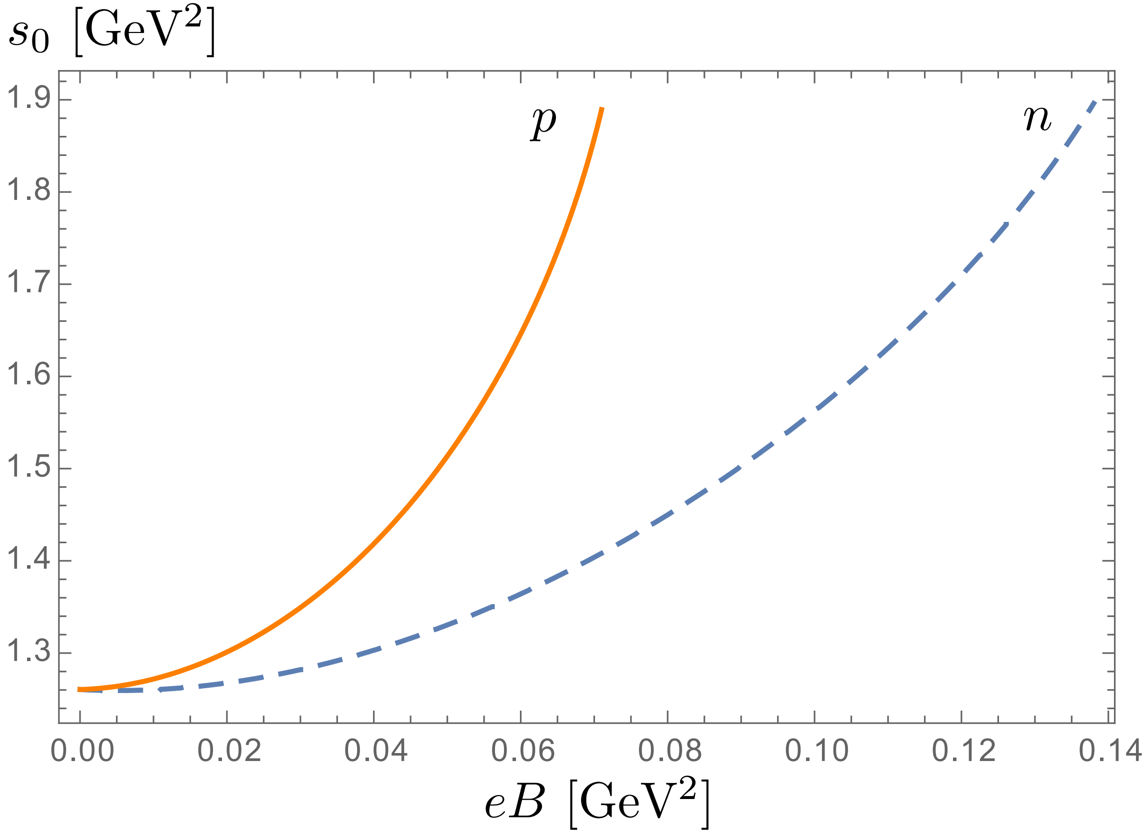
$$\Pi_V^\mu = p_\parallel^\mu \boxed{\Pi_V^\parallel} + p_\perp^\mu \boxed{\Pi_V^\perp} + \tilde{p}_\perp^\mu \tilde{\Pi}_V^\perp$$

$$\Pi_A^\mu = \tilde{p}_\parallel^\mu \Pi_A$$

$$\tilde{p}_\perp^\mu \equiv \epsilon_\perp^{\mu\alpha} p_\alpha$$

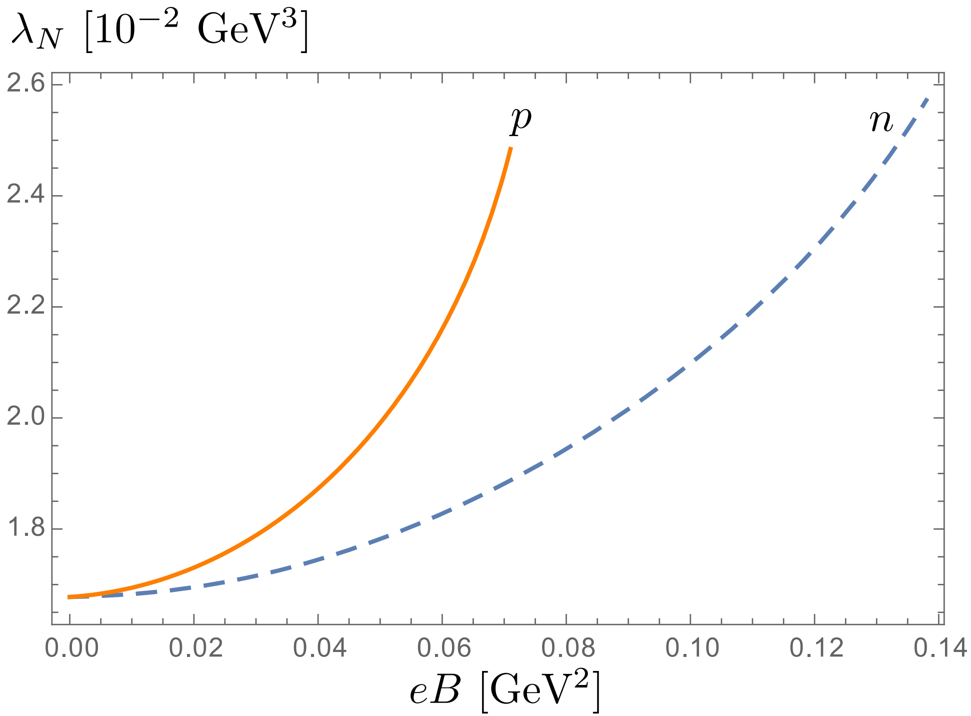
$$\tilde{p}_\parallel^\mu \equiv \epsilon_\parallel^{\mu\alpha} p_\alpha$$

$$\Pi_T^{\mu\nu} = \epsilon_\perp^{\mu\nu} \boxed{\Pi_T^\perp} + (p_\parallel^\mu p_\perp^\nu - p_\parallel^\nu p_\perp^\mu) \Pi_T^{\parallel\perp} + (p_\parallel^\mu \tilde{p}_\perp^\nu - p_\parallel^\nu \tilde{p}_\perp^\mu) \tilde{\Pi}_T^{\parallel\perp},$$

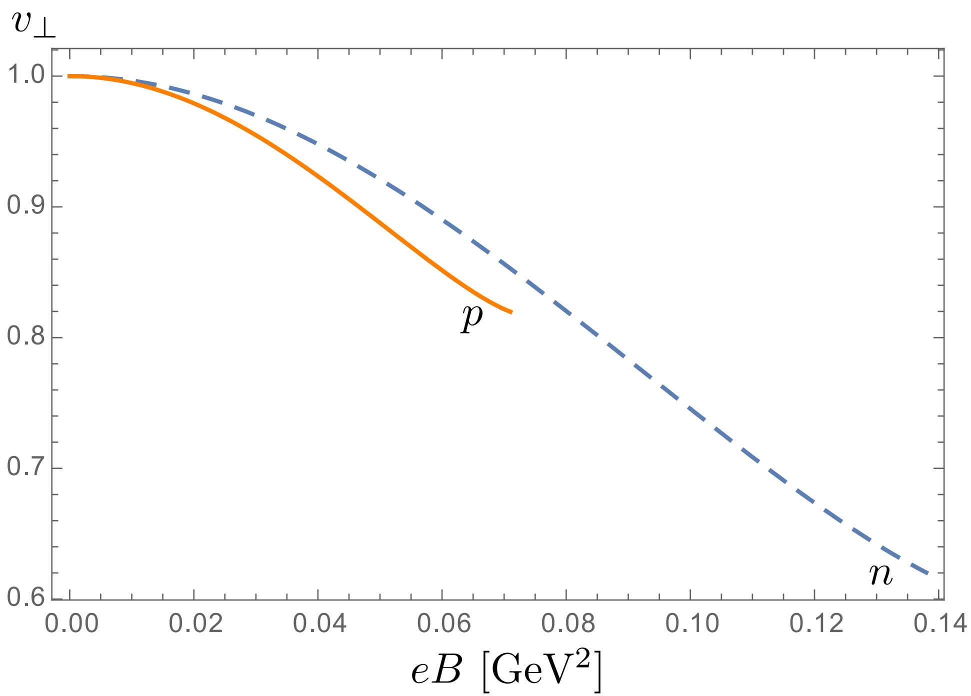


Inputs:

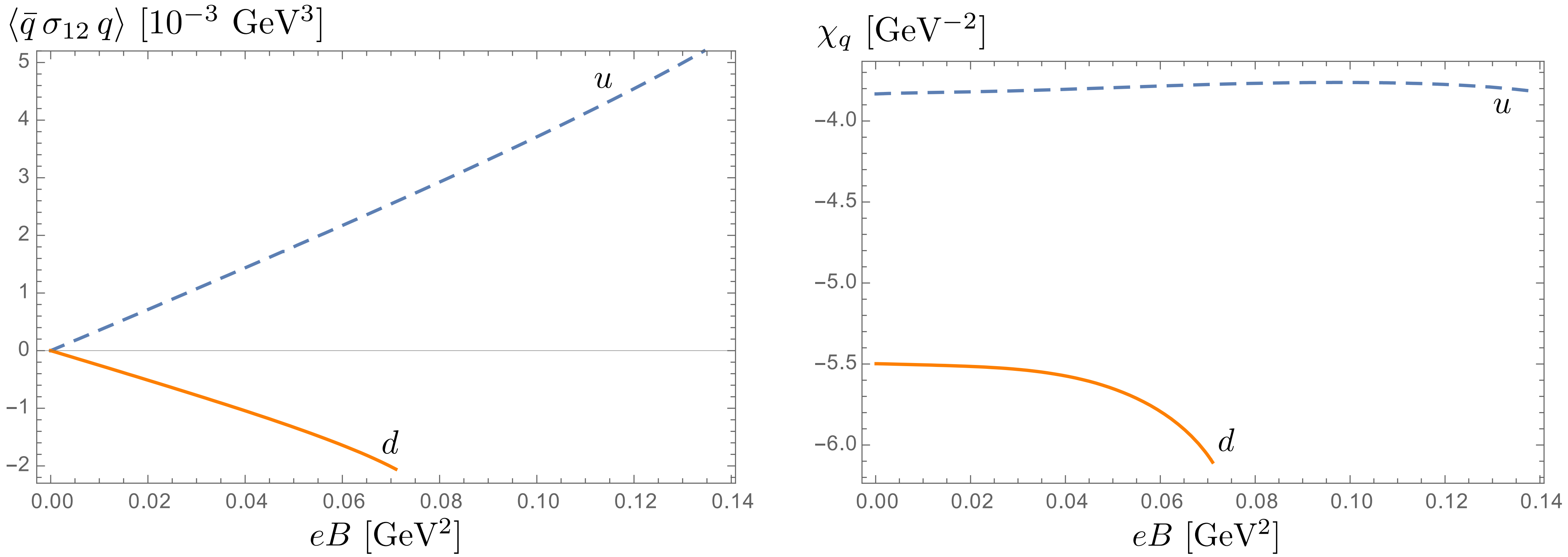
nucleon masses (B)
quark masses (B)
<GG> ~ constant



$\lambda_N\ [10^{-2}\ \mathrm{GeV}^3]$
2.6
2.4
2.2
2.0
1.8
p
n
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
$eB\ [\mathrm{GeV}^2]$



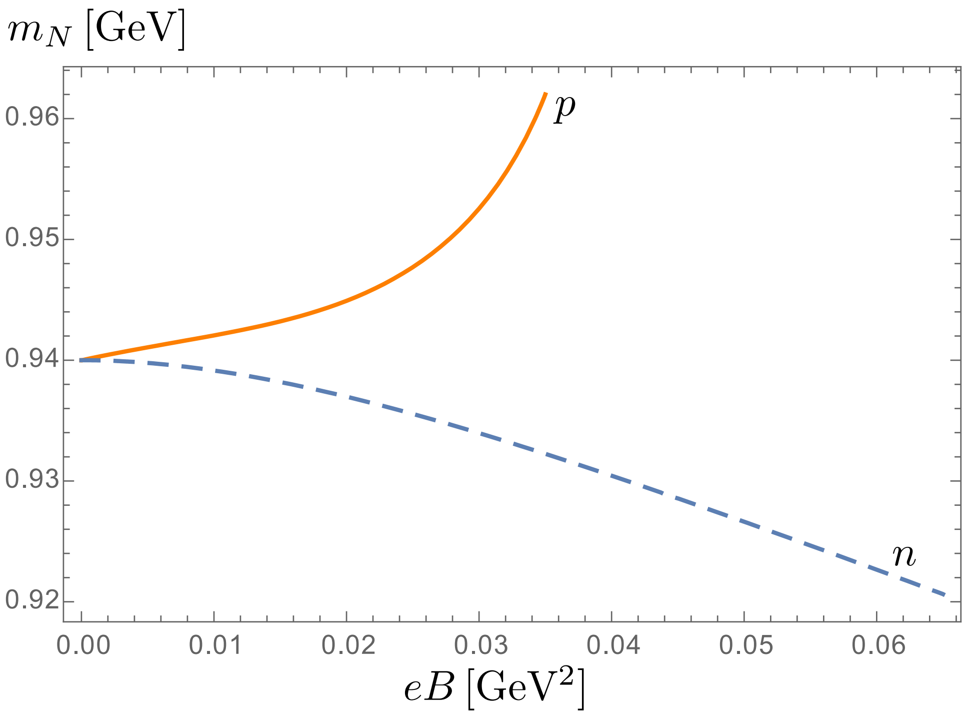
$v_\perp$
1.0
0.9
0.8
0.7
0.6
p
n
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
$eB\ [\mathrm{GeV}^2]$

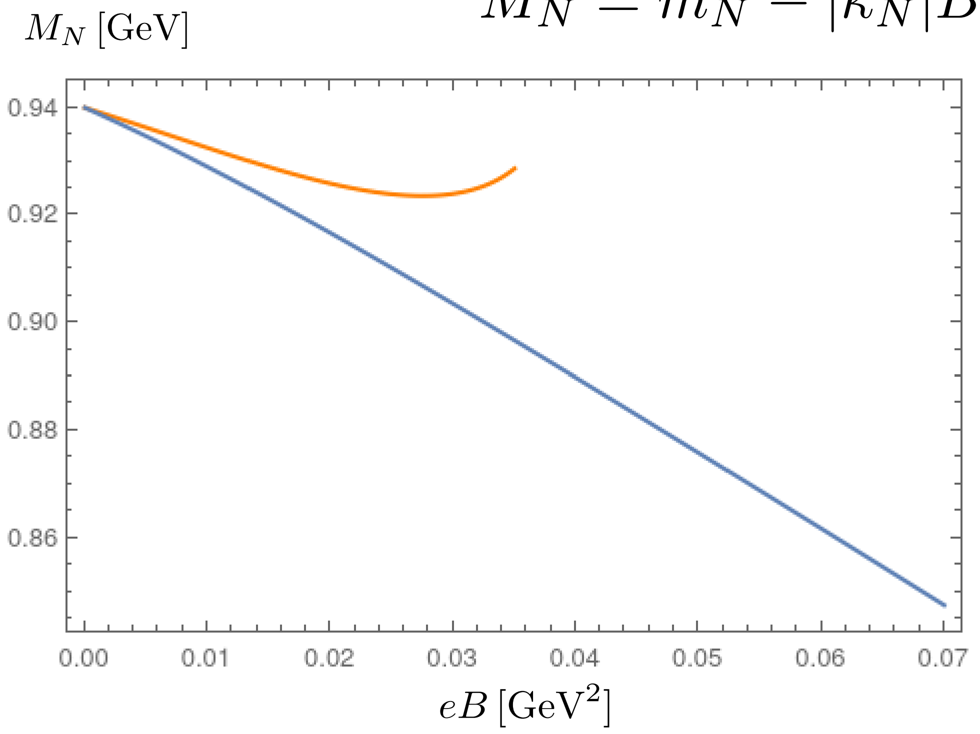


$$\chi_q = \langle \bar{q}\sigma_{12}q\rangle / e_q B \langle \bar{q}q\rangle$$

assuming $\chi_q = \text{constant}$ → $\langle \bar{q}\sigma_{12}q \rangle$ input

$$M_N = m_N - |\kappa_N| B$$





Nucleon - Axial – nucleon correlator

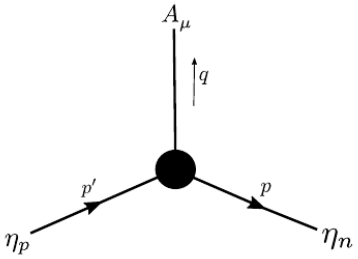
$$\Pi_\mu(x,y,z) = -\langle 0|\,\mathcal{T}\,\eta_p(x)\,A_\mu(y)\,\bar{\eta}_n(z)\,|0\rangle$$

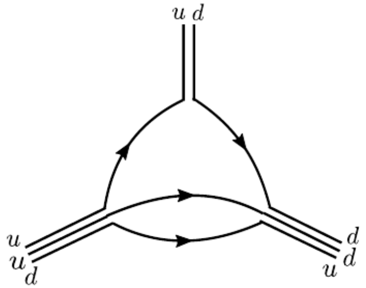
$$\langle p',s'|A_\mu(y)|p,s\rangle = \bar{u}_p^{s'}(p')\,T_\mu(q)\,u_n^s(p)\,e^{iq\cdot y}$$

$$T_\mu(q) = G_A(t)\gamma_\mu\gamma_5 + G_P(t)\gamma_5\frac{q_\mu}{2m_N} + G_T(t)\sigma_{\mu\nu}\gamma_5\frac{q_\nu}{2m_N}$$

$$t = (p'-p)^2$$

$$g_A = G_A(0)$$





$$\Pi_\mu(p,p') = \int d^4y\, d^4z\, e^{-i(q\cdot y + p\cdot z)}\, \Pi_\mu(0,y,z) \quad \Rightarrow \quad \Pi_\mu^{\rm had}(p,p') = \lambda_n\lambda_p \frac{(\not p + m_n) T_\mu(q)(\not p' + m_p)}{(p^2 - m_n^2)(p'^2 - m_p^2)}$$

$$\mathrm{tr}\,[\Pi_\mu(p,p')\,\gamma_\nu] = -4i\epsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta \Pi(s,s',t) \quad \Rightarrow \quad \Pi^{\rm had}(s,s',t) = \lambda_n\lambda_p \frac{G_A(t) + G_T(t)(m_n - m_p)/m_N}{(s - m_n^2)(s' - m_p^2)}$$

$$\Pi^{\rm pQCD}(s,s',0) = \frac{s^2 \ln(-s/\Lambda^2) - s'^2 \ln(-s'/\Lambda^2)}{(2\pi)^4\,(s' - s)} + \text{regular terms}$$

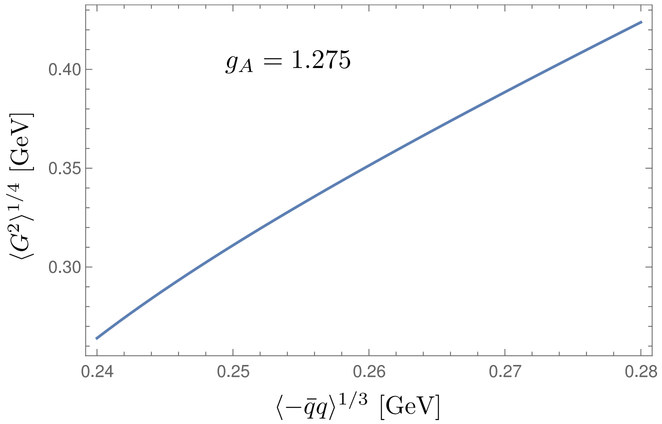
Double FESR

$$\int_0^{s_p} \frac{ds'}{\pi}\, \mathrm{Im}_{s'} \int_0^{s_n} \frac{ds}{\pi}\, \mathrm{Im}_s \Pi^{\rm had}(s,s',t) = \oint_{s_p} \frac{ds'}{2\pi i} \oint_{s_n} \frac{ds}{2\pi i}\, \Pi^{\rm QCD}(s,s',t)$$

$$\Rightarrow \quad g_A \lambda_n \lambda_p\, \theta(s_n - m_n^2)\theta(s_p - m_p^2) = \frac{1}{48\pi^4}\left[s_n^3\, \theta(s_p - s_n) + s_p^3\, \theta(s_n - s_p)\right].$$

$$g_A = \frac{1}{48\pi^4} \frac{s_0^3}{\lambda_N^2}$$

Obtaining λ_N and s_0 from NN-correlator

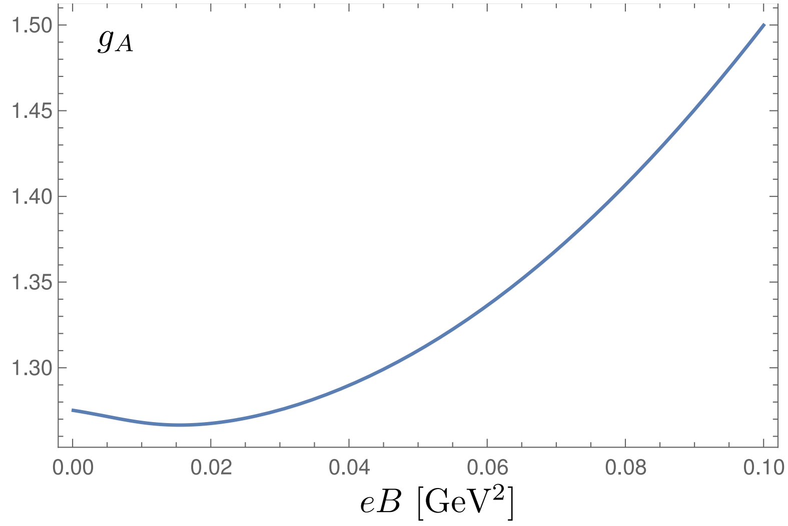


Finite B ➡ $\Pi_\mu(p', p) = -S_p^B(p')T_\mu(q)S_n(p)$

$$G_A\gamma_\mu \to G_A^{\|}\gamma_\mu^{\|} + G_A^{\perp}\gamma_\mu^{\perp} + \tilde{G}_A F_{\mu\nu}\gamma^\nu,$$

$$g_A = \frac{1}{48\pi^4}\frac{s_n^3}{\lambda_p\lambda_n}$$

Lowest order approximation ➡



Summary and outlook

- Charged pions, nucleon and QCD parameters → reasonable results
- Hadronic threshold rises → B is confinant

- Include new condensates and correlators → full description
- B-dependent non-normal ordered condensates
- Temperature and density

Summary and outlook

- Charged pions, nucleon and QCD parameters → reasonable results
- Hadronic threshold rises → B is confinant

- Include new condensates and correlators → full description
- B-dependent non-normal ordered condensates
- Temperature and density

OBRIGADO!