Lecture 2 General Intro to Topological Mechanics (finit ω) I

1. Basic concepts

- Topology : many definitions, basically, it studies

  properties of a geometric object that are preserved under continuous deformations

← number of handles (1) remain the same

- Topological states :

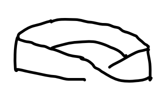
  Phases of matter characterized by nontrivial topologies

  Many categories.

  For what we are concerned with :

$\boxed{\text{State of the system e.g. } \psi(\vec{q})} \xleftrightarrow{\text{map}} \boxed{\text{Nontrial topological space}}$

e.g.



Mobius strip

Mathematical language natural for this mapping :

Fiber bundle theory

2. Vector bundle theory in 10 min

(disclaimer : ... as an average physicist)

- Vector bundle :

$$V = \{ V_p \mid p \in B \}$$

a point   base space

a vector space defined at point p

$\longrightarrow$ a "fiber" at $p \in B$

B must be a smooth geometric object

V_p varies smoothly on B

- Section $\vec{v}$: a smooth choice of $\vec{v}(p)$ on a region $p \in \Sigma \subset B$ (local)

If $\vec{v}(p)$ can be defined for all $p \in B$

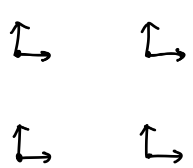
$\longrightarrow$ global section.

(often requires $\vec{v} \neq 0$)

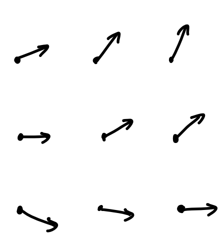
Simple examples :

① $T_p \mathbb{R}^2$

tangent bundle   flat plane



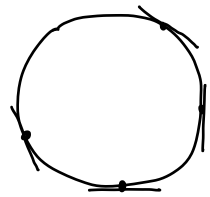
a section



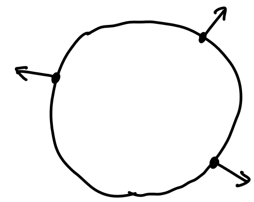
2D tangent space at each point p

② Bundles on S^1

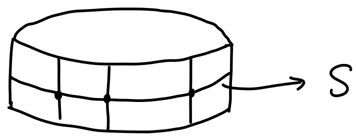
tangent:



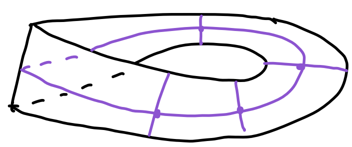
normal:



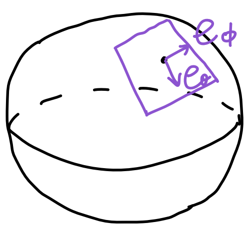
③ Line bundle on a cylinder



④ Line bundle on a Mobius strip



⑤ $T_p S^2$ $\quad (T_p \mathbb{R}^3|_{S^2})$



span a 2D plane

Is there a global section for each of these bundles?

"Obstruction" of defining a global section $\longleftrightarrow$ Nontrivial topological states or more specifically

$\psi(q) \to \psi(q+\delta q)$

"Connection"

"Curvature"

from one fiber to the next one

$\longleftrightarrow$

Berry connection

Berry curvature

wave function at neighboring points in $\vec{k}$ space

$\longleftrightarrow$

$\vec{A}$

$\vec{B} = \vec{\nabla} \times \vec{A}$

gauge field via minimal coupling

Integrate

genus of the surface

$$1-g = \frac{1}{4\pi} \int K \, ds$$

$\longleftrightarrow$

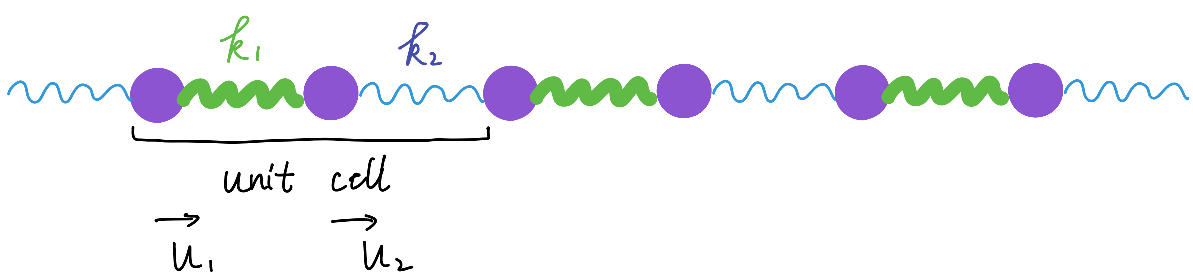
Chern number

$$C = \frac{1}{2\pi} \int_{BZ} f \, dk_x dk_y$$

obstruction to find a global gauge

3. Example : 1D mechanical chain (finite ω SSH chain)

(1) Bulk states



EOM in k space :

$$\begin{pmatrix} k_1+k_2 & -k_1-k_2 e^{-ika} \\ -k_1-k_2 e^{+ika} & k_1+k_2 \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} = m\omega^2 \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix}$$

Let $K = k_1 + k_2$

$\quad z = k_1 + k_2 e^{ika}$

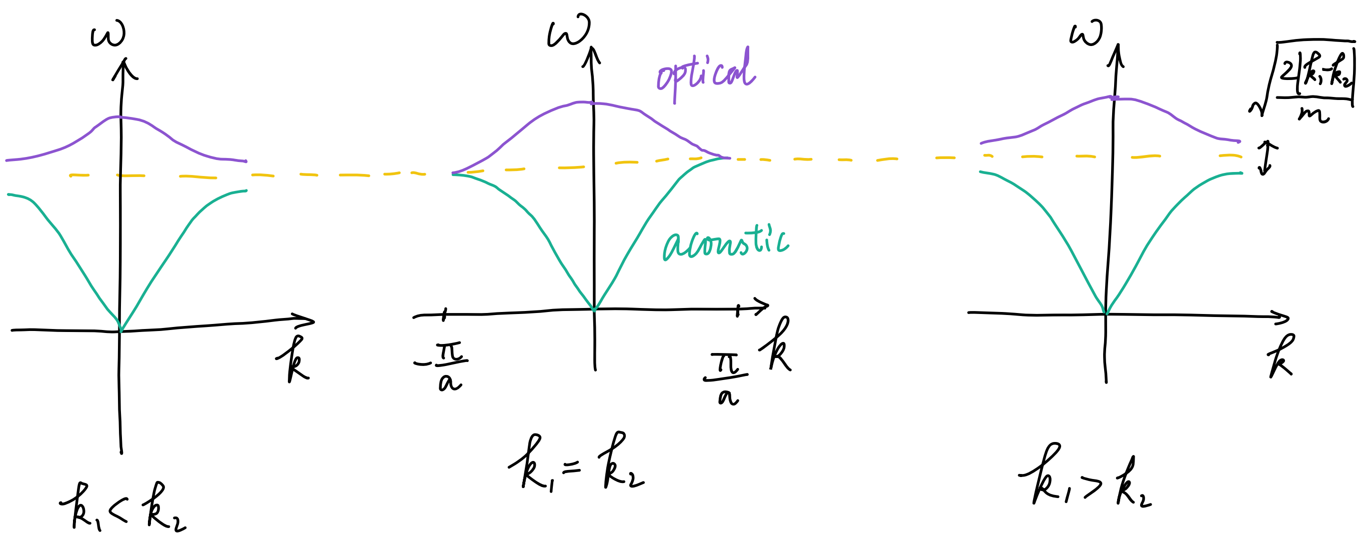
$\rightarrow$ eigenvalues :

$$(K - m\omega^2)^2 - |z|^2 = 0$$

$$K - m\omega^2 = \pm |z|$$

$$m\omega^2 = K \pm |z| = k_1 + k_2 \pm \left| k_1 + k_2 e^{ika} \right|$$

$$= k_1 + k_2 \pm \sqrt{(k_1+k_2)^2 - 4k_1 k_2 \sin^2\left(\frac{ka}{2}\right)}$$



$\rightarrow$ eigenmodes :

for $\omega = \omega_\pm$, let eigenmode $V_\pm = \begin{pmatrix} 1 \\ x_\pm \end{pmatrix}$

$$\begin{pmatrix} K & -\bar{z} \\ -z & K \end{pmatrix} \begin{pmatrix} 1 \\ x_\pm \end{pmatrix} = m\omega_\pm^2 \begin{pmatrix} 1 \\ x_\pm \end{pmatrix} \qquad (\text{here } K = k_1 + k_2)$$

$$K - \bar{z} x_\pm = m\omega_\pm^2 = K \pm |z|$$

$$\therefore \quad x_\pm = \mp \frac{|z|}{\bar{z}} = \mp \frac{z}{|z|}$$

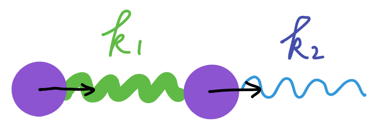
$\therefore$ optical (ω_+) : $\quad v_+(k) = \left(1, \frac{z}{|z|}\right) = \left(1, -\frac{k_1 + k_2 e^{ika}}{|k_1 + k_2 e^{ika}|}\right)$

acoustic (ω_-) : $\quad v_-(k) = \left(1, -\frac{z}{|z|}\right) = \left(1, +\frac{k_1 + k_2 e^{ika}}{|k_1 + k_2 e^{ika}|}\right)$

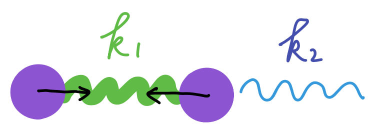
at $k = \pm\frac{\pi}{a}$, $\quad v_\pm\left(\frac{\pi}{a}\right) = \left(1, \mp\frac{k_1 - k_2}{|k_1 - k_2|}\right)$

- If $k_1 < k_2$,

v_+

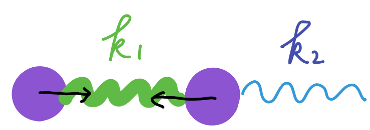


v_-

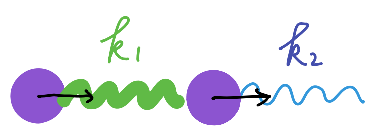


- If $k_1 > k_2$,

v_+

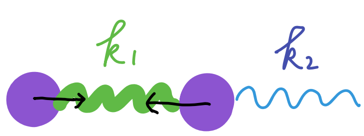


v_-



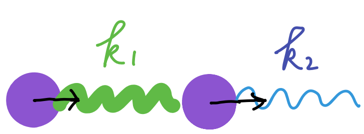
However, for both bands, at $k \to 0$:

v_+



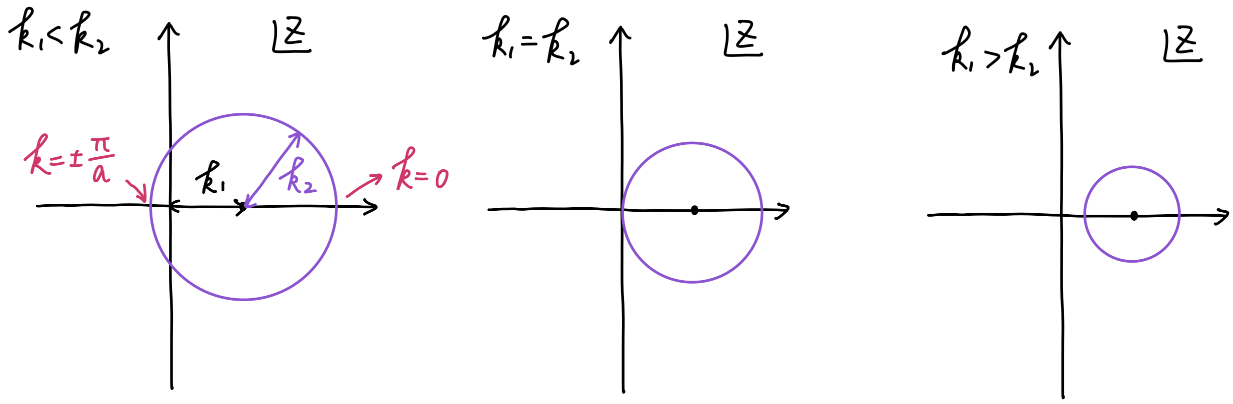
What happened for $k_1 < k_2$ case ?

v_-



"band inversion"

Complex plane of $z = k_1 + k_2 e^{ika}$



From $k = -\frac{\pi}{a}$ to $\frac{\pi}{a}$

$\Delta \operatorname{Arg}(z) = 2\pi$ $\qquad$ $\Delta \operatorname{Arg}(z)$ not well defined $\qquad$ $\Delta \operatorname{Arg}(z) = 0$

$\downarrow$

$$\boxed{\text{"winding number"} \quad \nu = 1}$$

More general definition of the winding number

"Zak phase" $\quad \theta = \operatorname{Im} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \underbrace{\langle \phi(q), \partial_q \phi(q) \rangle}_{\text{"Berry connection"}} dq$

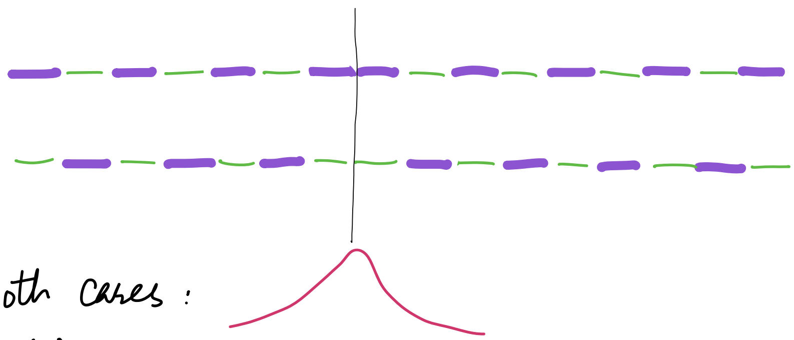
$$= \begin{cases} \pi, & k_1 < k_2 \\ 0, & k_1 > k_2 \end{cases}$$

Winding number $\nu = \frac{\theta}{\pi}$

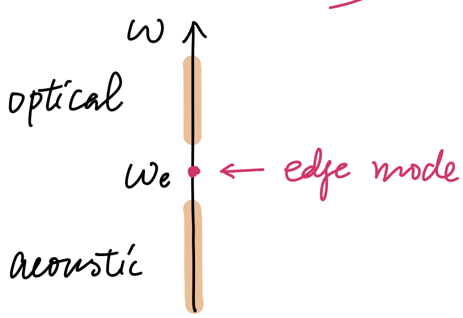
What's the consequence?

(2) Edge states

Consider interfaces : if calculate modes numerically



Both cases :



exponential localization

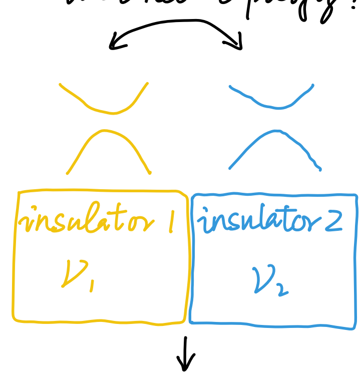
< dynamical analysis from L1 : signal at ω_e will only excite vibrations at interface >

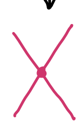
Does this edge state have anything to do with the topological winding number ?

"Bulk-edge correspondence" : many ways to prove

- General picture

distinct topology : no way to continuously connect





gap has to close at the interface : edge states

- Simple proof for this 1D chain : see ref.

  number of edge modes at interface $= |\nu_1 - \nu_2|$