1. **ID chain → general 2-band models**

   For the ID chain we had

   $$ D(\mathbf{k}) = \begin{pmatrix} k_1 + k_2 & -k_1 - k_2 e^{-i\mathbf{k} \cdot \mathbf{a}} \\ -k_1 - k_2 e^{i\mathbf{k} \cdot \mathbf{a}} & k_1 + k_2 \end{pmatrix} $$

   All hermitian $2 \times 2$ matrices can be written as

   $$ D(\mathbf{k}) = d_0(\mathbf{k}) \hat{I} + dx(\mathbf{k}) \sigma_x + dy(\mathbf{k}) \sigma_y + dz(\mathbf{k}) \sigma_z $$

   where

   $$ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} $$

   Here:

   $d_0 = k_1 + k_2$  \quad ← unimportant, shift

   $dx = -k_1 - k_2 \cos \mathbf{k} \cdot \mathbf{a}$

   $dy = -k_2 \sin \mathbf{k} \cdot \mathbf{a}$

   $dz = 0$

   **This form is sometimes called a “Dirac Hamiltonian”**

   *(only applicable to 2-band models)*

   Why this representation is useful: *general formula for Berry curvature*.

2. **The Berry phase**
Remember we defined Berry connection
\[
\vec{A} = i \langle \Psi(\vec{x}) | \nabla_{\vec{x}} | \Psi(\vec{x}) \rangle
\]

The Berry phase: the contour integral of \( \vec{A} \)
\[
\gamma = \oint \vec{d}\vec{x} \cdot \vec{A}(\vec{x}) \quad \text{(Phase winding of } \Psi(\vec{x}) \text{ over a closed path calculated for the 1D chain is an example of } \gamma \text{.)}
\]

Use Stoke's formula
\[
\gamma = \oint \vec{d}\vec{x} \cdot \vec{A}(\vec{x}) = \int d^2\vec{x} \cdot (\nabla \times \vec{A})
\]
\( \vec{B} \): Berry curvature

We can do this in \( \mathbb{R}^2 \) space
\[
\vec{A}(\vec{x}) = i \langle \Psi(\vec{x}) | \nabla_{\vec{x}} | \Psi(\vec{x}) \rangle
\]

With a little more math, a nice formula \( \Rightarrow \)
\[
\vec{B}(\vec{x}) = \pm \frac{\vec{d}}{2|\vec{d}|^3} \quad : \text{"magnetic monopole"}
\]

\( \Rightarrow \)
\[
\gamma = \int_{S^2} d^2\vec{x} \cdot \vec{B}(\vec{x}) = \frac{1}{2} \Omega(S)
\]

solid angle on \( S^2 \)
Physics picture of $\gamma \neq 0$:

Obstruction of finding a global gauge on $S_2$

Complex phase of $\Psi(\mathcal{J})$ → tangent bundle on $S_2$

If there is a global gauge → $\gamma = 0$

Big picture:

Closed path in $\mathcal{J}$ space (IBZ)

\[ \gamma = \int d^2 \mathcal{J} \cdot B(\mathcal{J}) = \frac{1}{2} \Omega(S) \]

\[ \approx \frac{1}{2} \int d\mathcal{J} \cdot \mathcal{A}(\mathcal{J}) \]

(Jacobian cancels)

\[ \gamma x = \int d\mathcal{R} \cdot \mathcal{A}(\mathcal{R}) \]

In the case of the 1D chain model

\[ d_2 = 0 \quad \Rightarrow \quad \mathcal{A}(\mathcal{R}) \text{ always on equator} \]
Note: this topological winding is "protected by symmetry"

\[ \gamma = 0 \rightarrow \nu = 0 \]

\[ \gamma = \pi \rightarrow \nu = 1 \]

If we allow \( d_z \neq 0 \)

This is an example of a "symmetry protected topological state"

Disorder that breaks this symmetry (e.g. \( m_z \neq m_x \)) will destroy the topological protection of the edge states.
The case of 2D lattices:

What controls the edge states is the “Chern number”

\[ C = \frac{1}{2\pi} \int_{2D \text{ BZ}} d^2 \mathbf{k} \text{B} (\mathbf{k}) \]

= degree of map \( \mathbf{k} \rightarrow \mathbf{\hat{d}(k)} \)

How many different ways (equivalence class) can we wrap \( T^2 \) onto \( S^2 \)?

This is described by “homotopy theory.”

In this case → all integers \( \mathbb{Z} \)

\( C \) of 2D systems: integers.
Maxwell lattice topological mechanics and topological soft modes in disordered materials

Xiaoming Mao
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Outline

• Recap from Lectures 2 & 3
• Selected examples of $\omega > 0$ TMMs (topological mechanical metamaterials)
• Maxwell lattice TMMs
  – Fundamental theories (recap from Tom’s lectures)
  – Selected new advances
• Aperiodic Maxwell network topological mechanics
  – Fiber networks
  – Reciprocal diagrams and a mechanical duality theorem
  – Quasicrystals
Recap from Lectures 2&3

• Basic concepts of topology and fiber bundle models
  
• 1D mechanical chain
  - Bulk spectra, gap
  - Band inversion and winding number
  - Edge states

• General 2-band models
  - Dirac Hamiltonian
  - The Berry phase and the Chern number

Main references:
• Topological insulators and geometry of vector bundles, Sergeev, arXiv:2011.05004 [cond-mat.mes-hall]

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Topological mechanics of gyroscopic metamaterials

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Topological Phononic Crystals with One-Way Elastic Edge Waves

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EOM of one gyroscope ($\vec{\ell}$ is the principal axis of gyroscope):

$$\dot{\vec{\ell}} = \frac{\ell^2}{I\omega} (\vec{\ell} \times \vec{F})$$

Complex notation:

- Site displacements: $\psi \rightarrow \delta_x + i\delta_y$
- Site forces: $F \rightarrow F_x + iF_y$

EOM for lattice site $p$:

$$i \frac{d\psi_p}{dt} = \Omega_g \psi_p + \frac{\Omega_k}{2} \sum_{q}^{nn} \left( (\psi_p - \psi_q) + e^{2i\theta_{pq}} (\psi_p^* - \psi_q^*) \right)$$

4*4 Hamiltonian $H(k)$

- 2 middle bands
- Gap permitted by broken time-reversal symmetry

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2D Valley Hall TMMs: without breaking TRS

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Observation of topological valley modes in an elastic hexagonal lattice

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(Received 14 May 2017; revised manuscript received 29 August 2017; published 17 October 2017)

Dirac cones open when $\gamma \neq 0$
AAH (Aubry-Andre-Harper) Model: Concept

Electron tight-binding model with a periodic potential

\[-t \psi_{n+1} - t \psi_{n-1} - E_0 \cos(n\theta + \phi)\psi_n = E\psi_n\]

Maps to 2D quantum Hall states on a lattice

\[g(m + 1) + g(m - 1) + 2 \cos(2\pi m \alpha - \nu)g(m) = \epsilon g(m)\]

Momentum in y

Magnetic field \(\alpha = a^2H / 2\pi(\hbar c/e)\)

“Hofstadter butterfly”:
AAH Model: Adiabatic Pumping

- Adiabatic pumping in AAH model \(-t\psi_{n+1} - t\psi_{n-1} - E_0 \cos(n\theta + \phi)\psi_n = E\psi_n\)

![Diagram showing adiabatic pumping in AAH model with associated equations and plots.

Cut at a given \(\theta\)

\(\alpha \leftrightarrow \theta\)

gapless edge states at the right edge

gapless edge states at the left edge

bulk modes

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Experimental Demonstration of Dynamic Topological Pumping across Incommensurate Bilayered Acoustic Metamaterials

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More TMMs

• TMM is an explosive field. I’ve left out $10^2 \sim 10^3$ papers
• I only showed you a tiny tiny fraction of all topological states
• Good reviews on topological mechanics:
Maxwell lattice TMMs

• Maxwell lattices
  – DOF = constraints in the bulk
  – $C(q)$ and $Q(q)$ matrices are square

$$\mathcal{H} = \begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix}; \quad \mathcal{H}^2 = \begin{pmatrix} QQ^T & 0 \\ 0 & Q^TQ \end{pmatrix}$$

Eigenstates: $\psi = \begin{pmatrix} t \\ u \end{pmatrix}$
  - Bond tensions
  - Site displacements

Chiral (particle-hole) symmetry: $\{\mathcal{H}, \tau^z\} = 0$

Time-reversal symmetry: $\mathcal{H} = \mathcal{H}^*$

Class BDI: topological invariant in 1D

$C \cdot \vec{u} = 0$
$Q \cdot \vec{t} = 0$

Kane-Lubensky topological index for Maxwell lattices

Corresponding edge modes: ZMs and SSSs

Maxwell-Calladine index theorem

$n_i = \frac{1}{2 \pi i} \oint_{C_i} dk \cdot \text{Tr} \left[ Q(k)^{-1} \nabla_k Q(k) \right] = \frac{1}{2 \pi} \oint_{C_i} dk \cdot \nabla_k \phi(k)$

$n \in \mathbb{Z}$

$$N_0 - N_S = N_d - N_b = 0$$

• C. L. Kane and T. C. Lubensky, Nat. Phys. 10, 39 (2014).

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