Lecture 3 General Intro to Topological Mechanics (finite ω) II

1. 1D chain $\rightarrow$ general 2-band models

For the 1D chain we had

$$D(\vec{k}) = \begin{pmatrix} k_1 + k_2 & -k_1 - k_2 e^{-ika} \\ -k_1 - k_2 e^{+ika} & k_1 + k_2 \end{pmatrix}$$

All hermitian 2×2 matrices can be written as

$$D(k) = d_0(k)\,\underline{\underline{I}} + d_x(k)\,\underline{\underline{\sigma}}_x + d_y(k)\,\underline{\underline{\sigma}}_y + d_z(k)\,\underline{\underline{\sigma}}_z$$

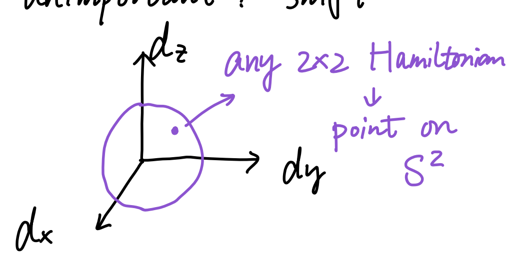
where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

here: $d_0 = k_1 + k_2$ $\leftarrow$ unimportant, shift

$d_x = -k_1 - k_2 \cos ka$

$d_y = -k_2 \sin ka$

$d_z = 0$



This form is sometimes called a "Dirac Hamiltonian"

(only applicable to 2-band models)

Why this representation is useful: general formula for Berry curvature.

2. The Berry phase

Remember we defined Berry connection

$$\vec{A} = i \langle \psi(q) | \vec{\nabla}_q | \psi(q) \rangle$$

The Berry phase : the contour integral of $\vec{A}$

$$\gamma = \oint d\vec{q} \cdot \vec{A}(q)$$

(Phase winding of $\psi(\vec{q})$ over a closed path
θ calculated for the 1D chain
is an example of γ)

Use Stoke's formula

$$\gamma = \oint d\vec{q} \cdot \vec{A}(q) = \int d^2\vec{q} \cdot \underbrace{(\vec{\nabla} \times \vec{A})}_{\equiv \vec{B}}$$

$\vec{B}$: Berry curvature

We can do this in $\vec{d}(\vec{k})$ space

map

$$\vec{A}(\vec{d}) = i \langle \psi(\vec{d}) | \nabla_{\vec{d}} \psi(\vec{d}) \rangle$$

With a little more math. a nice formula →

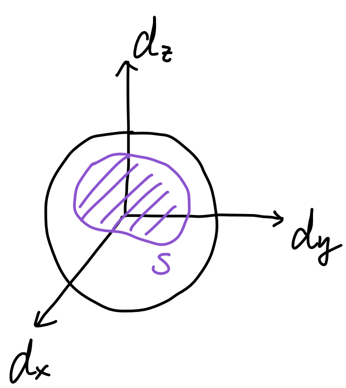
$$\vec{B}(d) = \pm \frac{\vec{d}}{2|d|^3}$$

: "magnetic monopole"

+ upper band
− lower band

⇒ $$\gamma = \int_S d^2\vec{d} \cdot \vec{B}(\vec{d}) = \frac{1}{2}\Omega(S)$$

solid angle on S^2



Physics picture of $\gamma \neq 0$:

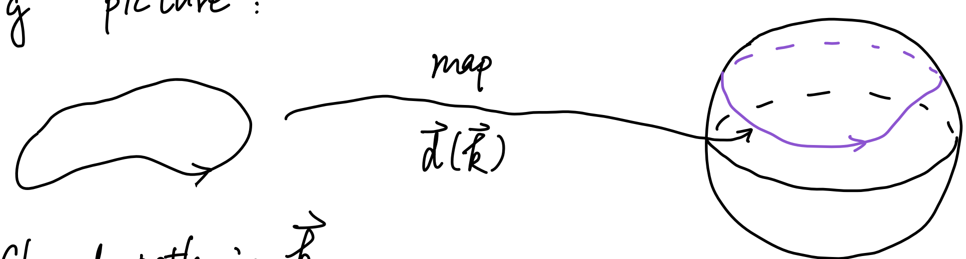
Obstruction of finding a global gauge on S_2

Complex phase of $\psi(\vec{d})$

$\rightarrow$ tangent bundle on S_2

If there is a global gauge $\rightarrow \gamma = 0$

Big picture :



$\vec{d}$ space

Closed path in $\vec{k}$ space (1BZ)

$$\gamma = \int_S d^2\vec{d} \cdot \vec{B}(\vec{d}) = \frac{1}{2}\Omega(S)$$

$$\| $$

$$\int d\vec{d} \cdot \vec{A}(\vec{d})$$

$$\| \qquad \text{(Jacobian cancels)}$$

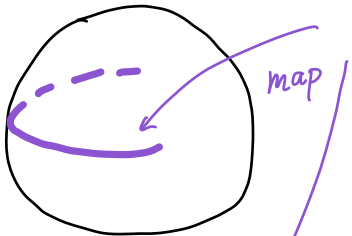
$$\gamma_k = \int d\vec{k} \cdot A(\vec{k})$$

- In the case of the 1D chain model

$\because d_z = 0 \quad \therefore \vec{d}(\vec{k})$ always on equator

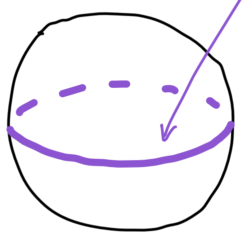
IBZ : $k = -\frac{\pi}{a}$ to $\frac{\pi}{a}$

$k_1 > k_2$



$\gamma = 0 \rightarrow \nu = 0$

$k_1 < k_2$



$\gamma = \pi \rightarrow \nu = 1$

Note: this topological winding is "protected by symmetry"

$d_z = 0$ (called "chiral symmetry"
or "sublattice symmetry")

If we allow $d_z \neq 0$

"pull the contour off"

$\rightarrow$ ν not robust

This is an example of a "symmetry protected topological state"

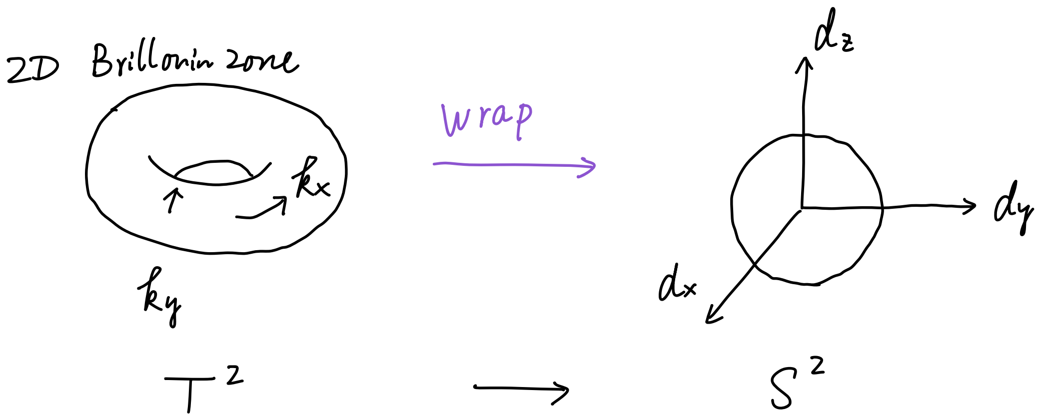
Disorder that breaks this symmetry (e.g. $m_1 \neq m_2$) will distroy the topological protection of the edge states.

- The case of 2D lattices:

What controls the edge states is the "Chern number"

$$C = \frac{1}{2\pi} \int_{2D\ BZ} d^2\vec{k}\ B(\vec{k})$$

$$= \text{degree of map } \vec{k} \to \vec{d}(\vec{k})$$



How many different ways (equivalence class) can we wrap T^2 onto S^2 ?

This is described by "homotopy theory"

In this case $\to$ all integers $\mathbb{Z}$

C of 2D systems : integers.

Maxwell lattice topological mechanics and topological soft modes in disordered materials

Xiaoming Mao
University of Michigan

Outline

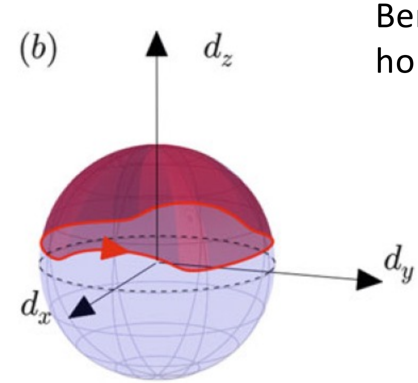
- Recap from Lectures2&3
- Selected examples of $\omega > 0$ TMMs (topological mechanical metamaterials)
- Maxwell lattice TMMs
  - Fundamental theories (recap from Tom's lectures)
  - Selected new advances
- Aperiodic Maxwell network topological mechanics
  - Fiber networks
  - Reciprocal diagrams and a mechanical duality theorem
  - Quasicrystals

Recap from Lectures 2&3

- Basic concepts of topology and fiber bundle models
- 1D mechanical chain
  - Bulk spectra, gap
  - Band inversion and winding number
  - Edge states
- General 2-band models
  - Dirac Hamiltonian
  - The Berry phase and the Chern number

$$\hat{H}(\mathbf{d}) = d_x\hat{\sigma}_x + d_y\hat{\sigma}_y + d_z\hat{\sigma}_z = \mathbf{d}\cdot\hat{\boldsymbol{\sigma}}$$

map



Berry curvature is homogeneous on this sphere

$$\mathbf{B}^{\pm}(\mathbf{d}) = \pm\frac{\mathbf{d}}{|\mathbf{d}|}\frac{1}{2\mathbf{d}^2}$$

$$\gamma_{-}(\mathscr{C}) = \frac{1}{2}\Omega_{\mathscr{C}}$$

Any 2-band model $\widehat{H}(\mathbf{k})$

Chern number = degree of mapping

Main references:

- Topological insulators and geometry of vector bundles, Sergeev, arXiv:2011.05004 [cond-mat.mes-hall]
- A short course on topological insulators, Asboth, Oroszlany, Palyi, Lecture Notes in Physics 919, Springer (2015)

2D TMM: gyroscope lattice

Topological mechanics of gyroscopic metamaterials

Lisa M. Nash[a], Dustin Kleckner[a], Alismari Read[a], Vincenzo Vitelli[b], Ari M. Turner[c,1], and William T. M. Irvine[a,1]

[a]James Franck Institute and Department of Physics, The University of Chicago, Chicago, IL 60637; [b]Instituut-Lorentz for Theoretical Physics, Universiteit Leiden, 2300 RA Leiden, The Netherlands; and [c]Department of Physics and Astronomy, Johns Hopkins University, Baltimore, MD 21218

Edited by David A. Weitz, Harvard University, Cambridge, MA, and approved October 13, 2015 (received for review April 17, 2015)

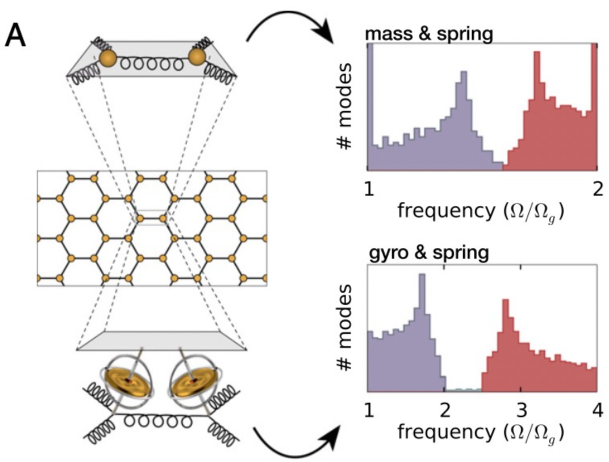
PRL **115,** 104302 (2015) PHYSICAL REVIEW LETTERS week ending 4 SEPTEMBER 2015

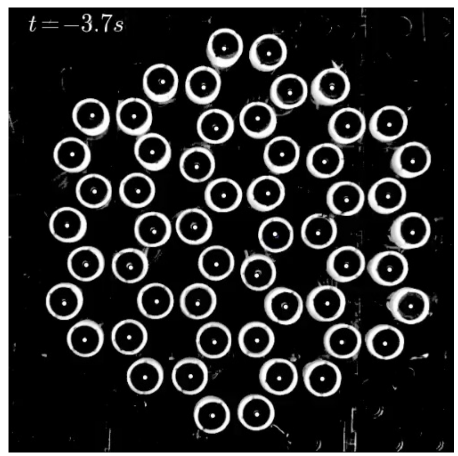
Topological Phononic Crystals with One-Way Elastic Edge Waves

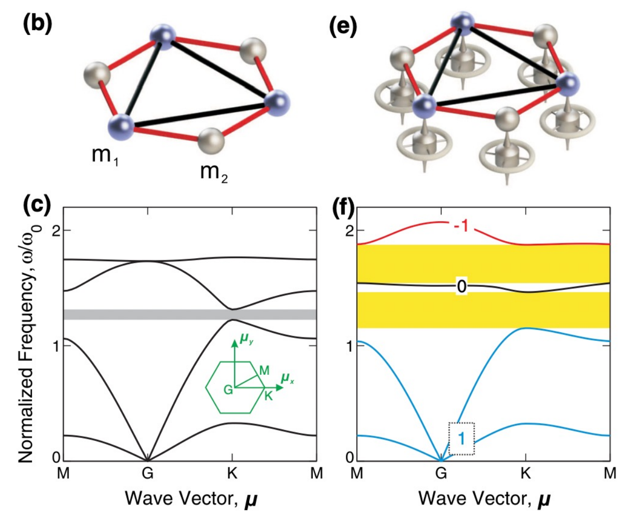
Pai Wang,[1] Ling Lu,[2] and Katia Bertoldi[1,3,*]

[1]Harvard John A. Paulson School of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts 02138, USA
[2]Department of Physics, MIT, Cambridge, Massachusetts 02139, USA
[3]Kavli Institute, Harvard University, Cambridge, Massachusetts 02138, USA

(Received 3 April 2015; published 4 September 2015)







2D TMM: gyroscope lattice

EOM of one gyroscope ($\vec{\ell}$ is the principal axis of gyroscope): $\dot{\vec{\ell}} = \frac{\ell^2}{I\omega}(\hat{\ell}\times\vec{F})$

Complex notation: $\psi \rightarrow \delta_x + i\delta_y$ Site displacements

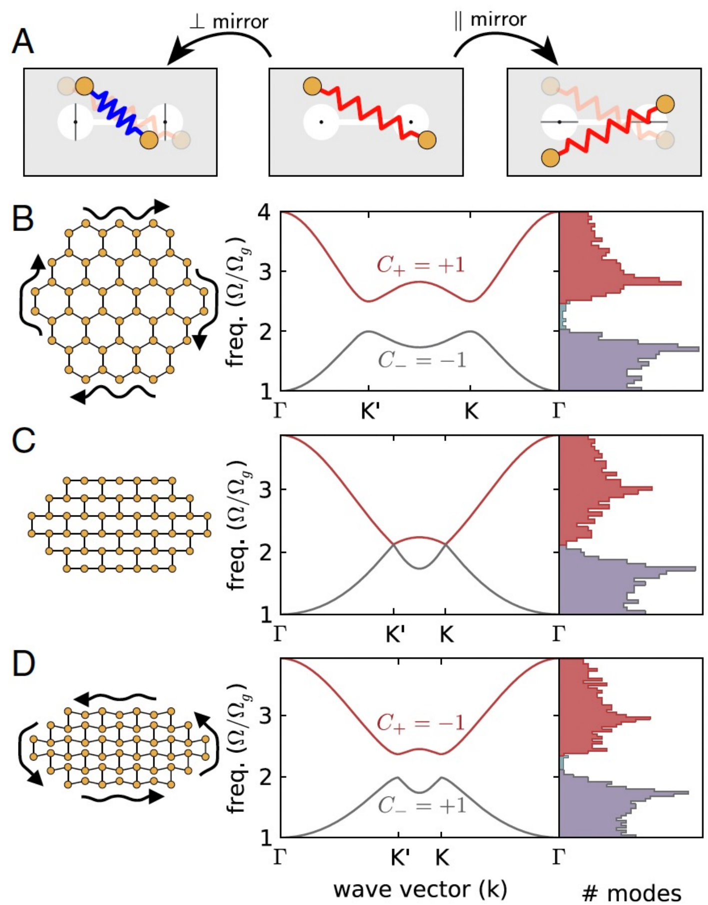
$F \rightarrow F_x + iF_y$ Site forces

EOM for lattice site p:

$$i\frac{d\psi_p}{dt} = \Omega_g\psi_p + \frac{\Omega_k}{2}\sum_q^{n.n.}\left[\left(\psi_p - \psi_q\right) + e^{2i\theta_{pq}}\left(\psi_p^* - \psi_q^*\right)\right]$$

4*4 Hamiltonian $H(k)$

2 middle bands



Gap permitted by broken time-reversal symmetry

2D Valley Hall TMMs: without breaking TRS

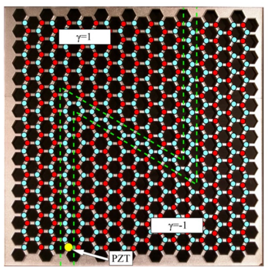
PHYSICAL REVIEW B **96**, 134307 (2017)

Observation of topological valley modes in an elastic hexagonal lattice

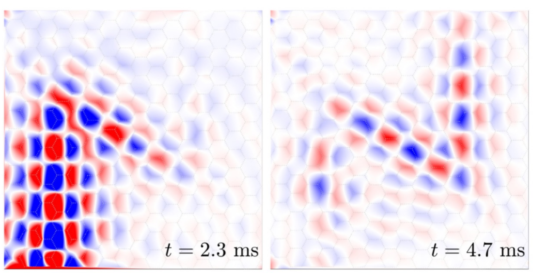
Javier Vila* and Raj Kumar Pal
School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

Massimo Ruzzene
School of Aerospace Engineering and School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

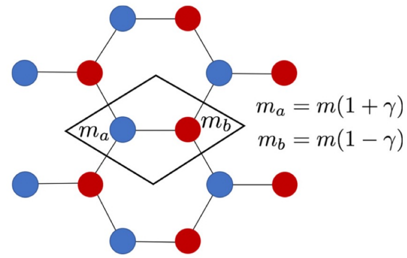
(Received 14 May 2017; revised manuscript received 29 August 2017; published 17 October 2017)



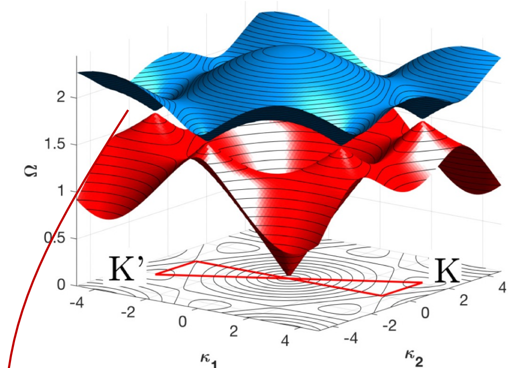
(a)



(b) (c)

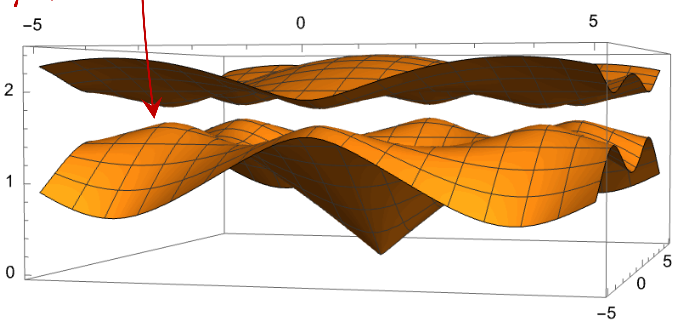


(a)



(b)

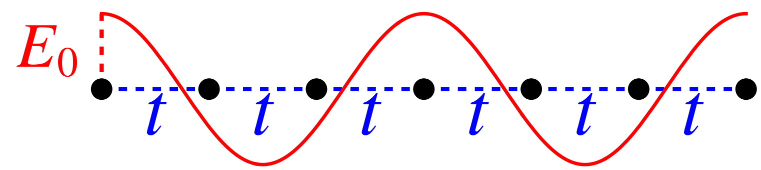
Dirac cones open when $\gamma \neq 0$



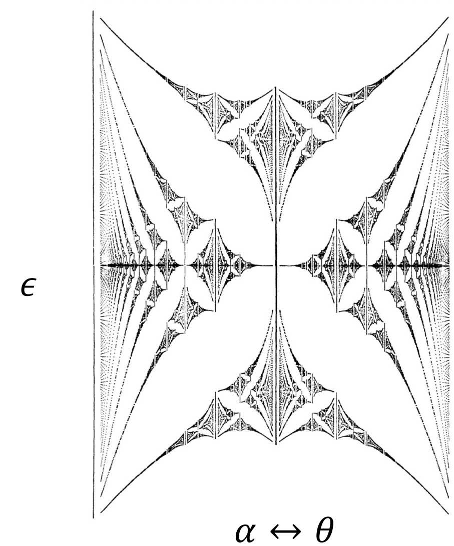
AAH (Aubry-Andre-Harper) Model: Concept

Electron tight-binding model with a periodic potential

$$-t\,\psi_{n+1} - t\,\psi_{n-1} - E_0\cos(n\theta+\phi)\psi_n = E\psi_n$$



“Hofstadter butterfly”:



Maps to 2D quantum Hall states on a lattice

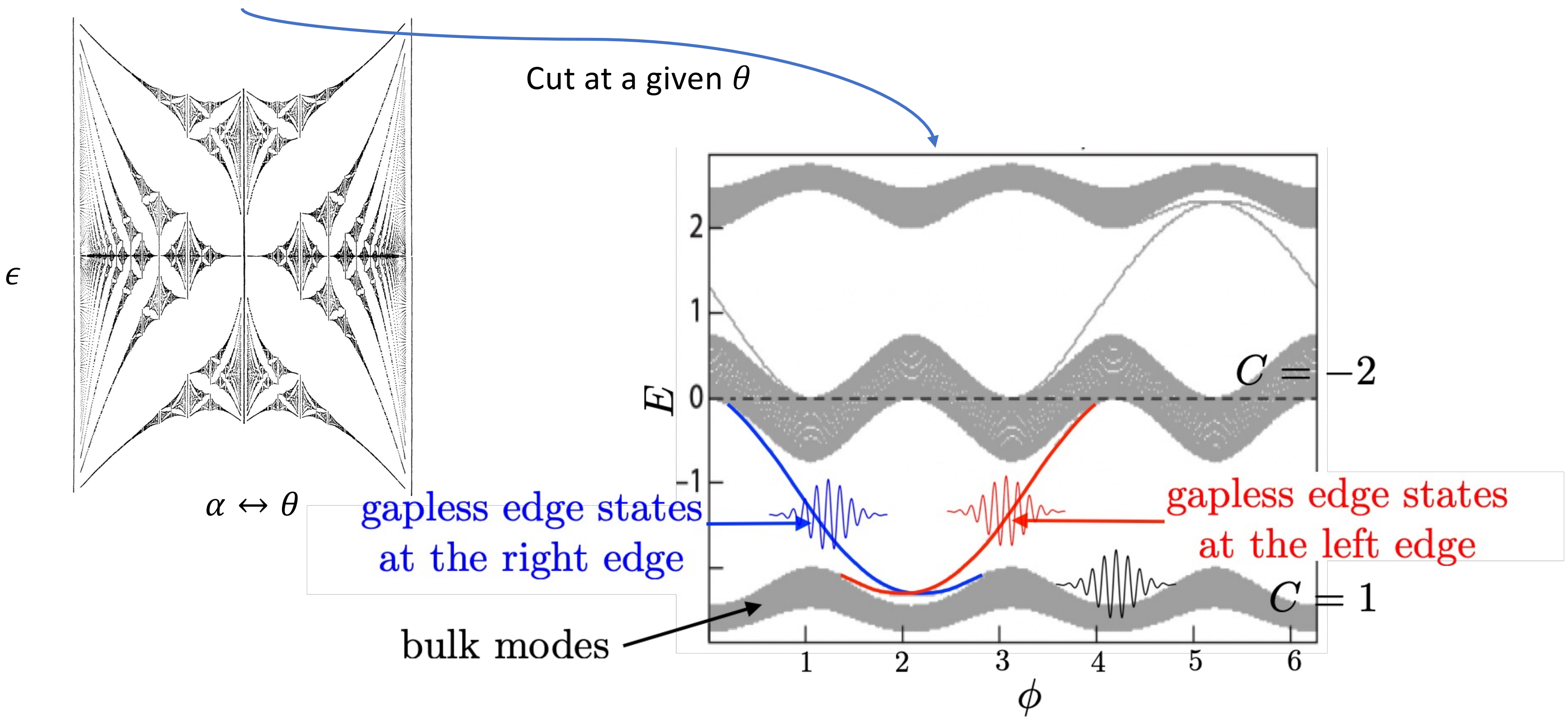
$$g(m+1) + g(m-1) + 2\cos(2\pi m\alpha - \nu)g(m) = \epsilon g(m)$$

Momentum in y

Magnetic field $\alpha = a^2 H/2\pi(\hbar c/e)$

AAH Model: Adiabatic Pumping

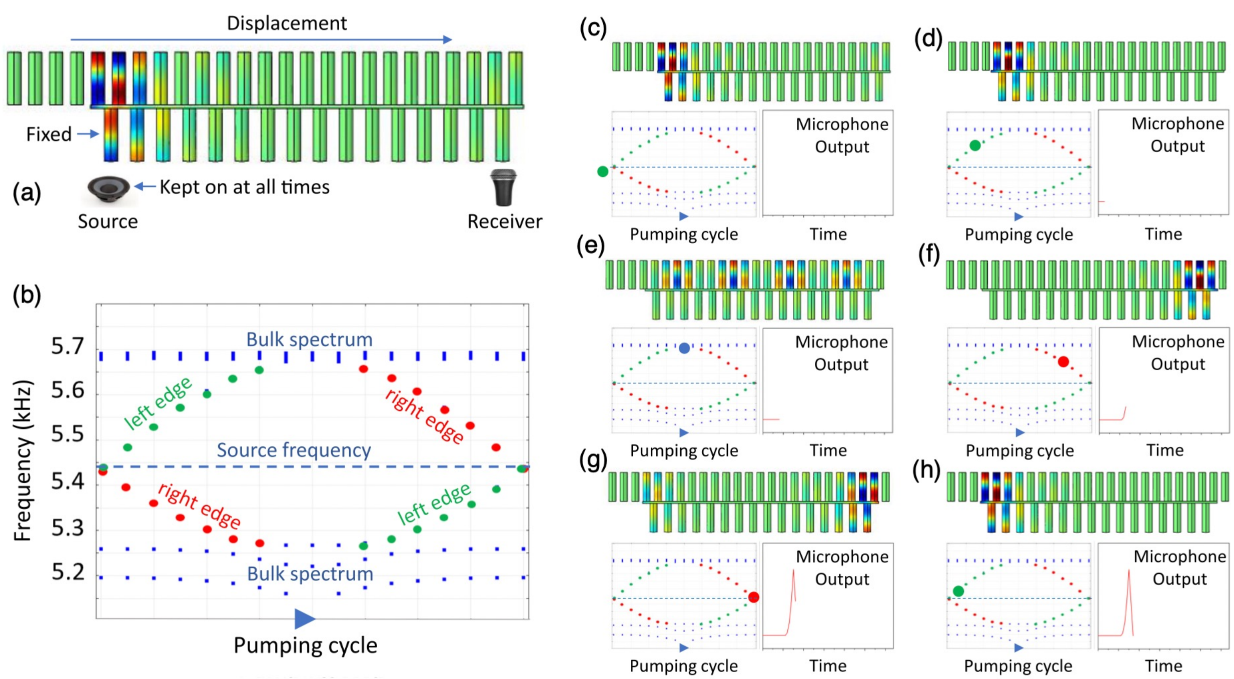
- Adiabatic pumping in AAH model $-t\,\psi_{n+1} - t\,\psi_{n-1} - E_0 \cos(n\theta + \phi)\psi_n = E\psi_n$



AAH Model: Experiment

Experimental Demonstration of Dynamic Topological Pumping across Incommensurate Bilayered Acoustic Metamaterials

Wenting Cheng,[1] Emil Prodan,[2,*] and Camelia Prodan[1,†]
[1]Department of Physics, New Jersey Institute of Technology, Newark, New Jersey 07102, USA
[2]Department of Physics, Yeshiva University, New York, New York 10016, USA



More TMMs

- TMM is an explosive field. I've left out $10^2 \sim 10^3$ papers
- I only showed you a tiny tiny fraction of all topological states
- Good reviews on topological mechanics:
  - Huber, Sebastian D. "Topological mechanics." *Nature Physics*12.7 (2016): 621-623.
  - Zheng, Shengjie, Guiju Duan, and Baizhan Xia. "Progress in Topological Mechanics." *Applied Sciences* 12.4 (2022): 1987.
  - Xin, Li, et al. "Topological mechanical metamaterials: A brief review." *Current Opinion in Solid State and Materials Science*24.5 (2020): 100853.
  - Yi, Chen, et al. "Research progress of elastic topological materials." *力学进展* 51.2 (2021): 189-256.

Maxwell lattice TMMs

- Maxwell lattices
  - DOF = constraints in the bulk
  - $C(q)$ and $Q(q)$ matrices are square

$$\boldsymbol{C} \cdot \vec{u} = 0$$
$$\boldsymbol{Q} \cdot \vec{t} = 0$$

$$\mathcal{H} = \begin{pmatrix} 0 & Q \\ Q^{\mathrm{T}} & 0 \end{pmatrix}; \quad \mathcal{H}^2 = \begin{pmatrix} QQ^{\mathrm{T}} & 0 \\ 0 & Q^{\mathrm{T}}Q \end{pmatrix}$$

Kane-Lubensky topological index for Maxwell lattices

$$\boxed{n_i = \frac{1}{2\pi i} \oint_{C_i} d\mathbf{k} \cdot \mathrm{Tr}[\mathrm{Q}(\mathbf{k})^{-1} \nabla_{\mathbf{k}} \mathrm{Q}(\mathbf{k})] = \frac{1}{2\pi} \oint_{C_i} d\mathbf{k} \cdot \nabla_{\mathbf{k}} \phi(\mathbf{k})}$$

Eigenstates: $\psi = \begin{pmatrix} t \\ u \end{pmatrix}$ Bond tensions / Site displacements

Corresponding edge modes: ZMs and SSSs

Chiral (particle-hole) symmetry: $\{\mathcal{H}, \tau^z\} = 0$ $\quad d_z = 0$, eigenvalues appear in $\pm$ pairs

Maxwell-Calladine index theorem

$$N_0 - N_S = Nd - N_b = 0$$

Time-reversal symmetry: $\mathcal{H} = \mathcal{H}^*$

Class BDI: topological invariant in 1D $\quad n \in \mathbb{Z}$

• C. L. Kane and T. C. Lubensky, Nat. Phys. 10, 39 (2014).